Weighted Bergman spaces and the integral means spectrum of conformal mappings

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1 Introduction

The class $\mathcal{S}$. The class of univalent functions φ from the open unit disk $\mathbb{D}$ into the complex plane $\mathbb{C}$, subject to the normalizations $\varphi(0) = 0$ and $\varphi'(0) = 1$, is denoted by $\mathcal{S}$. It is classical that for $\varphi \in \mathcal{S}$, we have the distortion estimates

$$\frac{1-|z|}{(1+|z|)^3} \le |\varphi'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \qquad z \in \mathbb{D}. \tag{1.1}$$

The above-mentioned estimates are sharp, as shows the example of a rotation of the *Kœbe function*

$$\kappa(z) = \frac{z}{(1-z)^2}, \qquad z \in \mathbb{D},$$

which is in $\mathcal{S}$ and maps the disk onto the plane minus the slit $]-\infty, -\frac{1}{4}]$. After all, a simple calculation shows that

$$\kappa'(z) = \frac{1+z}{(1-z)^3}, \qquad z \in \mathbb{D}.$$

It is of interest to better understand the sets in $\mathbb{D}$ where $|\varphi'(z)|$ is either large or small. For instance, $|\kappa'(z)|$ is big near the boundary point $z = 1$, and small near $z = -1$, and elsewhere, the size is quite modest. One way to measure the average growth or decrease is to consider the *integral means*

$$\mathbf{M}_t[\varphi'](r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\varphi'\left(re^{i\theta}\right)\right|^t d\theta, \qquad 0 < r < 1,$$

where t is a real parameter. It is clear from (1.1) that

$$\mathbf{M}_t[\varphi'](r) = O\left(\frac{1}{(1-r)^\beta}\right) \text{ as } r \to 1^-, \tag{1.2}$$

holds for some positive β that depends on t. The *infimum* of all values of β for which the estimate (1.2) is valid is denoted by $\beta_\varphi(t)$. This is known as the *integral means spectral function* for φ, or simply the *integral means spectrum* of φ. The *universal integral means spectrum* for the class $\mathcal{S}$ is then defined by

$$\mathrm{B}_{\mathcal{S}}(t) = \sup_{\varphi \in \mathcal{S}} \beta_\varphi(t).$$

Each $\beta_\varphi(t)$ is a convex function of t, and therefore, $\mathrm{B}_{\mathcal{S}}(t)$ is a convex function of t as well. It is a consequence of (1.1) plus testing with $\varphi(z) = z$ that

$$0 \le \mathrm{B}_{\mathcal{S}}(t) \le \max\left\{3t, -t\right\}, \qquad t \in \mathbb{R}. \tag{1.3}$$

We call this the *trivial bound*.

For certain t, the exact values of $\mathrm{B}_{\mathcal{S}}(t)$ are known. Namely, (see [6])

$$\mathrm{B}_{\mathcal{S}}(t) = 3t - 1 \qquad \text{for} \quad \frac{2}{5} \le t < +\infty,$$

and there exists a critical value R_{CM}, $2 \le R_{\mathrm{CM}} < +\infty$ such that

$$\mathrm{B}_{\mathcal{S}}(t) = -t - 1 \qquad \text{for} \quad -\infty < t \le -R_{\mathrm{CM}},$$

whereas $-t-1 < \mathrm{B}_{\mathcal{S}}(t)$ for $-R_{\mathrm{CM}} < t < +\infty$ (see [4]). The exact value of the universal constant R_{CM} is not known. The well-known Brennan conjecture is equivalent to the statement that $R_{\mathrm{CM}} = 2$, which may also be expressed as $\mathrm{B}_{\mathcal{S}}(-2) = 1$.

The class Σ. We should also mention the related class Σ of conformal maps φ which map the external disk

$$\mathbb{D}_e = \big\{z \in \mathbb{C}_\infty : 1 < |z| \le +\infty\big\}$$

into the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ in such a way that

$$\varphi(z) = z + O(1), \qquad |z| \to +\infty.$$

It is classical that for $\varphi \in \Sigma$, we have the distortion estimates

$$\frac{|z|^2 - 1}{|z|^2} \le |\varphi'(z)| \le \frac{|z|^2}{|z|^2 - 1}, \qquad z \in \mathbb{D}_e. \tag{1.4}$$

For $\varphi \in \Sigma$, we consider the integral means

$$\mathbf{M}_t[\varphi'](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big|\varphi'\big(re^{i\theta}\big)\big|^t \, d\theta, \qquad 1 < r < +\infty,$$

and in this case, we are interested in the behavior of this quantity as $r \to 1^+$. The infimum of all β such that

$$\mathbf{M}_t[\varphi'](r) = O\left(\frac{1}{(r-1)^\beta}\right) \quad \text{as} \quad r \to 1^+$$

holds is denoted by $\beta_\varphi(t)$. And $\mathrm{B}_\Sigma(t)$ – the universal spectral function for the class Σ – is defined as the supremum of all $\beta_\varphi(t)$, where φ ranges over all elements of Σ. This function $\mathrm{B}_\Sigma(t)$ is a convex function of t, for essentially the same reasons that $\mathrm{B}_{\mathcal{S}}(t)$ is. The trivial bound of $\mathrm{B}_\Sigma(t)$ based on the pointwise estimate (1.4) is

$$0 \le \mathrm{B}_\Sigma(t) \le |t|, \qquad t \in \mathbb{R}. \tag{1.5}$$

It is known that

$$\mathrm{B}_\Sigma(t) = |t| - 1, \qquad t \in]-\infty, -R_{\mathrm{CM}}] \cup [2, +\infty[,$$

where the constant R_{CM} is the same as before, so the remaining interval $[-R_{\mathrm{CM}}, 2]$ is what should be investigated.

Comparison of spectra. By analyzing the harmonic measure of the set of points where the boundary of a simply connected set is close to the origin, Nikolai Makarov found in [11] the following relation between the two spectral functions:

$$\mathrm{B}_{\mathcal{S}}(t) = \max\big\{\mathrm{B}_\Sigma(t),\, 3t - 1\big\}, \qquad t \in \mathbb{R}. \tag{1.6}$$

We should tell the reader that Makarov's original statement deals with $\mathcal{S}_b$, the class of *bounded* conformal maps from $\mathbb{D}$ into $\mathbb{C}$ that preserve the origin, in place of the class Σ, but that these classes are sufficiently similar for the argument to carry over.

Here, we intend to study mainly the spectral function $\mathrm{B}_{\mathcal{S}}(t)$. We shall obtain estimates that are considerably better than what has been known up to this point. However, we have not been able to settle the part of the so-called Kraetzer conjecture [10] that applies to $\mathrm{B}_{\mathcal{S}}$; this conjecture claims that

$$\mathrm{B}_{\Sigma}(t) = \frac{t^2}{4}, \qquad -2 \le t \le 2.$$

Bergman space methods. We prefer to obtain a reformulation of the definition of $\beta_\varphi(t)$ for $\varphi \in \mathcal{S}$. It is easy to see that, for $-1 < \alpha < +\infty$,

$$\int_0^1 \mathbf{M}_t[\varphi'](r)\,(1-r)^\alpha \mathrm{d}r < +\infty \quad \Longrightarrow \quad \mathbf{M}_t[\varphi'](r) = O\biggl(\frac{1}{(1-r)^{\alpha+1}}\biggr) \text{ as } r \to 1^-,$$
$$\mathbf{M}_t[\varphi'](r) = O\biggl(\frac{1}{(1-r)^{\alpha+1}}\biggr) \text{ as } r \to 1^- \quad \Longrightarrow \quad \int_0^1 \mathbf{M}_t[\varphi'](r)\,(1-r)^{\alpha+\varepsilon} \mathrm{d}r < +\infty,$$

for each positive ε. For a given parameter α with $-1 < \alpha < +\infty$, we now introduce the Bergman space $\mathcal{H}_\alpha(\mathbb{D})$, consisting of those holomorphic functions f on $\mathbb{D}$ with

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z) < +\infty,$$

where we use the notation

$$\mathrm{d}A_\alpha(z) = (\alpha+1)\bigl(1-|z|^2\bigr)^\alpha \,\mathrm{d}A(z), \qquad \mathrm{d}A(z) = \frac{\mathrm{d}x\mathrm{d}y}{\pi} \ \ (z = x+iy). \tag{1.7}$$

The above expression defines a norm on $\mathcal{H}_\alpha(\mathbb{D})$ which makes it a Hilbert space. In view of the above relationships, we have the identity

$$\beta_\varphi(t) = \inf\Bigl\{\alpha+1 : \bigl(\varphi'\bigr)^{t/2} \in \mathcal{H}_\alpha(\mathbb{D})\Bigr\}. \tag{1.8}$$

We think of this as a kind of "Hilbertization" of the problem.

In this paper, we obtain estimates of the norms

$$\Bigl\|\bigl(\varphi'\bigr)^{t/2}\Bigr\|_\alpha$$

which are uniform in $\varphi \in \mathcal{S}$; in particular, this leads to estimates of the function $\mathrm{B}_{\mathcal{S}}(t)$. Our methods are Bergman space techniques in combination with the classical tools of Geometric Function Theory, such as Grönwall's area theorem. To be more precise, we exploit a generalization of the area theorem, due to Prawitz. The advantage of our method is that it permits us to encode essentially the full strength of the area-based results, rather than just a single aspect thereof, such as the classical estimate ($\varphi \in \mathcal{S}$)

$$\left|\frac{\varphi''(z)}{\varphi'(z)} - \frac{2\bar{z}}{1-|z|^2}\right| \le \frac{4}{1-|z|^2}, \qquad z \in \mathbb{D}, \tag{1.9}$$

which is a consequence of Bieberbach's inequality $\frac{1}{2}|\varphi''(0)| = |\widehat{\varphi}(2)| \le 2$.

Complex parameters in the spectral function. It is natural to consider the integral means spectral functions also for complex arguments. For complex $\tau \in \mathbb{C}$, we define the associated τ-integral means of φ' by

$$\mathbf{M}_t[\varphi'](r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Bigl|\bigl[\varphi'\bigl(re^{i\theta}\bigr)\bigr]^\tau\Bigr| \,d\theta, \qquad 0 < r < 1,$$

for $\varphi \in \mathcal{S}$, and by the same formula with $1 < r < +\infty$ for $\varphi \in \Sigma$. The definition of the power is more delicate this time, but we are saved by the fact that $\varphi'(z)$ is zero-free in the disk, and we choose – as a matter of convenience – the branch of $[\varphi'(z)]^\tau$ which gives the value 1 for $z = 0$. This allows us to define $\beta_\varphi(\tau)$ just as before, and taking the suprema over the two classes $\mathcal{S}$ and Σ, we obtain the universal integral means spectral functions $\mathrm{B}_{\mathcal{S}}(\tau)$ and $\mathrm{B}_\Sigma(\tau)$ defined over $\tau \in \mathbb{C}$. A simple analysis of these two functions shows that each is convex in the whole complex plane. Our method will supply estimates of the function $\mathrm{B}_{\mathcal{S}}(\tau)$ for complex τ, but we usually do not stress this fact.

Underlying ideas. We outline the underlying philosophy of the paper. As we began this study of integral means spectral functions, we got increasingly convinced that the topic is related to the smallness of certain operators associated to a given conformal mapping φ. For instance, we should mention that in [7], it was observed that Brennan's conjecture may be phrased as the statement that the function $|\varphi'|^{(2-p)/p}$ is an area (A_p) Muckenhoupt weight for $4/3 < p < 4$. We would then like this to mean that the logarithm of $|\varphi'|^{(2-p)/p}$ is small in a sense, like what is true for the arc length (A_p) Muckenhoupt weights, at least for $p = 2$; this is the celebrated Helson-Szegö theorem [9]. The smallness is in that case measured in terms of (part of) the BMO norm of the logarithm of the weight. What corresponds to the subspace $\mathrm{BMOA}(\mathbb{D})$ of $\mathrm{BMO}(\mathbb{T})$ (consisting of all functions whose Poisson extensions to the interior are holomorphic) in the case when arc length of replaced by area measure is the *Bloch space* $\mathcal{B}(\mathbb{D})$ (see, for instance, [8]) of all holomorphic functions f in $\mathbb{D}$ with

$$\|f\|_{\mathcal{B}} = \sup \Big\{ \big(1 - |z|^2\big)\, |f'(z)| : \ z \in \mathbb{D} \Big\} < +\infty;$$

the above expression is known as the *Bloch norm*. So, ideologically, we would hope to find some estimates of the Bloch norm of $\log \varphi'$ which should imply all the desired integrability properties of powers of φ'. We are of course groping in the dark here, as there is no known theorem of Helson-Szegö type that would apply in the area measure case. However, it is true that if a function $f \in \mathcal{B}(\mathbb{D})$ has sufficiently small Bloch norm, then e^f belongs to any fixed Bergman space $\mathcal{H}_\alpha(\mathbb{D})$ with $-1 < \alpha < +\infty$. It is also true that $\log \varphi' \in \mathcal{B}(\mathbb{D})$ for $\varphi \in \mathcal{S}$; this is an easy consequence of (1.9). The problem is that there is a genuine gap between the constants for the necessary and the sufficient conditions, and the only way to bridge that gap is to find an appropriate substitute for the Bloch norm as defined above. In [7], it was suggested by the first-named author, Hedenmalm, that spectral properties of a Volterra-type operator associated with $\log \varphi'$ should be relevant for the problem at hand; inspiration for this came from conversations with Alexandru Aleman. Then the second-named author, Shimorin, found that the multiplier norm of the Schwarzian derivative from the space $\mathcal{H}_\alpha(\mathbb{D})$ to $\mathcal{H}_{\alpha+4}(\mathbb{D})$ could be estimated effectively by using the area methods directly rather than going via the classical pointwise estimate

$$\left| \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2} \left[\frac{\varphi''(z)}{\varphi'(z)} \right]^2 \right| \le \frac{6}{(1 - |z|^2)^2}, \qquad z \in \mathbb{D}, \tag{1.10}$$

and that this led to a better estimate of $\mathrm{B}_{\mathcal{S}}(-1)$ and $\mathrm{B}_{\mathcal{S}}(-2)$ than what was previously known. We should mention that (1.10) also expresses in a way that $\log \varphi' \in \mathcal{B}(\mathbb{D})$, and that the multiplier norm estimate implies an estimate of the spectral radius of a Volterra-type operator associated with the Schwarzian derivative. Shimorin's work suggests that the multiplier norm of the derivative of $\log \varphi'$ from $\mathcal{H}_\alpha(\mathbb{D})$ to $\mathcal{H}_{\alpha+2}(\mathbb{D})$ is a more appropriate way to measure the size of $\log \varphi'$ than applying the usual Bloch norm. Then, by dissecting a theorem by Prawitz, which generalizes the Grönwall area theorem, we found a collection of estimates of multiplier norm type, parametrized by a real parameter θ, $0 < \theta \le 1$. Generally speaking, these estimates were the result of the application of the diagonal restriction operator on the bidisk $\mathbb{D}^2$ and the use of sharp constants in norm estimates. By adding higher order terms corresponding to the multiplicity of the zero along the diagonal, we found an estimate that was in fact an equality for all full mappings φ. Unfortunately, the vast majority of these additional terms carry information of which it is, generally speaking, hard to make *effective* use as regards the study of integral means spectra. The details of the method are presented in Sections 2, 3, and 4.

2 Area theorem type estimates

The theorem of Prawitz. Our point of departure is a theorem of Prawitz, which generalizes Grönwall's famous area theorem.

THEOREM 2.1 *Let $\varphi \in \mathcal{S}$. Then, for $0 < \theta \leq 1$, we have*

$$\int_{\mathbb{D}} \left| \varphi'(z) \left(\frac{z}{\varphi(z)} \right)^{\theta+1} - 1 \right|^2 \frac{\mathrm{d}A(z)}{|z|^{2\theta+2}} \leq \frac{1}{\theta},$$

with equality precisely for the full mappings φ.

Proof. The inequality follows from a classical inequality due to Prawitz, see [13, p. 13] (the inequality in [13] is formulated for functions of the class Σ, but a standard passage from Σ to $\mathcal{S}$ leads to the above inequality). The fact that we have an equality precisely for the full mappings is a part of Prawitz' theorem. ■

An alternative proof of this theorem is presented at the end of this section.

In Theorem 2.1,

$$\left(\frac{z}{\varphi(z)} \right)^{\theta+1} = \exp\left((\theta+1) \log \frac{z}{\varphi(z)} \right),$$

where the logarithm expression is determined uniquely by the requirements that it be holomorphic in $\mathbb{D}$ and that it assume the value 0 at $z = 0$.

A two-variable version of Prawitz' theorem. We shall try to move the special point $z = 0$ about in the disk, by the following procedure. We start with a given $\varphi \in \mathcal{S}$, and put

$$\psi(\zeta) = \frac{\varphi\left(\frac{\zeta+w}{1+\bar{w}\zeta}\right) - \varphi(w)}{(1-|w|^2)\,\varphi'(w)}, \qquad \zeta \in \mathbb{D},$$

for fixed $w \in \mathbb{D}$, which then is another element of $\mathcal{S}$. Now, we insert this ψ in place of φ in Theorem 2.1,

$$\int_{\mathbb{D}} \left| \frac{1}{\varphi'(w)} (1+\bar{w}\zeta)^{-2} \varphi'\left(\frac{\zeta+w}{1+\bar{w}\zeta} \right) \left(\frac{(1-|w|^2)\varphi'(w)\,\zeta}{\varphi\left(\frac{\zeta+w}{1+\bar{w}\zeta}\right) - \varphi(w)} \right)^{\theta+1} - 1 \right|^2 \frac{\mathrm{d}A(\zeta)}{|\zeta|^{2\theta+2}} \leq \frac{1}{\theta},$$

and we make the change of variables

$$z = \frac{\zeta+w}{1+\bar{w}\zeta} \quad \Longleftrightarrow \quad \zeta = \frac{z-w}{1-\bar{w}z}$$

in the integral. We obtain then after simplification

$$\int_{\mathbb{D}} \left| \frac{\varphi'(z)}{\varphi'(w)} \left(\frac{\varphi'(w)\,(z-w)}{\varphi(z)-\varphi(w)} \right)^{\theta+1} - \left(\frac{1-|w|^2}{1-\bar{w}z} \right)^{1-\theta} \right|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta+2}} \leq \frac{1}{\theta}\,(1-|w|^2)^{-2\theta}, \quad (2.1)$$

valid for all θ in the interval $0 < \theta \leq 1$. Let

$$\Phi_\theta(z,w) = \frac{1}{z-w}\left\{\frac{\varphi'(z)}{\varphi'(w)}\left(\frac{\varphi'(w)\,(z-w)}{\varphi(z)-\varphi(w)}\right)^{\theta+1} - 1\right\}, \qquad (z,w)\in\mathbb{D}^2,\ \ z\neq w,$$

and

$$L_\theta(z,w) = \frac{1}{z-w}\left\{1 - \left(\frac{1-|w|^2}{1-\bar{w}z}\right)^{1-\theta}\right\}, \qquad (z,w)\in\mathbb{D}^2,\ \ z\neq w.$$

We note that Φ_θ extends analytically to the whole bidisk $\mathbb{D}^2$, and that its diagonal restriction is

$$\Phi_\theta(z,z) = \frac{1-\theta}{2}\,\frac{\varphi''(z)}{\varphi'(z)}.$$

The function L_θ extends real analytically to $\mathbb{D}^2$. In view of (2.1), we have the following.

THEOREM 2.2 *Fix θ, $0 < \theta \leq 1$, and let $\varphi \in \mathcal{S}$ be arbitrary. Then, for all $w \in \mathbb{D}$,*

$$\int_{\mathbb{D}}\Big|\Phi_\theta(z,w) + L_\theta(z,w)\Big|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta}} \leq \frac{1}{\theta}\,(1-|w|^2)^{-2\theta},$$

with equality if and only if φ is a full mapping.

There are (at least) two ways to generalize Theorem 2.2. First, let $\mu \in \mathbb{C}\setminus\varphi(\mathbb{D})$. Then the function

$$\varphi_\mu(z) = \frac{\mu\varphi(z)}{\mu-\varphi(z)}$$

is again in $\mathcal{S}$ and replacing φ by φ_μ in (2.1) leads to

$$\int_{\mathbb{D}}\left|\frac{\varphi'(z)}{\varphi'(w)}\left(\frac{\mu-\varphi(w)}{\mu-\varphi(z)}\right)^{1-\theta}\left(\frac{\varphi'(w)\,(z-w)}{\varphi(z)-\varphi(w)}\right)^{\theta+1} - \left(\frac{1-|w|^2}{1-\bar{w}z}\right)^{1-\theta}\right|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta+2}} \leq \frac{1}{\theta}\,(1-|w|^2)^{-2\theta}. \quad (2.2)$$

We introduce the notation

$$\Phi_{\theta,\mu}(z,w) = \frac{1}{z-w}\left\{\frac{\varphi'(z)}{\varphi'(w)}\left(\frac{\mu-\varphi(w)}{\mu-\varphi(z)}\right)^{1-\theta}\left(\frac{\varphi'(w)\,(z-w)}{\varphi(z)-\varphi(w)}\right)^{\theta+1} - 1\right\},$$

so that

$$\Phi_{\theta,\mu}(z,w) = \left(\frac{\mu-\varphi(w)}{\mu-\varphi(z)}\right)^{1-\theta}\Phi_\theta(z,w) + \frac{1}{z-w}\left\{\left(\frac{\mu-\varphi(w)}{\mu-\varphi(z)}\right)^{1-\theta} - 1\right\}.$$

Note that as μ tends to ∞ from inside the complement of $\varphi(\mathbb{D})$,

$$\Phi_{\theta,\mu}(z,w) \to \Phi_\theta(z,w).$$

Also, $\Phi_{1,\mu}(z,w) \equiv \Phi_1(z,w)$. In terms of $\Phi_{\theta,\mu}$, estimate (2.2) can be written as follows.

THEOREM 2.3 *Fix θ, $0 < \theta \leq 1$. Let $\varphi \in \mathcal{S}$ be arbitrary, and suppose $\mu \in \mathbb{C}\setminus\varphi(\mathbb{D})$. Then, for all $w \in \mathbb{D}$,*

$$\int_{\mathbb{D}}\Big|\Phi_{\theta,\mu}(z,w) + L_\theta(z,w)\Big|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta}} \leq \frac{1}{\theta}\,(1-|w|^2)^{-2\theta},$$

with equality if and only if φ is a full mapping.

One way to spread out the effect of the point μ in Theorem 2.3 is to integrate both sides of the inequality with respect to a probability measure in the variable μ, supported on $\mathbb{C} \setminus \varphi(\mathbb{D})$. A particularly attractive choice of such a measure would be the harmonic measure for the point at the origin.

The diagonal restriction of the function $\Phi_{\theta,\mu}$ equals

$$\Phi_{\theta,\mu}(z,z) = \frac{1-\theta}{2}\left(\frac{\varphi''(z)}{\varphi'(z)} + \frac{2\,\varphi'(z)}{\mu - \varphi(z)}\right)$$

Note that the μ-average of this function with respect to the harmonic measure for the origin equals

$$\frac{1-\theta}{2}\left\{\frac{\varphi''(z)}{\varphi'(z)} + \frac{2}{z} - \frac{2\,\varphi'(z)}{\varphi(z)}\right\} = \frac{1-\theta}{2}\,\frac{d}{\mathrm{d}z}\log\frac{z^2\,\varphi'(z)}{(\varphi(z))^2}.$$

The expression $z^2\varphi'(z)/(\varphi(z))^2$ is essentially the derivative of a function from Σ, if we use the inversion map to go from $\mathbb{D}$ to $\mathbb{D}_e$. So, averaging with respect to μ in this way may lead to interesting properties for the class Σ.

Ways to extend the method. Another direction which offers a way to generalize Theorem 2.2 is obtained by starting with other initial inequalities in place of Prawitz' estimate (see Theorem 2.1). A family of such inequalities can be derived from the following theorem of de Branges (see [2]): *if ψ is a conformal map of the unit disk into itself with $\psi(0) = 0$, then the composition operator*

$$C_\psi \,:\, f \mapsto f \circ \psi$$

is contractive in the space $\mathcal{G}^\nu$ of formal Laurent-type series

$$f(z) = \sum_{n=0}^{+\infty} c_n z^{n+\nu}$$

supplied with the indefinite norm

$$\|f\|^2_{\mathcal{G}^\nu} = \sum_{n=0}^{+\infty}(n+\nu)|c_n|^2.$$

Here, ν is an arbitrary real number.

To deduce the inequality of Theorem 2.1 from this theorem of de Branges, we take an arbitrary *bounded* univalent function $\varphi \in \mathcal{S}$. Then $\phi(z) = \varphi(z)/R$ is a conformal self-map of $\mathbb{D}$ for sufficiently big values of the positive real parameter R. We pick $\nu = -\theta$ with $0 < \theta < 1$, and apply de Branges' theorem to the function $f(z) = z^{-\theta}$, while comparing ϕ to the identity mapping. The result is

$$R^{2\theta}\sum_{n=0}^{+\infty}(n-\theta)\left|\widehat{\psi}(n)\right|^2 \le -\theta \le 0,$$

where the $\widehat{\psi}(n)$ are the Taylor coefficients of the associated function ψ, defined by

$$\psi(z) = \left(\frac{\varphi(z)}{z}\right)^{-\theta}, \qquad z \in \mathbb{D}.$$

This implies that

$$\sum_{n=1}^{+\infty}\frac{1}{n-\theta}\left|(n-\theta)\,\widehat{\psi}(n)\right|^2 \le \theta\left[1 - R^{-2\theta}\right] \le \theta. \tag{2.3}$$

The estimate (2.3) can be found in a paper of Nehari [14], where it is shown that the first inequality is an equality, provided ϕ maps $\mathbb{D}$ to a region whose complement in $\mathbb{D}$ has zero area. The function

$$g(z)=\sum_{n=0}^{+\infty}(n-\theta)\,\widehat{\psi}(n)\,z^n, \qquad z\in\mathbb{D},$$

may be written in the form

$$g(z)=-\theta\,\varphi'(z)\left(\frac{\varphi(z)}{z}\right)^{-1-\theta}, \qquad z\in\mathbb{D},$$

and, moreover, we have that

$$\sum_{n=1}^{+\infty}\frac{|\widehat{g}(n)|^2}{n-\theta}=\int_{\mathbb{D}}\left|\frac{g(z)-\widehat{g}(0)}{z}\right|^2\frac{\mathrm{d}A(z)}{|z|^{2\theta}}.$$

The estimate of Theorem 2.1 for bounded $\varphi\in\mathcal{S}$ now is an easy consequence of (2.3). The general case follows by a standard approximation argument involving dilations.

A similar argument with θ in the interval $0<\theta\le 2$ leads to the following inequality:

$$\int_{\mathbb{D}}\left|\varphi'(z)\left(\frac{z}{\varphi(z)}\right)^{1+\theta}-1-(1-\theta)\,\widehat{\varphi}(2)\,z\right|^2\frac{\mathrm{d}A(z)}{|z|^{2+2\theta}}\le\frac{1}{\theta}+(\theta-1)\left|\widehat{\varphi}(2)\right|^2. \tag{2.4}$$

Here, $\widehat{\varphi}(2)=\frac{1}{2}\varphi''(0)$ is the second Taylor coefficient of φ at the origin. We expect that equality occurs in (2.4) if and only if φ is a full mapping, like it was with Prawitz' inequality (see Theorem 2.1). One may now apply the same transformations as in the proof of Theorem 2.2, and obtain another, more complicated, inequality in the spirit of (2.1). It is of course also possible to obtain inequalities of the same type for bigger values of θ, at the expense of the compactness of the expression.

3 Bergman spaces on the bidisk

For $-\infty<\alpha,\beta<+\infty$, we consider the Hilbert space $\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2)$ of all Lebesgue measurable functions on the bidisk $\mathbb{D}^2$ (modulo null functions), subject to the norm boundedness condition

$$\|f\|_{\alpha,\beta}=\left(\int_{\mathbb{D}}\int_{\mathbb{D}}|f(z,w)|^2|z-w|^{2\beta}\mathrm{d}A(z)\,\mathrm{d}A_\alpha(w)\right)^{1/2}<+\infty,$$

where $\mathrm{d}A_\alpha$ is as in (1.7). We also need the closed subspace $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ of $\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2)$ that consists of functions holomorphic in $\mathbb{D}^2$. *The space $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ is trivial unless $-1<\alpha<+\infty$.* The reproducing kernel for the space $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ will be denoted by

$$P_{\alpha,\beta}\big((z,w);(z',w')\big), \qquad (z,w),\,(z',w')\in\mathbb{D}^2;$$

it is holomorphic in (z,w), and anti-holomorphic in (z',w'). It is defined by the reproducing property

$$f(z,w)=\int_{\mathbb{D}}\int_{\mathbb{D}}P_{\alpha,\beta}\big((z,w);(z',w')\big)\,f(z',w')\,|z'-w'|^{2\beta}\mathrm{d}A(z')\,\mathrm{d}A_\alpha(w'),$$

for all $(z,w)\in\mathbb{D}^2$ and $f\in\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$. In case $\beta=0$, it is given by the explicit formula

$$P_{\alpha,0}\big((z,w);(z',w')\big)=\frac{1}{(1-z\bar{z}')^2(1-w\bar{w}')^{\alpha+2}}, \qquad (z,w),\,(z',w')\in\mathbb{D}^2.$$

Associated with a kernel $T = T_{\alpha,\beta}$ of the variables $((z,w);(z',w')) \in \mathbb{D}^2 \times \mathbb{D}^2$, we have an operator on $\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2)$ defined by

$$\mathbf{T}_{\alpha,\beta} f(z,w) = \int_{\mathbb{D}}\int_{\mathbb{D}} T_{\alpha,\beta}\big((z,w);(z',w')\big)\, f(z',w')\, |z'-w'|^{2\beta}\, \mathrm{d}A(z')\, \mathrm{d}A_\alpha(w'),$$

for $(z,w) \in \mathbb{D}^2$, which is going to be bounded in all cases we shall consider. For instance, associated with the kernel $P_{\alpha,\beta}$ is the operator $\mathbf{P}_{\alpha,\beta}$ which effects the orthogonal projection $\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2) \to \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$.

Let $N = 0,1,2,3,\ldots$ be a nonnegative integer, and consider the closed subspace $\mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2)$ of $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ consisting of functions with

$$f(z,w) = O\big(|z-w|^N\big)$$

near the diagonal. These functions vanish up to degree N along the diagonal, and are holomorphically divisible by $(z-w)^N$. For $N=0$, we have

$$\mathcal{H}_{\alpha,\beta;0}(\mathbb{D}^2) = \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2);$$

more generally, for $N = 1,2,3,\ldots$,

$$\mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2) = \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2) \qquad \text{if} \quad -\infty < \beta + N \le 0.$$

Being a closed subspace of the Hilbert space $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$, the subspace $\mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2)$ has a reproducing kernel function, denoted

$$P_{\alpha,\beta;N}\big((z,w);(z',w')\big), \qquad (z,w),\,(z',w') \in \mathbb{D}^2.$$

Associated to the kernel is the orthogonal projection

$$\mathbf{P}_{\alpha,\beta;N} : \mathcal{L}_{\alpha,\beta}(\mathbb{D}^2) \to \mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2).$$

The following is an important observation.

PROPOSITION 3.1 *For $-1 < \alpha, \beta < +\infty$, we have*

$$P_{\alpha,\beta;N}\big((z,w);(z',w')\big) = (z-w)^N(\bar{z}'-\bar{w}')^N\, P_{\alpha,\beta+N}\big((z,w);(z',w')\big),$$

for $(z,w),\,(z',w') \in \mathbb{D}^2$.

Proof. We note that multiplication by $(z-w)^N$ is an isometric isomorphism

$$\mathcal{H}_{\alpha,\beta+N}(\mathbb{D}^2) \to \mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2);$$

from this, the conclusion is immediate. ■

For $N = 0,1,2,3,\ldots$, consider the Hilbert space

$$\mathcal{I}_{\alpha,\beta;N}(\mathbb{D}^2) = \mathcal{H}_{\alpha,\beta;N}(\mathbb{D}^2) \ominus \mathcal{H}_{\alpha,\beta;N+1}(\mathbb{D}^2).$$

Its reproducing kernel has the form

$$Q_{\alpha,\beta;N}\big((z,w);(z',w')\big) = P_{\alpha,\beta;N}\big((z,w);(z',w')\big) - P_{\alpha,\beta;N+1}\big((z,w);(z',w')\big), \tag{3.1}$$

and the associated operator projects orthogonally

$$\mathbf{Q}_{\alpha,\beta;N} : \mathcal{L}_{\alpha,\beta}(\mathbb{D}^2) \to \mathcal{I}_{\alpha,\beta;N}(\mathbb{D}^2).$$

We write $Q_{\alpha,\beta}$ for the special kernel $Q_{\alpha,\beta;0}$. It then follows from Proposition 3.1 that

$$Q_{\alpha,\beta;N}\big((z,w);(z',w')\big) = (z-w)^N(\bar{z}'-\bar{w}')^N\,Q_{\alpha,\beta+N}\big((z,w);(z',w')\big). \qquad (3.2)$$

The fact that the only function that vanishes to an infinite degree along the diagonal is the zero function implies the orthogonal decomposition

$$\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2) = \bigoplus_{N=0}^{+\infty} \mathcal{I}_{\alpha,\beta;N}(\mathbb{D}^2).$$

As a consequence, we have the decomposition of the kernel

$$\begin{aligned} P_{\alpha,\beta}\big((z,w);(z',w')\big) &= \sum_{N=0}^{+\infty} Q_{\alpha,\beta;N}\big((z,w);(z',w')\big) \\ &= \sum_{N=0}^{+\infty} (z-w)^N(\bar{z}'-\bar{w}')^N\,Q_{\alpha,\beta+N}\big((z,w);(z',w')\big). \qquad (3.3) \end{aligned}$$

and the norm decomposition

$$\big\|\mathbf{P}_{\alpha,\beta}\,f\big\|^2_{\alpha,\beta} = \sum_{N=0}^{+\infty} \big\|\mathbf{Q}_{\alpha,\beta;N}\,f\big\|^2_{\alpha,\beta}, \qquad f \in \mathcal{L}_{\alpha,\beta}(\mathbb{D}^2). \qquad (3.4)$$

There are some natural families of unitary operators acting in spaces $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$. First, we can perform simultaneous rotations of variables z and w:

$$R_\theta[f](z,w) = f(e^{i\theta}z, e^{i\theta}w); \quad f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2); \quad \theta \in \mathbb{R}.$$

The next family of unitary operators is given by the lemma below.

LEMMA 3.2 *For each $\lambda \in \mathbb{D}$, the operator*

$$\mathbf{U}_\lambda[f](z,w) = \frac{(1-|\lambda|^2)^{\alpha/2+\beta+2}}{(1-\bar{\lambda}z)^{\beta+2}(1-\bar{\lambda}w)^{\alpha+\beta+2}}\; f\left(\frac{\lambda-z}{1-\bar{\lambda}z}, \frac{\lambda-w}{1-\bar{\lambda}w}\right)$$

is unitary on $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$, and its square is the identity: $\mathbf{U}_\lambda^2[f] = f$ for all $f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$.

Proof. This amounts to an elementary change of variables calculation. ∎

In fact, if both α and β are even integers then for each Möbius automorphism ψ of the disk $\mathbb{D}$ one can define the operator $\mathbf{U}_\psi$:

$$\mathbf{U}_\psi[f](z,w) = f\big(\psi(z),\psi(w)\big)\cdot\big(\psi'(z)\big)^{1+\beta/2}\cdot\big(\psi'(w)\big)^{1+\alpha/2+\beta/2}.$$

Then all operators $\mathbf{U}_\psi$ are unitary in $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ and the map $\psi \mapsto \mathbf{U}_\psi$ is a unitary representation of the group of Möbius automorphisms of $\mathbb{D}$.

We proceed by analyzing the reproducing kernel $P_{\alpha,\beta}$ along the diagonal.

LEMMA 3.3 *Fix $-1 < \alpha, \beta < +\infty$. We then have*

$$P_{\alpha,\beta}\big((z,w);(z',z')\big) = Q_{\alpha,\beta}\big((z,w);(z',z')\big) = \frac{\sigma(\alpha,\beta)}{(1-z\bar{z}')^{\beta+2}(1-w\bar{z}')^{\alpha+\beta+2}},$$

where the constant $\sigma(\alpha,\beta)$ is given by

$$\frac{1}{\sigma(\alpha,\beta)} = \int_{\mathbb{D}}\int_{\mathbb{D}} |z-w|^{2\beta}\,\mathrm{d}A(z)\,\mathrm{d}A_\alpha(w).$$

Proof. We note first that the fact that rotation operators R_θ are unitary in $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ implies that

$$P_{\alpha,\beta}\big((e^{i\theta}z, e^{i\theta}w); (0,0)\big) = P_{\alpha,\beta}\big((z,w); (0,0)\big).$$

Now, we observe that the only functions analytic in $\mathbb{D}^2$ and having this property are the constant functions, which follows at once by considering double power series expansions. Hence, $P_{\alpha,\beta}\big((z,w); (0,0)\big)$ is constant in (z,w), and we write

$$\sigma(\alpha,\beta) = P_{\alpha,\beta}\big((z,w); (0,0)\big) \tag{3.5}$$

for this constant. The above integral formula for $\sigma(\alpha,\beta)$ follows from the reproducing property of the kernel $P_{\alpha,\beta}\big((\cdot,\cdot); (0,0)\big)$ applied to the constant function 1.

Now, let $\lambda \in \mathbb{D}$. We pick $f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$, and note that in view of (3.5) and Lemma 3.2,

$$\begin{aligned}\big(1-|\lambda|^2\big)^{\alpha/2+\beta+2} f(\lambda,\lambda) = \mathbf{U}_\lambda[f](0,0) &= \sigma(\alpha,\beta)\left\langle \mathbf{U}_\lambda^2[f], \mathbf{U}_\lambda[1]\right\rangle_{\alpha,\beta} \\ &= \sigma(\alpha,\beta)\left\langle f, \mathbf{U}_\lambda[1]\right\rangle_{\alpha,\beta}.\end{aligned}$$

This formula expresses the reproducing identity at the diagonal point (λ,λ), which shows that

$$P_{\alpha,\beta}\big((z,w); (\lambda,\lambda)\big) = \sigma(\alpha,\beta)(1-|\lambda|^2)^{-\alpha/2-\beta-2}\mathbf{U}_\lambda[1](z,w),$$

which after some simplification gives the desired expression. ■

In view of Lemma 3.3,

$$P_{\alpha,\beta}\big((z,z); (z',z')\big) = \frac{\sigma(\alpha,\beta)}{(1-z\bar{z}')^{\alpha+2\beta+4}},$$

which we identify as the reproducing kernel for the Hilbert space coinciding as a set with the space $\mathcal{H}_{\alpha+2\beta+2}(\mathbb{D})$ from the introduction and supplied with the norm

$$\|f\|^2 = \frac{1}{\sigma(\alpha,\beta)}\int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A_{\alpha+2\beta+2}(z) = \frac{1}{\sigma(\alpha,\beta)}\|f\|^2_{\alpha+2\beta+2}.$$

Let $\oslash$ denote the operation of taking the diagonal restriction:

$$(\oslash f)(z) = f(z,z), \qquad z \in \mathbb{D}.$$

In view of the general theory of reproducing kernels (see [1] and [16]), we have the sharp estimate

$$\frac{1}{\sigma(\alpha,\beta)}\|\oslash f\|^2_{\alpha+2\beta+2} \le \|f\|^2_{\alpha,\beta}, \qquad f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2). \tag{3.6}$$

In fact, we can even determine the corresponding norm identity.

LEMMA 3.4 *We have the equality of norms*

$$\frac{1}{\sigma(\alpha,\beta)}\big\|\oslash f\big\|^2_{\alpha+2\beta+2} = \big\|\mathbf{Q}_{\alpha,\beta}f\big\|^2_{\alpha,\beta}, \qquad f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2).$$

Proof. The analysis of reproducing kernel functions that leads up to the estimate (3.6) also shows that to each $f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ there exists a $g \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ such that $\oslash g = \oslash f$ and

$$\frac{1}{\sigma(\alpha,\beta)}\|\oslash f\|^2_{\alpha+2\beta+2} = \|g\|^2_{\alpha,\beta}, \qquad f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2).$$

We decompose this g as follows:

$$g = \mathbf{Q}_{\alpha,\beta} f + \big(g - \mathbf{Q}_{\alpha,\beta} f\big) \in \mathcal{I}_{\alpha,\beta;0}(\mathbb{D}^2) + \mathcal{H}_{\alpha,\beta;1}(\mathbb{D}^2).$$

As this decomposition is orthogonal, we get

$$\big\|\mathbf{Q}_{\alpha,\beta} f\big\|^2_{\alpha,\beta} \le \big\|\mathbf{Q}_{\alpha,\beta} f\big\|^2_{\alpha,\beta} + \big\|g - \mathbf{Q}_{\alpha,\beta} f\big\|^2_{\alpha,\beta} = \Big\|\mathbf{Q}_{\alpha,\beta} f + \big(g - \mathbf{Q}_{\alpha,\beta} f\big)\Big\|^2_{\alpha,\beta} = \|g\|^2_{\alpha,\beta}.$$

The assertion now follows from the above estimates together with (3.6). ■

The constant $\sigma(\alpha,\beta)$ can be evaluated explicitly.

LEMMA 3.5 *Fix $-1 < \alpha, \beta < +\infty$. Then*

$$\frac{1}{\sigma(\alpha,\beta)} = \int_{\mathbb{D}}\int_{\mathbb{D}} |z-w|^{2\beta}\, \mathrm{d}A(z)\, \mathrm{d}A_\alpha(w) = \frac{1}{1+\beta}\, \frac{\Gamma(\alpha+2)\,\Gamma(\alpha+2\beta+3)}{\Gamma(\alpha+\beta+2)\,\Gamma(\alpha+\beta+3)}.$$

Proof. We perform the change of variables

$$\zeta = \frac{w-z}{1-\bar{w}\,z}, \qquad z = \frac{w-\zeta}{1-\bar{w}\,\zeta},$$

and replace the pair (z,w) by (ζ,w). The result is, after simplification,

$$\begin{aligned}\frac{1}{\sigma(\alpha,\beta)} &= \frac{\alpha+1}{(1+\beta)(\alpha+2\beta+3)} \sum_{n=0}^{+\infty} \frac{(\beta+2)_n(\beta+1)_n}{n!\,(\alpha+2\beta+4)_n} \\ &= \frac{\alpha+1}{(1+\beta)(\alpha+2\beta+3)}\, {}_2F_1\big(\beta+2,\beta+1;\alpha+2\beta+4;1\big),\end{aligned}$$

where ${}_2F_1$ denotes Gauss' hypergeometric function. Here, we use the standard Pochhammer notation

$$(a)_n = a(a+1)(a+2)\dots(a+n-1).$$

The assertion now follows from the well-known identity

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.7}$$

The proof is complete. ■

REMARK 3.6 It follows from Lemma 3.5 that

$$\frac{\sigma(\alpha,\beta+n)}{\sigma(\alpha,\beta)} = \frac{n+1+\beta}{1+\beta}\, \frac{(\alpha+\beta+2)_n(\alpha+\beta+3)_n}{(\alpha+2\beta+3)_{2n}}, \qquad n = 1,2,3,\dots.$$

We obtain an integral representation of the kernel $Q_{\alpha,\beta}$.

LEMMA 3.7 *Fix $-1 < \alpha, \beta < +\infty$. The kernel $Q_{\alpha,\beta}$ is given by the integral formula*

$$Q_{\alpha,\beta}\big((z,w);(z',w')\big) = \sigma(\alpha,\beta) \int_{\mathbb{D}} \frac{\mathrm{d}A_{\alpha+2\beta+2}(\xi)}{(1-\bar{\xi}z)^{\beta+2}(1-\bar{\xi}w)^{\alpha+\beta+2}(1-\xi\bar{z}')^{\beta+2}(1-\xi\bar{w}')^{\alpha+\beta+2}},$$

for $(z,w),(z',w') \in \mathbb{D}^2$.

Proof. It is enough to establish that if $\widetilde{Q}_{\alpha,\beta}$ denotes the kernel defined by the above integral formula, then it coincides with the reproducing kernel function for the space $\mathcal{I}_{\alpha,\beta;0}(\mathbb{D}^2) = \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2) \ominus \mathcal{H}_{\alpha,\beta;1}(\mathbb{D}^2)$. To this end, we first check that for each individual point $(z', w') \in \mathbb{D}^2$, the function

$$
\begin{aligned}
(z,w) \mapsto \widetilde{Q}_{\alpha,\beta}\big((z,w);(z',w')\big)& \\
= \sigma(\alpha,\beta) \int_{\mathbb{D}} &\frac{\mathrm{d}A_{\alpha+2\beta+2}(\xi)}{(1-\bar{\xi}z)^{\beta+2}(1-\bar{\xi}w)^{\alpha+\beta+2}(1-\xi\bar{z}')^{\beta+2}(1-\xi\bar{w}')^{\alpha+\beta+2}},
\end{aligned}
$$

belongs to $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2) \ominus \mathcal{H}_{\alpha,\beta;1}(\mathbb{D}^2)$. As a first step, we see that if we use the methods of Chapter 1 in [8], we can show that this function belongs to $\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2)$, and then, by inspection, it is also analytic in $\mathbb{D}^2$, and hence an element of $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$. To prove that it is orthogonal to $\mathcal{H}_{\alpha,\beta;1}(\mathbb{D}^2)$, we note that each "term"

$$
(z,w) \mapsto \frac{1}{(1-\bar{\xi}z)^{\beta+2}(1-\bar{\xi}w)^{\alpha+\beta+2}(1-\xi\bar{z}')^{\beta+2}(1-\xi\bar{w}')^{\alpha+\beta+2}},
$$

is a multiple of the element that achieves the point evaluation at the diagonal point (ξ,ξ), and therefore it is orthogonal to the subspace $\mathcal{H}_{\alpha,\beta;1}(\mathbb{D}^2)$, as these functions vanish at all diagonal points.

Now, we see, by inspection, that

$$
\widetilde{Q}_{\alpha,\beta}\big((z,z);(z',w')\big) = \frac{\sigma(\alpha,\beta)}{(1-z\bar{z}')^{\beta+2}(1-z\bar{w}')^{\alpha+\beta+2}};
$$

this follows from the reproducing property of the well-known kernel function in the space $\mathcal{H}_{\alpha+2\beta+2}(\mathbb{D})$. We note that this is the same as $Q_{\alpha,\beta}\big((z,z);(z',w')\big)$, according to Lemma 3.3. And since functions from $\mathcal{I}_{\alpha,\beta;0}$ are uniquely determined by their diagonal restrictions, we obtain $\widetilde{Q}_{\alpha,\beta} = Q_{\alpha,\beta}$. ■

PROPOSITION 3.8 *Fix $-1 < \alpha, \beta < +\infty$. Then, for $N = 0, 1, 2, 3, \ldots$, we have*

$$
\big\|\mathbf{Q}_{\alpha,\beta;N}\, f\big\|^2_{\alpha,\beta} = \frac{1}{\sigma(\alpha,\beta+N)} \left\| \oslash \left[\frac{\mathbf{P}_{\alpha,\beta;N}\, f(z,w)}{(z-w)^N}\right]\right\|^2_{\alpha+2\beta+2N+2}, \qquad f \in \mathcal{L}_{\alpha,\beta}(\mathbb{D}^2).
$$

Proof. This follows from a combination of Proposition 3.1 and Lemma 3.4. ■

In view of Lemma 3.3, we have, for $z \in \mathbb{D}$,

$$
\begin{aligned}
\oslash \left[\frac{\mathbf{P}_{\alpha,\beta;N}\, f(z,w)}{(z-w)^N}\right](z) &= \sigma(\alpha,\beta+N) \\
\times \int_{\mathbb{D}}\int_{\mathbb{D}} &\frac{(\bar{z}'-\bar{w}')^N}{(1-z\bar{z}')^{\beta+N+2}(1-z\bar{w}')^{\alpha+\beta+N+2}}\, f(z',w')\,|z'-w'|^{2\beta}\mathrm{d}A(z')\,\mathrm{d}A_\alpha(w'). \quad (3.8)
\end{aligned}
$$

We want to express this in terms of derivatives of order N of f. To this end, we note that the series expansion in (3.3) leads to

$$
\big[\boldsymbol{\partial}_z^k\, P_{\alpha,\beta}\big]\big((z,z);(z',w')\big) = \sum_{n=0}^{k} n!\,\big(\bar{z}'-\bar{w}'\big)^n \binom{k}{n} \Big[\boldsymbol{\partial}_z^{k-n}\, Q_{\alpha,\beta+n}\Big]\big((z,z);(z',w')\big), \quad (3.9)
$$

where $\boldsymbol{\partial}_z$ stands for the (partial) derivative with respect to z. Moreover, in view of Lemma 3.7,

$$\boldsymbol{\partial}_z^{k-n}\, Q_{\alpha,\beta}\big((z,w);(z',w')\big) = \sigma(\alpha,\beta)\,(\beta+2)_{k-n} \\ \times \int_{\mathbb{D}} \frac{\bar{\xi}^{k-n}\,\mathrm{d}A_{\alpha+2\beta+2}(\xi)}{(1-\bar{\xi}z)^{\beta+k-n+2}(1-\bar{\xi}w)^{\alpha+\beta+2}(1-\xi\bar{z}')^{\beta+2}(1-\xi\bar{w}')^{\alpha+\beta+2}},$$

which, when restricted to the diagonal, becomes

$$\big[\boldsymbol{\partial}_z^{k-n}\, Q_{\alpha,\beta}\big]\big((z,z);(z',w')\big) = \sigma(\alpha,\beta)\,(\beta+2)_{k-n} \\ \times \int_{\mathbb{D}} \frac{\bar{\xi}^{k-n}\,\mathrm{d}A_{\alpha+2\beta+2}(\xi)}{(1-\bar{\xi}z)^{\alpha+2\beta+k-n+4}(1-\xi\bar{z}')^{\beta+2}(1-\xi\bar{w}')^{\alpha+\beta+2}} \\ = \frac{(\beta+2)_{k-n}}{(\alpha+2\beta+4)_{k-n}}\;\boldsymbol{\partial}_z^{k-n}\,\frac{\sigma(\alpha,\beta)}{(1-z\bar{z}')^{\beta+2}(1-z\bar{w}')^{\alpha+\beta+2}}.$$

By changing β to $\beta+n$, we obtain, in view of Lemma 3.3, that

$$\big[\boldsymbol{\partial}_z^{k-n}\, Q_{\alpha,\beta+n}\big]\big((z,z);(z',w')\big) \\ = \frac{(\beta+n+2)_{k-n}}{(\alpha+2\beta+2n+4)_{k-n}}\;\boldsymbol{\partial}_z^{k-n}\Big[P_{\alpha,\beta+n}\big((z,z);(z',w')\big)\Big]. \tag{3.10}$$

Now, applying (3.9) to a function $f\in\mathcal{L}_{\alpha,\beta}(\mathbb{D}^2)$, while taking (3.10) into account, we find that

$$\oslash\big[\boldsymbol{\partial}_z^k\mathbf{P}_{\alpha,\beta}f\big](z) = \sum_{n=0}^{k} n!\begin{pmatrix}k\\n\end{pmatrix}\frac{(\beta+n+2)_{k-n}}{(\alpha+2\beta+2n+4)_{k-n}}\;\boldsymbol{\partial}_z^{k-n}\oslash\left[\frac{\mathbf{P}_{\alpha,\beta;n}\,f(z,w)}{(z-w)^n}\right](z).$$

We differentiate the above relation $N-k$ times with respect to z, and obtain

$$\boldsymbol{\partial}_z^{N-k}\oslash\big[\boldsymbol{\partial}_z^k\mathbf{P}_{\alpha,\beta}f\big](z) \\ = \sum_{n=0}^{k} n!\begin{pmatrix}k\\n\end{pmatrix}\frac{(\beta+n+2)_{k-n}}{(\alpha+2\beta+2n+4)_{k-n}}\;\boldsymbol{\partial}_z^{N-n}\oslash\left[\frac{\mathbf{P}_{\alpha,\beta;n}\,f(z,w)}{(z-w)^n}\right](z). \tag{3.11}$$

We now formulate the desired relation.

PROPOSITION 3.9 *Fix $-1<\alpha,\beta<+\infty$. For each $N=0,1,2,3,\ldots$, we have*

$$\oslash\left[\frac{\mathbf{P}_{\alpha,\beta;N}\,f(z,w)}{(z-w)^N}\right] = \sum_{k=0}^{N} a_{k,N}\;\boldsymbol{\partial}_z^{N-k}\oslash\big[\boldsymbol{\partial}_z^k\mathbf{P}_{\alpha,\beta}f\big],$$

where

$$a_{k,N} = \frac{(-1)^{N-k}}{k!(N-k)!}\;\frac{(\beta+k+2)_{N-k}}{(\alpha+2\beta+N+k+3)_{N-k}}.$$

Proof. In view of (3.11), we should verify that

$$\sum_{k=0}^{N} a_{k,N}\;\boldsymbol{\partial}_z^{N-k}\oslash\big[\boldsymbol{\partial}_z^k\mathbf{P}_{\alpha,\beta}f\big](z) \\ = \sum_{k=0}^{N}\sum_{n=0}^{k} a_{k,N}\;n!\begin{pmatrix}k\\n\end{pmatrix}\frac{(\beta+n+2)_{k-n}}{(\alpha+2\beta+2n+4)_{k-n}}\;\boldsymbol{\partial}_z^{N-n}\oslash\left[\frac{\mathbf{P}_{\alpha,\beta;n}\,f(z,w)}{(z-w)^n}\right](z) \\ = \oslash\left[\frac{\mathbf{P}_{\alpha,\beta;N}\,f(z,w)}{(z-w)^N}\right](z), \tag{3.12}$$

where $a_{k,N}$ is as above. We realize that it is enough to show that

$$\sum_{k=n}^{N} a_{k,N}\; n! \begin{pmatrix} k \\ n \end{pmatrix} \frac{(\beta+n+2)_{k-n}}{(\alpha+2\beta+2n+4)_{k-n}} = \delta_{n,N}, \qquad n=0,1,2,3,\ldots,N,$$

where the delta is the usual Kronecker symbol; as we implement the given values of the constants $a_{k,N}$, this amounts to

$$\sum_{k=n}^{N} \frac{(-1)^{N-k}}{(k-n)!\,(N-k)!} \frac{(\beta+n+2)_{k-n}(\beta+k+2)_{N-k}}{(\alpha+2\beta+2n+4)_{k-n}(\alpha+2\beta+N+k+3)_{N-k}} = \delta_{n,N},$$

for $n=0,1,2,3,\ldots,N$. We quickly verify that this is correct for $n=N$. To deal with smaller values of n, we first note that

$$(\beta+n+2)_{k-n}(\beta+k+2)_{N-k} = (\beta+n+2)_{N-n},$$

which is independent of k, so that we may factor it out, and reduce the problem to showing that

$$\sum_{k=n}^{N} \frac{(-1)^{N-k}}{(k-n)!\,(N-k)!} \frac{1}{(\alpha+2\beta+2n+4)_{k-n}(\alpha+2\beta+N+k+3)_{N-k}} = 0,$$

for $n=0,1,2,\ldots,N-1$. We compute that

$$(\alpha+2\beta+2n+4)_{k-n}(\alpha+2\beta+N+k+3)_{N-k} = \frac{(\alpha+2\beta+2n+4)_{2N-2n-1}}{(\alpha+2\beta+n+k+4)_{N-n-1}},$$

which reduces our task further to showing that

$$\sum_{k=n}^{N} \frac{(-1)^{N-k}}{(k-n)!\,(N-k)!} (\alpha+2\beta+n+k+4)_{N-n-1} = 0,$$

for $n=0,1,2,\ldots,N-1$. We introduce the variables $j=k-n$ and $N'=N-n$, and rewrite the above:

$$\sum_{j=0}^{N'} \frac{(-1)^{N'-j}}{j!\,(N'-j)!} (\alpha+2\beta+2n+j+4)_{N'-1} = 0,$$

for $n=0,1,2,\ldots$ and $N'=1,2,3,\ldots$. Next, we consider the variable

$$\lambda = \alpha+2\beta+2n+4,$$

which we shall think of as an independent variable, and we once more rewrite the above assertion:

$$\sum_{j=0}^{N'} (-1)^j \begin{pmatrix} N' \\ j \end{pmatrix} (\lambda+j)_{N'-1} = 0,$$

for $N'=1,2,3,\ldots$. The expression $q(\lambda)=(\lambda)_{N'-1}$ is a polynomial of degree $N'-1$ in λ, and

$$\sum_{j=0}^{N'} (-1)^j \begin{pmatrix} N' \\ j \end{pmatrix} q(\lambda+j)$$

is an N'-th order iterated difference, which automatically produces 0 on polynomials of degree less than N'. The assertion follows. ■

We finally obtain an expansion of the norm in $\mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$ on the bidisk in terms of "one-dimensional" norms, taken over the unit disk, analogous to the Taylor expansion along the diagonal.

COROLLARY 3.10 *For $f \in \mathcal{H}_{\alpha,\beta}(\mathbb{D}^2)$, we have the norm expansion*

$$\|f\|^2_{\alpha,\beta} = \sum_{N=0}^{+\infty} \frac{1}{\sigma(\alpha,\beta+N)} \left\| \sum_{k=0}^{N} a_{k,N}\, \boldsymbol{\partial}_z^{N-k} \oslash \left[\boldsymbol{\partial}_z^k f\right] \right\|^2_{\alpha+2\beta+2N+2},$$

where the constants are as in Lemma 3.5 and Proposition 3.9.

Proof. This results from a combination of (3.4) and Propositions 3.8 and 3.9. ■

4 The main inequality

Integration with respect to the second variable. Fix θ, $0 < \theta \le 1$, and let $\varphi \in \mathcal{S}$ be arbitrary. At times, the calculations below will be valid only for $0 < \theta < 1$, but the validity for $\theta = 1$ can usually be established easily by a simple limit argument. By Theorem 2.2, we have

$$\int_{\mathbb{D}} \left|\Phi_\theta(z,w) + L_\theta(z,w)\right|^2 \frac{\mathrm{d}A(z)}{|z-w|^{2\theta}} \le \frac{1}{\theta}\,(1-|w|^2)^{-2\theta}, \tag{4.1}$$

Let g be a function that is holomorphic in $\mathbb{D}$. Then, in view of (4.1),

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}} \left|\Phi_\theta(z,w)\, g(w) + L_\theta(z,w)\, g(w)\right|^2 |z-w|^{-2\theta}\, \mathrm{d}A(z)\, \mathrm{d}A_\alpha(w) \\ \le \frac{1}{\theta} \int_{\mathbb{D}} |g(w)|^2\, (1-|w|^2)^{-2\theta}\, \mathrm{d}A_\alpha(w) = \frac{\alpha+1}{\theta(\alpha-2\theta+1)} \|g\|^2_{\alpha-2\theta}\end{aligned} \tag{4.2}$$

(the last equality holds provided that $-1+2\theta < \alpha < +\infty$).

In what follows, we assume that $g \in \mathcal{H}_{\alpha-2\theta}(\mathbb{D})$ and $-1 < \alpha - 2\theta < +\infty$. The left hand side of (4.2) expresses the square of the norm of the function $\Phi_\theta(z,w)\, g(w) + L_\theta(z,w)\, g(w)$ in the space $\mathcal{L}_{\alpha,-\theta}(\mathbb{D}^2)$. It will be shown later that both terms of this sum belong to $\mathcal{L}_{\alpha,-\theta}(\mathbb{D}^2)$ and hence one has the following decomposition:

$$\begin{aligned}\Phi_\theta(z,w)\, g(w) + L_\theta(z,w)\, g(w) \\ = \Big\{\Phi_\theta(z,w)\, g(w) + \mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big]\Big\} + \mathbf{P}^\perp_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big],\end{aligned} \tag{4.3}$$

with the corresponding decomposition of the norm

$$\begin{aligned}\big\|\Phi_\theta(z,w)\, g(w) + L_\theta(z,w)\, g(w)\big\|^2_{\alpha,-\theta} \\ = \big\|\Phi_\theta(z,w)\, g(w) + \mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big]\big\|^2_{\alpha,-\theta} + \big\|\mathbf{P}^\perp_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big]\big\|^2_{\alpha,-\theta}.\end{aligned} \tag{4.4}$$

Here, $\mathbf{P}^\perp_{\alpha,-\theta}$ is the projection complementary to $\mathbf{P}_{\alpha,-\theta}$:

$$\mathbf{P}^\perp_{\alpha,-\theta} = \mathbf{I} - \mathbf{P}_{\alpha,-\theta} \text{ in } \mathcal{L}_{\alpha,-\theta}(\mathbb{D}^2),$$

where $\mathbf{I}$ stands for the identity operator. It follows that the inequality (4.2) assumes the form

$$\begin{aligned}\big\|\Phi_\theta(z,w)\, g(w) + \mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big]\big\|^2_{\alpha,-\theta} \\ \le \frac{\alpha+1}{\theta(\alpha-2\theta+1)} \|g\|^2_{\alpha-2\theta} - \big\|\mathbf{P}^\perp_{\alpha,-\theta}\big[L_\theta(z,w)\, g(w)\big]\big\|^2_{\alpha,-\theta}.\end{aligned} \tag{4.5}$$

The norm of a projected term. We shall find an explicit expression for the squared norm

$$\big\|\mathbf{P}^{\perp}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\big\|^2_{\alpha,-\theta}.$$

We first note that

$$\big\|\mathbf{P}^{\perp}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\big\|^2_{\alpha,-\theta} = \big\|L_\theta(z,w)\,g(w)\big\|^2_{\alpha,-\theta} - \big\|\mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\big\|^2_{\alpha,-\theta} \tag{4.6}$$

We recall the classical definition of the Gauss hypergeometric function:

$${}_2F_1(a,b;c;x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n (b)_n}{(c)_n\, n!}\, x^n,$$

where the series converges at least for complex $x \in \mathbb{D}$, unless we accidentally divide by zero in any of the terms.

LEMMA 4.1 *For fixed $w \in \mathbb{D}$, we have the identity*

$$\int_{\mathbb{D}} \big|L_\theta(z,w)\big|^2\, |z-w|^{-2\theta}\,\mathrm{d}A(z) = \frac{1}{\theta}\Big[1 - {}_2F_1\big(1-\theta,-\theta;1;|w|^2\big)\Big]\big(1-|w|^2\big)^{-2\theta}.$$

Proof. We make the change of variables

$$z = \frac{w-\zeta}{1-\bar{w}\zeta}, \qquad \zeta = \frac{w-z}{1-\bar{w}z},$$

which gives

$$\int_{\mathbb{D}} \big|L_\theta(z,w)\big|^2\, |z-w|^{-2\theta}\,\mathrm{d}A(z) = \big(1-|w|^2\big)^{-2\theta} \int_{\mathbb{D}} \left|\frac{1-(1-\bar{w}\zeta)^{\theta-1}}{\zeta}\right|^2 |\zeta|^{-2\theta}\,\mathrm{d}A(\zeta).$$

We expand the power appearing in the integrand on the right hand side as a Taylor series, and use that z^j and z^k are orthogonal in a radially weighted Bergman space whenever $j \neq k$. The expression involving the Gauss hypergeometric function then results from this. ∎

LEMMA 4.2 *For $w \in \mathbb{D}$, we have*

$${}_2F_1\big(1-\theta,-\theta;1;|w|^2\big) \geq {}_2F_1\big(1-\theta,-\theta;1;1\big) = \frac{\Gamma(2\theta+1)}{2[\Gamma(\theta+1)]^2}.$$

Proof. The inequality follows if we see that the coefficients of the Taylor series for ${}_2F_1(1-\theta,-\theta;1;x)$ are all negative except for the first one. The evaluation of ${}_2F_1(1-\theta,-\theta;1;1)$ is classical (see any book on special functions). ∎

Combining these two lemmas, we obtain the following.

PROPOSITION 4.3 *For $g \in \mathcal{H}_{\alpha-2\theta}(\mathbb{D})$, we have*

$$\begin{aligned}
&\int_{\mathbb{D}} \big|L_\theta(z,w)\,g(w)\big|^2\, |z-w|^{-2\theta}\,\mathrm{d}A(z)\,\mathrm{d}A_\alpha(w) \\
&\quad = \frac{\alpha+1}{\theta\,(\alpha-2\theta+1)} \int_{\mathbb{D}} \Big[1 - {}_2F_1\big(1-\theta,-\theta;1;|w|^2\big)\Big]\, |g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w) \\
&\qquad \leq \frac{\alpha+1}{\theta\,(\alpha-2\theta+1)} \left[1 - \frac{\Gamma(2\theta+1)}{2[\Gamma(\theta+1)]^2}\right] \|g\|^2_{\alpha-2\theta}.
\end{aligned}$$

In particular, we see that the function $L_\theta(z,w)g(w)$ is in the space $\mathcal{L}_{\alpha,-\theta}(\mathbb{D}^2)$. For later use, we need the following representation of the square of its norm:

$$\big\|L_\theta(z,w)g(w)\big\|^2_{\alpha,-\theta} = \frac{\alpha+1}{\theta(\alpha-2\theta+1)}\left(1-\frac{\Gamma(2\theta+1)}{2[\Gamma(\theta+1)]^2}\right)\|g\|^2_{\alpha-2\theta} + \frac{\alpha+1}{\theta(\alpha-2\theta+1)}$$
$$\times\int_{\mathbb{D}}\Big[{}_2F_1(1-\theta,-\theta;1;1) - {}_2F_1(1-\theta,-\theta;1;|w|^2)\Big]\,|g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w). \quad (4.7)$$

Only the first term of this sum is essential for our purposes, and the second may be considered as a contribution of "higher order terms". This is made explicit in the following lemma.

LEMMA 4.4 *There exists a positive constant $C_1 = C_1(\alpha,\theta)$ depending only on α and θ such that*

$$0 \le \int_{\mathbb{D}}\Big[{}_2F_1(1-\theta,-\theta;1;|w|^2) - {}_2F_1(1-\theta,-\theta;1;1)\Big]\,|g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w) \le C_1\,\|g\|^2_{\alpha-\theta}.$$

Proof. We use the inequality

$$1-x^n \le n^\theta(1-x)^\theta, \qquad 0\le x\le 1,$$

and the well-known asymptotics of the Pochhammer symbol

$$\frac{(1-\theta)_n}{n!} \sim \frac{n^{-\theta}}{\Gamma(1-\theta)} \quad \text{as } \ n\to+\infty,$$

to obtain

$$\begin{aligned}
0 &\le \int_{\mathbb{D}}\Big[{}_2F_1(1-\theta,-\theta;1;|w|^2) - {}_2F_1(1-\theta,-\theta;1;1)\Big]\,|g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w)\\
&= \int_{\mathbb{D}}\sum_{n=1}^{+\infty}\frac{|(-\theta)_n|(1-\theta)_n}{(n!)^2}(1-|w|^{2n})\,|g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w)\\
&\le \theta\int_{\mathbb{D}}\sum_{n=1}^{+\infty}\frac{\big[(1-\theta)_n\big]^2}{(n-\theta)(n!)^2}n^\theta\,(1-|w|^2)^\theta\,|g(w)|^2\,\mathrm{d}A_{\alpha-2\theta}(w)\\
&\le C_2(\alpha,\theta)\left(\sum_{n=1}^{+\infty}\frac{n^{-\theta}}{n-\theta}\right)\|g\|^2_{\alpha-\theta},
\end{aligned}$$

for some appropriate positive constant $C_2(\alpha,\theta)$. By putting

$$C_1(\alpha,\theta) = C_2(\alpha,\theta)\sum_{n=1}^{+\infty}\frac{n^{-\theta}}{n-\theta},$$

the assertion follows, at least for $0<\theta<1$. The remaining case $\theta=1$ is trivial. ■

It follows from Lemma 4.4 that (4.7) can be written as

$$\|L_\theta(z,w)g(w)\|^2_{\alpha,-\theta} = \frac{\alpha+1}{\theta(\alpha-2\theta+1)}\left[1-\frac{\Gamma(2\theta+1)}{2[\Gamma(\theta+1)]^2}\right]\|g\|^2_{\alpha-2\theta} + O\big(\|g\|^2_{\alpha-\theta}\big), \quad (4.8)$$

where the constant in the big "Oh" term only depends on α and θ. To proceed in our calculation of the norm of

$$\big\|\mathbf{P}^\perp_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\big\|^2_{\alpha,-\theta},$$

we should like to know the norm of the analytic projection of the function $L_\theta(z,w)\,g(w)$. We do this by calculating the norm of of each contribution in the expansion of the function around the diagonal, in accordance with (3.4) and Proposition 3.8.

PROPOSITION 4.5 *For $g \in \mathcal{H}_{\alpha-2\theta}(\mathbb{D})$, we have*

$$\oslash\left[\frac{\mathbf{P}_{\alpha,-\theta;N}\big[L_\theta(z,w)\,g(w)\big]}{(z-w)^N}\right](z) = \frac{(-1)^{N+1}(1-\theta)_{N+1}}{(N+1)!\,(\alpha+N+2-2\theta)_{N+1}}\; g^{(N+1)}(z), \qquad z\in\mathbb{D}.$$

Proof. In view of (3.8),

$$\oslash\left[\frac{\mathbf{P}_{\alpha,-\theta;N}\big[L_\theta(z,w)\,g(w)\big]}{(z-w)^N}\right](z) = \sigma(\alpha,-\theta+N)$$
$$\times\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{(\bar z'-\bar w')^N}{(1-z\bar z')^{-\theta+N+2}(1-z\bar w')^{\alpha-\theta+N+2}}\,L_\theta(z',w')\,g(w')\,|z'-w'|^{-2\theta}\mathrm{d}A(z')\,\mathrm{d}A_\alpha(w').$$

We first integrate with respect to z', that is, we compute

$$\int_{\mathbb{D}}\frac{(\bar z'-\bar w')^N}{(1-z\bar z')^{-\theta+N+2}(1-z\bar w')^{\alpha-\theta+N+2}}\,L_\theta(z',w')\,|z'-w'|^{-2\theta}\mathrm{d}A(z').$$

The change of variables

$$z'=\frac{w'+\zeta}{1+\bar w'\zeta},\qquad \zeta=\frac{w'-z'}{1-\bar w' z'},$$

leads to

$$\int_{\mathbb{D}}\frac{(\bar z'-\bar w')^N}{(1-z\bar z')^{-\theta+N+2}(1-z\bar w')^{\alpha-\theta+N+2}}\,L_\theta(z',w')\,|z'-w'|^{-2\theta}\mathrm{d}A(z')$$
$$=\bar w'\,\frac{(1-|w'|^2)^{N+1-2\theta}}{(1-z\bar w')^{\alpha-2\theta+2N+4}}\int_{\mathbb{D}}\frac{1}{\bar w'\zeta}\left[\left(1+\bar w'\zeta\right)^{\theta-1}-1\right]\left(1+\bar\zeta\,\frac{w'-z}{1-z\bar w'}\right)^{\theta-N-2}\bar\zeta^N\,\frac{\mathrm{d}A(\zeta)}{|\zeta|^{2\theta}}$$
$$=\bar w'\,\frac{(1-|w'|^2)^{N+1-2\theta}}{(1-z\bar w')^{\alpha-2\theta+2N+4}}\sum_{n=0}^{+\infty}\frac{\begin{pmatrix}\theta-1\\ N+n+1\end{pmatrix}\begin{pmatrix}\theta-N-2\\ n\end{pmatrix}}{N+n+1-\theta}\,(\bar w')^{N+n}\left(\frac{w'-z}{1-z\bar w'}\right)^n.$$

The integration with respect to w' then gives

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{(\bar z'-\bar w')^N}{(1-z\bar z')^{-\theta+N+2}(1-z\bar w')^{\alpha-\theta+N+2}}\,L_\theta(z',w')\,g(w')\,|z'-w'|^{-2\theta}\mathrm{d}A(z')\,\mathrm{d}A_\alpha(w')$$
$$=(\alpha+1)\sum_{n=0}^{+\infty}\frac{\begin{pmatrix}\theta-1\\ N+n+1\end{pmatrix}\begin{pmatrix}\theta-N-2\\ n\end{pmatrix}}{N+n+1-\theta}$$
$$\times\int_{\mathbb{D}}(\bar w')^{N+n+1}\,(w'-z)^n\,g(w')\,\frac{(1-|w'|^2)^{N+1+\alpha-2\theta}}{(1-z\bar w')^{\alpha-2\theta+2N+n+4}}\,\mathrm{d}A(w').$$

Next, we notice that by differentiating the reproducing identity for the weighted Bergman kernel k times, we obtain

$$\int_{\mathbb{D}}\frac{(\bar w')^k}{(1-z\,\bar w')^{\gamma+k+2}}\,f(w')\,\mathrm{d}A_\gamma(w')=\frac{1}{(\gamma+2)_k}\,f^{(k)}(z);$$

as we implement this into the above identity, the result is

$$\oslash\left[\frac{\mathbf{P}_{\alpha,-\theta;N}\big[L_\theta(z,w)\,g(w)\big]}{(z-w)^N}\right](z)$$
$$=(-1)^{N+1}\,(\alpha+1)\,\frac{\sigma(\alpha,-\theta+N)}{(N+1)!}\,g^{(N+1)}(z)\sum_{n=0}^{+\infty}\frac{(1-\theta)_{N+n+1}(N+1-\theta)_n}{(N+1-\theta)\,n!\,(\alpha+N-2\theta+2)_{N+n+2}},$$

so that

$$\begin{aligned}\oslash&\left[\frac{\mathbf{P}_{\alpha,-\theta;N}\big[L_\theta(z,w)\,g(w)\big]}{(z-w)^N}\right](z)\\ &=(-1)^{N+1}\,(\alpha+1)\,\frac{\sigma(\alpha,-\theta+N)}{(N+1)!}\,g^{(N+1)}(z)\frac{(1-\theta)_N}{(\alpha+N-2\theta+2)_{N+2}}\\ &\qquad\times\sum_{n=0}^{+\infty}\frac{(N+1-\theta)_n(N+2-\theta)_n}{n!\,(\alpha+2N-2\theta+4)_n}\\ &=(-1)^{N+1}\,(\alpha+1)\,\frac{\sigma(\alpha,-\theta+N)}{(N+1)!}\,g^{(N+1)}(z)\frac{(1-\theta)_N}{(\alpha+N-2\theta+2)_{N+2}}\\ &\qquad\times\;{}_2F_1\big(N+1-\theta,N+2-\theta;\alpha+2N-2\theta+4;1\big).\end{aligned}$$

If we use (3.7) as well as Lemma 3.5, the proof is completed. ■

COROLLARY 4.6 *For $g\in\mathcal{H}_{\alpha-2\theta}(\mathbb{D})$, we have*

$$\begin{aligned}&\Big\|\mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\Big\|^2_{\alpha,-\theta}\\ &\quad=\sum_{N=0}^{+\infty}\frac{1}{\sigma(\alpha,-\theta+N)}\left[\frac{(1-\theta)_{N+1}}{(N+1)!\,(\alpha+N+2-2\theta)_{N+1}}\right]^2\,\big\|g^{(N+1)}\big\|^2_{\alpha-2\theta+2N+2},\end{aligned}\tag{4.9}$$

where the constant $\sigma(\alpha,-\theta+N)$ is as in Lemma 3.5.

The next proposition is crucial for our further analysis.

PROPOSITION 4.7 $(-1<\alpha<+\infty)$ *Fix the real parameter ν, with $0<\nu\le1$. Then there exists a positive constant $C_3(\alpha,\nu)$ such that for each function $g\in\mathcal{H}_\alpha(\mathbb{D})$ and every integer $n=1,2,3,\ldots,$*

$$0\le(\alpha+2)_{2n}\|g\|^2_\alpha-\big\|g^{(n)}\big\|^2_{\alpha+2n}\le C_3(\alpha,\nu)\,n^{2\nu}(\alpha+2)_{2n}\|g\|^2_{\alpha+\nu}.$$

Proof. The first step is to note that the norm in $\mathcal{H}_\alpha(\mathbb{D})$ can be expressed as follows in terms of the Taylor coefficients:

$$\|g\|^2_\alpha=\sum_{k=0}^{+\infty}\frac{k!}{(\alpha+2)_k}\,|\widehat{g}(k)|^2.$$

We then have

$$
\begin{aligned}
&(\alpha+2)_{2n}\|g\|_\alpha^2-\left\|g^{(n)}\right\|_{\alpha+2n}^2\\
&\quad=(\alpha+2)_{2n}\sum_{k=0}^{+\infty}\frac{k!}{(\alpha+2)_k}\,|\widehat{g}(k)|^2-\sum_{k=n}^{+\infty}\frac{(k-n)!}{(\alpha+2+2n)_{k-n}}\,\big[(k-n+1)_n\big]^2|\widehat{g}(k)|^2=\\
&\qquad\qquad=(\alpha+2)_{2n}\sum_{k=0}^{+\infty}\left(1-\frac{(k-n+1)_n}{(k+\alpha+2)_n}\right)\frac{k!}{(\alpha+2)_k}|\widehat{g}(k)|^2.
\end{aligned}
$$

The assertion of the proposition follows from this identity together with the following technical inequality:

$$
0\le 1-\frac{(k-n+1)_n}{(k+\alpha+2)_n}\le C_4(\alpha)\frac{n^{2\nu}}{(k+1)^\nu},\qquad k=0,1,2,3,\dots,\;\; n=1,2,3,\dots. \tag{4.10}
$$

The left hand side of this inequality is obvious. The right hand side is also more or less obvious (with $C_4(\alpha)=1$) for $k\le n^2-1$. So, we assume that $k\ge n^2$. Then we have, by the standard properties of the logarithm function,

$$
\begin{aligned}
&1-\frac{(k-n+1)_n}{(k+\alpha+2)_n}\le\log\left[\frac{(k+\alpha+2)_n}{(k-n+1)_n}\right]=\\
&\quad=\sum_{l=1}^{n}\left[\log\left(1+\frac{\alpha+1+l}{k}\right)-\log\left(1-\frac{n-l}{k}\right)\right]\le\sum_{l=1}^{n}\left[\frac{\alpha+1+l}{k}+C_5\frac{n-l}{k}\right]\le\\
&\qquad\qquad\le C_4(\alpha)\,\frac{n^2}{k+1}\le C_4(\alpha)\left[\frac{n^2}{k+1}\right]^\nu,
\end{aligned}
$$

for appropriate values of the positive constants $C_4(\alpha)$ and C_5. We are done. ■

We are now allowed to replace $\|g^{(N+1)}\|^2_{\alpha-2\theta+2N+2}$ in each term of (4.9) by the expression

$$
(\alpha-2\theta+2)_{2N+2}\,\|g\|^2_{\alpha-2\theta},
$$

while estimating the remainder as prescribed by Proposition 4.7 with $\nu=\theta$. After some algebraic manipulations, we then arrive at

$$
\Big\|\mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\Big\|^2_{\alpha,-\theta}=\varkappa(\alpha,\theta)\|g\|^2_{\alpha-2\theta}+O\big(\|g\|^2_{\alpha-\theta}\big), \tag{4.11}
$$

where

$$
\begin{aligned}
\varkappa(\alpha,\theta)=&\ \frac{(1-\theta)\Gamma(\alpha+2)\Gamma(\alpha+2-2\theta)}{\Gamma(\alpha+2-\theta)\Gamma(\alpha+3-\theta)}\\
&\times\sum_{N=0}^{+\infty}\big(\alpha+3-2\theta+2N\big)\,\frac{(1-\theta)_N(2-\theta)_N\big[(\alpha+2-2\theta)_N\big]^2}{(\alpha+2-\theta)_N(\alpha+3-\theta)_N\big[(N+1)!\big]^2}.
\end{aligned} \tag{4.12}
$$

The series which comes from summing the estimates for the remainders converges, by the standard asymptotics of the Pochhammer symbol.

The constant $\varkappa(\alpha,\theta)$ can be expressed in terms of the generalized hypergeometric function ${}_4F_3$. We recall its definition:

$$
{}_4F_3\left(\begin{matrix}a_1 & a_2 & a_3 & a_4\\ & b_1 & b_2 & b_3\end{matrix}\,\middle|\,x\right)=1+\sum_{n=1}^{+\infty}\frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n\,n!}\,x^n,
$$

wherever the series converges. By splitting the last factor in the right hand side of (4.12) as the sum $\alpha+3-2\theta+2N=(\alpha+2-2\theta+N)+(N+1)$, we obtain

$$\varkappa(\alpha,\theta)=\frac{(1-\theta)\Gamma(\alpha+2)\Gamma(\alpha+2-2\theta)}{\Gamma(\alpha+2-\theta)\Gamma(\alpha+3-\theta)}\bigg\{\frac{(\alpha+1-\theta)(\alpha+2-\theta)}{\theta(1-\theta)(\alpha+1-2\theta)}-$$
$$-\frac{(\alpha+1-\theta)(\alpha+2-\theta)}{\theta(1-\theta)(\alpha+1-2\theta)}\,{}_4F_3\left(\begin{matrix}-\theta & 1-\theta & \alpha-2\theta+1 & \alpha-2\theta+2\\ & 1 & \alpha-\theta+1 & \alpha-\theta+2\end{matrix}\,\middle|\,1\right)$$
$$+\,{}_4F_3\left(\begin{matrix}1-\theta & 2-\theta & \alpha-2\theta+2 & \alpha-2\theta+2\\ & 2 & \alpha-\theta+2 & \alpha-\theta+3\end{matrix}\,\middle|\,1\right)\bigg\}. \tag{4.13}$$

We combine (4.6), (4.8), and (4.11), to obtain the following expression for the right hand side of (4.5):

$$\left\{\frac{(\alpha+1)\Gamma(2\theta+1)}{2\theta\,(\alpha-2\theta+1)\big[\Gamma(\theta+1)\big]^2}+\varkappa(\alpha,\theta)\right\}\|g\|^2_{\alpha-2\theta}+O\big(\|g\|^2_{\alpha-\theta}\big).$$

On the other hand, the left hand side of (4.5) may be likewise decomposed into a series by the use of Corollary 3.10 and Proposition 4.5. For $k=0,1,2,3,\ldots$, we introduce the analytic functions $\Phi_{k,\theta}$ by the formula

$$\Phi_{k,\theta}(z)=\oslash\big[\boldsymbol{\partial}_z^k\Phi_\theta\big](z),\qquad z\in\mathbb{D}.$$

We arrive at the following statement.

PROPOSITION 4.8 $(-1+2\theta<\alpha<+\infty)$ *For $g\in\mathcal{H}_{\alpha-2\theta}(\mathbb{D})$, we have*

$$\Big\|\Phi_\theta(z,w)\,g(w)+\mathbf{P}_{\alpha,-\theta}\big[L_\theta(z,w)\,g(w)\big]\Big\|^2_{\alpha,-\theta}$$
$$=\sum_{N=0}^{+\infty}\frac{1}{\sigma(\alpha,-\theta+N)}\left\|b_N\,g^{(N+1)}(z)+\sum_{k=0}^{N}a_{k,N}\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\,g(z)\big]\right\|^2_{\alpha-2\theta+2N+2},$$

where the constant $\sigma(\alpha,-\theta+N)$ is as in Lemma 3.5, and the other constants are given by

$$b_N=\frac{(-1)^{N+1}(1-\theta)_{N+1}}{(N+1)!\,(\alpha-2\theta+N+2)_{N+1}} \tag{4.14}$$

and

$$a_{k,N}=\frac{(-1)^{N-k}}{k!(N-k)!}\;\frac{(-\theta+k+2)_{N-k}}{(\alpha-2\theta+N+k+3)_{N-k}}. \tag{4.15}$$

Finally, we express the main inequality (4.5) in the following guise.

THEOREM 4.9 $(-1+2\theta<\alpha<+\infty)$ *There exists a constant $C_6(\alpha,\theta)$ depending only on θ,α, with $0<\theta\le 1$, such that for any $g\in\mathcal{H}_\alpha(\mathbb{D})$,*

$$\sum_{N=0}^{+\infty}\frac{1}{\sigma(\alpha,-\theta+N)}\left\|b_N\,g^{(N+1)}(z)+\sum_{k=0}^{N}a_{k,N}\,\boldsymbol{\partial}_z^{N-k}\big[\Phi_{k,\theta}(z)\,g(z)\big]\right\|^2_{\alpha-2\theta+2N+2}\le$$
$$\le\left[\frac{(\alpha+1)\Gamma(2\theta+1)}{2\theta(\alpha-2\theta+1)\big[\Gamma(\theta+1)\big]^2}+\varkappa(\alpha,\theta)\right]\|g\|^2_{\alpha-2\theta}+C_6(\alpha,\theta)\,\|g\|^2_{\alpha-\theta},$$

where the constants $\sigma(\alpha,N-\theta)$, b_N, $a_{k,N}$, and $\varkappa(\alpha,\theta)$ are given by Lemma 3.5 and equations (4.14), (4.15), *and* (4.13), *respectively.*

5 The algebra of φ-forms

In the classical theory of univalent functions, we frequently encounter expressions like

$$\frac{\varphi''(z)}{\varphi'(z)} \qquad \text{and} \qquad \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2}\left[\frac{\varphi''(z)}{\varphi'(z)}\right]^2,$$

where the first is known as the *logarithmic derivative of the derivative* (or the *pre-Schwarzian derivative*), and the second is known as the *Schwarzian derivative* of the given univalent function $\varphi \in \mathcal{S}$. There are higher-order expressions of a similar nature, and it seems reasonable to try to classify them.

An expression of the form

$$\frac{\varphi^{(n+1)}(z)}{\varphi'(z)},$$

with n a positive integer, is said to be a *monomial φ-form of degree n and bidegree* 1. The degree and bidegree are additive under multiplication, which means that, for instance,

$$\frac{\varphi'''(z)\varphi''(z)}{[\varphi'(z)]^2}$$

is a monomial φ-form of degree 3 and bidegree 2. We form linear combinations of φ-forms of the same degree n and the same bidegree k, and say that the resulting expression is a monomial φ-form of degree n and bidegree k. We may also form linear combinations of monomial φ-forms of the same degree n but of different bidegrees, and speak of the result as a φ-form of degree n (without a bidegree). As we form sums of monomial φ-forms of various degrees, the maximum of which is n, we get a *φ-form* with the degree n. This way, we get an *algebra of φ-forms*. As far as we are concerned, only monomial φ-forms will be of any interest.

Explicit calculation of the functions $\Phi_{k,\theta}$. We recall the formula

$$\Phi_\theta(z,w) = \frac{1}{z-w}\left\{\frac{\varphi'(z)}{\varphi'(w)}\left(\frac{\varphi(z)-\varphi(w)}{\varphi'(w)\,(z-w)}\right)^{-\theta-1} - 1\right\}, \qquad (z,w)\in\mathbb{D}^2,\ \ z\neq w.$$

We expand $\varphi(z)$ in a Taylor series about $z = w$:

$$\varphi(z) = \varphi(w) + \sum_{j=1}^{+\infty}\frac{\varphi^{(j)(w)}}{j!}\,(z-w)^j.$$

This means that

$$\frac{\varphi(z)-\varphi(w)}{\varphi'(w)\,(z-w)} = \sum_{j=1}^{+\infty}\frac{1}{j!}\,\varphi^{(j)(w)}\varphi'(w)\,(z-w)^{j-1} = 1 + \sum_{j=2}^{+\infty}\frac{1}{j!}\,\frac{\varphi^{(j)}(w)}{\varphi'(w)}\,(z-w)^{j-1},$$

which leads to

$$\begin{aligned}
\left(\frac{\varphi(z)-\varphi(w)}{\varphi'(w)\,(z-w)}\right)^{-\theta-1} &= \left[1 + \sum_{j=2}^{+\infty}\frac{1}{j!}\,\frac{\varphi^{(j)}(w)}{\varphi'(w)}\,(z-w)^{j-1}\right]^{-\theta-1} \\
&= \sum_{n=0}^{+\infty}\begin{pmatrix} -\theta-1 \\ n \end{pmatrix}\left(\sum_{j=2}^{+\infty}\frac{1}{j!}\,\frac{\varphi^{(j)}(w)}{\varphi'(w)}\,(z-w)^{j-1}\right)^n \\
&= 1 + \sum_{n=1}^{+\infty}\begin{pmatrix} -\theta-1 \\ n \end{pmatrix}(z-w)^n\left(\sum_{j=2}^{+\infty}\frac{1}{j!}\,\frac{\varphi^{(j)}(w)}{\varphi'(w)}\,(z-w)^{j-2}\right)^n.
\end{aligned}$$

We also have the Taylor series expansion for φ', which leads to

$$\frac{\varphi'(z)}{\varphi'(w)} = 1 + \sum_{k=2}^{+\infty} \frac{1}{(k-1)!} \frac{\varphi^{(k)}(w)}{\varphi'(w)} (z-w)^{k-1}.$$

As we multiply these expressions together, we obtain

$$\begin{aligned}\frac{\varphi'(z)}{\varphi'(w)} \left(\frac{\varphi(z)-\varphi(w)}{\varphi'(w)\,(z-w)}\right)^{-\theta-1} &= 1 + \sum_{k=2}^{+\infty} \frac{1}{(k-1)!} \frac{\varphi^{(k)}(w)}{\varphi'(w)} (z-w)^{k-1} \\ &\quad + \sum_{k=1}^{+\infty} \frac{1}{(k-1)!} \frac{\varphi^{(k)}(w)}{\varphi'(w)} (z-w)^{k-1} \\ &\quad \times \sum_{n=1}^{+\infty} \binom{-\theta-1}{n} (z-w)^n \left(\sum_{j=2}^{+\infty} \frac{1}{j!} \frac{\varphi^{(j)}(w)}{\varphi'(w)} (z-w)^{j-2}\right)^n,\end{aligned}$$

so that

$$\begin{aligned}\Phi_\theta(z,w) &= \sum_{k=2}^{+\infty} \frac{1}{(k-1)!} \frac{\varphi^{(k)}(w)}{\varphi'(w)} (z-w)^{k-2} \\ &\quad + \sum_{k=1}^{+\infty} \frac{1}{(k-1)!} \frac{\varphi^{(k)}(w)}{\varphi'(w)} (z-w)^{k-1} \\ &\quad \times \sum_{n=1}^{+\infty} \binom{-\theta-1}{n} (z-w)^{n-1} \left(\sum_{j=2}^{+\infty} \frac{1}{j!} \frac{\varphi^{(j)}(w)}{\varphi'(w)} (z-w)^{j-2}\right)^n.\end{aligned}$$

The next step is to note that

$$\begin{aligned}&\left(\sum_{j=2}^{+\infty} \frac{1}{j!} \frac{\varphi^{(j)}(w)}{\varphi'(w)} (z-w)^{j-2}\right)^n \\ &\qquad = \sum_{j_1,\ldots,j_n=1}^{+\infty} \frac{\varphi^{(j_1+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_1+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n} (z-w)^{j_1+\ldots+j_n-n},\end{aligned}$$

so that we get

$$\begin{aligned}\Phi_\theta(z,w) &= \sum_{l=0}^{+\infty} \frac{1}{(l+1)!} \frac{\varphi^{(l+2)}(w)}{\varphi'(w)} (z-w)^l \\ &\quad + \left\{1 + \sum_{l=1}^{+\infty} \frac{1}{l!} \frac{\varphi^{(l+1)}(w)}{\varphi'(w)} (z-w)^l\right\} \\ &\quad \times \sum_{n=1}^{+\infty} \binom{-\theta-1}{n} \sum_{j_1,\ldots,j_n=1}^{+\infty} \frac{\varphi^{(j_1+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_1+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n} (z-w)^{j_1+\ldots+j_n-1}.\end{aligned} \tag{5.1}$$

For integers k, n, with $1 \le n \le k$, we introduce the function

$$\Psi_{k,n}(z) = \sum_{(j_1,\ldots,j_n)\in I(k,n)} \frac{\varphi^{(j_1+1)}(z)\cdots\varphi^{(j_n+1)}(z)}{(j_1+1)!\cdots(j_n+1)!\,[\varphi'(z)]^n},$$

where $I(k,n)$ is the set of all n-tuples $(j_1,\ldots,j_n)$ of positive integers with $j_1+\ldots+j_n=k$. We realize that $\Psi_{k,n}(z)$ is a monomial φ-form of degree k and bidegree n. We calculate that, for instance,

$$\Psi_{k,1}(z)=\frac{\varphi^{(k+1)}(z)}{(k+1)!\,\varphi'(z)},\qquad \Psi_{k,2}(z)=\sum_{l=1}^{k-1}\frac{\varphi^{(l+1)}(z)\,\varphi^{(k-l+1)}(z)}{(l+1)!(k-l+1)!\,[\varphi'(z)]^2}.$$

PROPOSITION 5.1 *For $k=0,1,2,\ldots$, we have*

$$\Phi_{k,\theta}(z)=\oslash\big[\boldsymbol{\partial}_z^k\Phi_\theta\big](z)=(k+1-\theta)\,k!\sum_{n=1}^{k+1}\frac{(-1)^{n-1}(\theta+1)_{n-1}}{n!}\Psi_{k+1,n}(z).$$

Proof. We calculate that

$$\begin{aligned}
&\sum_{l=1}^{+\infty}\frac{1}{l!}\,\frac{\varphi^{(l+1)}(w)}{\varphi'(w)}\,(z-w)^l\\
&\qquad\times\sum_{n=1}^{+\infty}\binom{-\theta-1}{n}\sum_{j_1,\ldots,j_n=1}^{+\infty}\frac{\varphi^{(j_1+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_1+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}\,(z-w)^{j_1+\ldots+j_n}\\
&=\sum_{n=1}^{+\infty}\binom{-\theta-1}{n}\sum_{j_0,j_1,\ldots,j_n=1}^{+\infty}(j_0+1)\,\frac{\varphi^{(j_0+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_0+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}\,(z-w)^{j_0+\ldots+j_n},
\end{aligned}$$

and realize that the expression involving the sum over $j_0,\ldots,j_n$ is essentially of the same type as the sum appearing on the previous line which was over $j_1,\ldots,j_n$. By (5.1), then, the k-th order Taylor coefficient is

$$\begin{aligned}
\frac{1}{k!}\,\Phi_{k,\theta}(w)=\frac{1}{k!}\,\oslash\big[\boldsymbol{\partial}_z^k\Phi_\theta\big](w)&=\frac{1}{(k+1)!}\,\frac{\varphi^{(k+2)}(w)}{\varphi'(w)}\\
&+\sum_{n=1}^{k+1}\binom{-\theta-1}{n}\sum_{(j_1,\ldots,j_n)\in I(k+1,n)}\frac{\varphi^{(j_1+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_1+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}\\
&+\sum_{n=1}^{k}\binom{-\theta-1}{n}\sum_{(j_0,\ldots,j_n)\in I(k+1,n+1)}(j_0+1)\,\frac{\varphi^{(j_0+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_0+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}.
\end{aligned}$$

We see that

$$\begin{aligned}
&\sum_{(j_0,\ldots,j_n)\in I(k+1,n+1)}(j_0+1)\,\frac{\varphi^{(j_0+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_0+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}\\
&\qquad=\frac{n+k+2}{n+1}\sum_{(j_0,\ldots,j_n)\in I(k+1,n+1)}\frac{\varphi^{(j_0+1)}(w)\cdots\varphi^{(j_n+1)}(w)}{(j_0+1)!\cdots(j_n+1)!\,[\varphi'(w)]^n}\\
&\qquad=\frac{n+k+2}{n+1}\,\Psi_{k+1,n+1}(w),
\end{aligned}$$

which leads to the simplification

$$\begin{aligned}
\frac{1}{k!}\,\Phi_{k,\theta}(w)=\frac{1}{(k+1)!}\,\frac{\varphi^{(k+2)}(w)}{\varphi'(w)}&+\sum_{n=1}^{k+1}\binom{-\theta-1}{n}\Psi_{k+1,n}(w)\\
&+\sum_{n=1}^{k}\binom{-\theta-1}{n}\frac{n+k+2}{n+1}\,\Psi_{k+1,n+1}(w).
\end{aligned}$$

As we change the order of summation a bit, and change variables from w to z, the assertion of the proposition follows. ■

REMARK 5.2 It follows that the expression $\Phi_{k,\theta}(z)$ is a monomial φ-form of degree $k+1$.

Derivatives of powers of φ'. Let λ be a complex parameter, and consider the function

$$g_\lambda(z) = [\varphi'(z)]^\lambda = \exp\left[\lambda \log \varphi'(z)\right], \qquad z \in \mathbb{D},$$

where $\log \varphi'(z)$ takes the value 0 at $z = 0$, and is analytic throughout the disk $\mathbb{D}$. We compute that

$$g'_\lambda(z) = \lambda\, \frac{\varphi''(z)}{\varphi'(z)}\, g_\lambda(z), \tag{5.2}$$

and

$$g''_\lambda(z) = \lambda \left(\frac{\varphi'''(z)}{\varphi'(z)} + (\lambda-1)\left[\frac{\varphi''(z)}{\varphi'(z)}\right]^2 \right) g_\lambda(z). \tag{5.3}$$

Let $\Omega_{k,\lambda}(z)$ be the function defined by

$$g^{(k)}_\lambda(z) = \Omega_{k,\lambda}(z)\, g_\lambda(z), \tag{5.4}$$

which means that

$$\Omega_{1,\lambda}(z) = \lambda\, \frac{\varphi''(z)}{\varphi'(z)}, \qquad \Omega_{2,\lambda}(z) = \lambda \frac{\varphi'''(z)}{\varphi'(z)} + \lambda(\lambda-1)\left[\frac{\varphi''(z)}{\varphi'(z)}\right]^2.$$

From the rules of differentiation, we have that

$$\Omega_{k+1,\lambda}(z) = \Omega'_{k,\lambda}(z) + \lambda\, \frac{\varphi''(z)}{\varphi'(z)}\, \Omega_{k,\lambda}(z).$$

This allows us to successively calculate a few higher order factors $\Omega_{k,\lambda}(z)$, such as $\Omega_{3,\lambda}(z)$:

$$\Omega_{3,\lambda}(z) = \lambda \frac{\varphi^{(4)}(z)}{\varphi'(z)} + 3\lambda(\lambda-1)\, \frac{\varphi'''(z)\varphi''(z)}{[\varphi'(z)]^2} + \lambda(\lambda-1)(\lambda-2)\left[\frac{\varphi''(z)}{\varphi'(z)}\right]^3, \tag{5.5}$$

To obtain the formula for the general case, we use the tentative representation

$$\Omega_{k,\lambda}(z) = \sum_{n=1}^{k} (\lambda-n+1)_n \sum_{(j_1,\ldots,j_n)\in I(k,n)} c(j_1,\ldots,j_n)\, \frac{\varphi^{(j_1+1)}(z)\cdots\varphi^{(j_n+1)}(z)}{[\varphi'(z)]^n}, \tag{5.6}$$

where as before, $I(k,n)$ is the set of all n-tuples $(j_1,\ldots,j_n)$ of positive integers with $j_1+\ldots+j_n = k$. Also, we assume that the as of yet undetermined coefficients $c(j_1,\ldots,j_n)$ are invariant under permutations, so that, for instance, $c(j_1,\ldots,j_n) = c(j_n,\ldots,j_1)$. Let $\mathfrak{P}(j_1,\ldots,j_n)$ denote the collection of all (different) permutations of the given n-tuple $(j_1,\ldots,j_n)$. We begin by setting $c(1) = 1$, and we define

$$c(j_1,\ldots,j_{n-1},0) = \frac{1}{n}\, c(j_1,\ldots,j_{n-1}),$$

for positive integers $j_1,\ldots,j_{n-1}$. All the other values of the constants appearing in (5.6) are obtained iteratively from the formula

$$c(j_1,\ldots,j_n) = \frac{n}{\left|\mathfrak{P}(j_1,\ldots,j_n)\right|} \sum_{(J_1,\ldots,J_n)\in\mathfrak{P}(j_1,\ldots,j_n)} c(J_1,\ldots,J_{n-1},J_n-1),$$

where the absolute value sign is used to denote the number of elements.

REMARK 5.3 For all $k = 1,2,3,\ldots$, the expression $\Omega_{k,\lambda}$ is a monomial φ-form of degree k.

6 Estimates of the integral means spectrum

An estimate based on the first diagonal term. In this section, we shall use the first term on the left hand side of the inequality of Theorem 4.9 to obtain an estimate of the universal integral means spectrum $B_{\mathcal{S}}(\tau)$, which is of interest mainly for $\tau \in \mathbb{C}$ near the origin.

Throughout this section, we assume that φ is a sufficiently smooth function of the class $\mathcal{S}$; to make this precise, we shall suppose that φ is analytic and univalent in slightly larger disk than $\mathbb{D}$. For appropriate values of the real parameter β (β is allowed to depend on τ), we shall obtain estimates of the norms $\left\|(\varphi')^{\tau/2}\right\|_{\beta-1}$ that are uniform in φ. By a standard dilation argument, we then get the same uniform norm estimate for general $\varphi \in \mathcal{S}$ as well. In view of (1.8), this leads to the estimate $B_{\mathcal{S}}(\tau) \leq \beta$.

The following proposition is based on Theorem 4.9, with only the first term on the left hand side counted. It uses a fixed value for the parameter θ. For the formulation, we need the expression

$$K(\beta,\theta) = \frac{(\beta+2\theta)\,\Gamma(2\theta+1)}{2\,\theta\beta\,[\Gamma(\theta+1)]^2} + \varkappa(\beta+2\theta-1,\theta), \tag{6.1}$$

where the function $\varkappa$ is as in (4.12) or (4.13).

PROPOSITION 6.1 *Fix $\tau \in \mathbb{C}\setminus\{0\}$ and θ with $0<\theta<1$. Suppose that for some positive real β, the following inequality holds:*

$$K(\beta,\theta) < (1-\theta)(\beta+1)(\beta+2)\left|\frac{1}{\beta+1}-\frac{1}{\tau}\right|^2 \frac{\Gamma(\beta+1+2\theta)\,\Gamma(\beta+2)}{\Gamma(\beta+1+\theta)\,\Gamma(\beta+2+\theta)}, \tag{6.2}$$

where the function K is as above. Suppose, in addition, that

$$\left\|(\varphi')^{\tau/2}\right\|_{\beta-1+\theta} = O(1)$$

holds uniformly in $\varphi \in \mathcal{S}$. Then we also have

$$\left\|(\varphi')^{\tau/2}\right\|_{\beta-1} = O(1)$$

uniformly in $\varphi \in \mathcal{S}$. In particular, $\mathrm{B}_{\mathcal{S}}(\tau) \leq \beta$.

Proof. If we take into account only the first term of the sum on the left hand side of the inequality in Theorem 4.9, and pick $\alpha = \beta+2\theta-1$, we obtain

$$\frac{1}{\sigma(\beta+2\theta-1,-\theta)}\left\|-\frac{1-\theta}{\beta+1}\,g' + \frac{1-\theta}{2}\,\frac{\varphi''}{\varphi'}\,g\right\|^2_{\beta+1} \leq K(\beta,\theta)\,\|g\|^2_{\beta-1} + O\Big(\|g\|^2_{\beta-1+\theta}\Big), \tag{6.3}$$

for an arbitrary $g \in \mathcal{H}_{\beta-1}(\mathbb{D})$. Here, we used the fact that

$$\Phi_{0,\theta}(z) = \oslash\Phi_\theta(z) = \frac{1-\theta}{2}\,\frac{\varphi''(z)}{\varphi'(z)}, \qquad z \in \mathbb{D},$$

which is an almost trivial case of Proposition 5.1.

The next step is to apply the estimate (6.3) to the functions

$$g(z) = g_\tau(z) = \big[\varphi'(z)\big]^{\tau/2},$$

and to make the observation that

$$\frac{\varphi''(z)}{\varphi'(z)}\,g_\tau(z)=\frac{2}{\tau}\,g_\tau'(z),\qquad z\in\mathbb{D}. \tag{6.4}$$

By Proposition 4.7 (with $\nu=\theta$), we have

$$\|g'\|^2_{\beta+1}=(\beta+1)(\beta+2)\,\|g\|^2_{\beta-1}+O\big(\|g\|^2_{\beta-1+\theta}\big)$$

holds generally, so that if we combine it with the above observation and recall the formula of Lemma 3.5, we obtain from (6.3) that

$$\left\{(1-\theta)\frac{\Gamma(\beta+1+2\theta)\Gamma(\beta+2)}{\Gamma(\beta+1+\theta)\Gamma(\beta+2+\theta)}(\beta+1)(\beta+2)\left|\frac{1}{\beta+1}-\frac{1}{\tau}\right|^2-K(\beta,\theta)\right\}\|g_\tau\|^2_{\beta-1} \\ =O\big(\|g_\tau\|^2_{\beta-1+\theta}\big),$$

which implies the assertion of the proposition. ■

REMARK 6.2 A part of the assertion of Proposition 6.1, namely $\mathrm{B}_{\mathcal{S}}(\tau)\le\beta$, remains true under the weaker assumption of "$\le$" in (6.2). This is so because in the case of equality in (6.2) for given θ, β, and τ, we may move τ slightly so as to achieve "$<$". Using the continuity of the function $\mathrm{B}_{\mathcal{S}}$, the asserted inequality follows by taking the limit.

We may use the above proposition iteratively to obtain successively better bounds for the function $B_{\mathcal{S}}(\tau)$ starting from some some trivial bound, like what follows from the pointwise Kœbe-Bieberbach estimate (1.1). A more general estimate is

$$\left|\big[\varphi'(z)\big]^\tau\right|\le\frac{(1+|z|)^{2|\tau|-\operatorname{Re}\tau}}{(1-|z|)^{2|\tau|+\operatorname{Re}\tau}},\qquad z\in\mathbb{D}, \tag{6.5}$$

which works for general $\tau\in\mathbb{C}$; it is obtained if we integrate (1.9), to get

$$\left|\log\varphi'(z)+\log\big(1-|z|^2\big)\right|\le 2\log\frac{1+|z|}{1-|z|},\qquad z\in\mathbb{D},$$

and perform the appropriate algebraic manipulations. It follows from (6.5) that for fixed $\tau\in\mathbb{C}$,

$$\left\|\big[\varphi'\big]^{\tau/2}\right\|^2_{2|\tau|+\operatorname{Re}\tau-1+\varepsilon}=O(1) \tag{6.6}$$

holds uniformly in $\varphi\in\mathcal{S}$, for all positive values of ε.

A first estimate of the integral means spectrum near the origin. We apply Proposition 6.1 to obtain asymptotic bounds for the function $B_{\mathcal{S}}(t)$ for t near the origin.

PROPOSITION 6.3 *Fix a θ with $0<\theta<1$. We then have*

$$\limsup_{\mathbb{C}\ni\tau\to0}\frac{B_{\mathcal{S}}(\tau)}{|\tau|^2}\le\frac{1+\theta}{2(1-\theta)}. \tag{6.7}$$

Proof. Pick a positive ε, and let

$$\beta=\beta(\tau)=\left[\frac{1+\theta}{2(1-\theta)}+\varepsilon\right]|\tau|^2.$$

We plug this β into both sides of (6.2), and observe that the left hand side behaves like

$$\frac{2(1-\theta)\Gamma(2\theta+1)}{\Gamma(1+\theta)\Gamma(2+\theta)}\,\frac{1}{|\tau|^2}+o\left(\frac{1}{|\tau|^2}\right) \qquad \text{as} \quad \tau\to 0,$$

while the right hand side behaves like

$$\left[\frac{1+\theta}{2(1-\theta)}+\varepsilon\right]^{-1}\frac{\Gamma(2\theta+1)}{[\Gamma(\theta+1)]^2}\,\frac{1}{|\tau|^2}+o\left(\frac{1}{|\tau|^2}\right) \qquad \text{as} \quad \tau\to 0,$$

which shows that condition (6.2) is fulfilled for sufficiently small values of $|\tau|$. As the trivial estimate (6.6) show that

$$\left\|\left[\varphi'\right]^{\tau/2}\right\|^2_{\beta(\tau)-1+\theta}=O(1)$$

for sufficiently small $|\tau|$, we may apply Proposition 6.1 to deduce that

$$\left\|\left[\varphi'\right]^{\tau/2}\right\|^2_{\beta(\tau)-1}=O(1)$$

holds uniformly in φ for sufficiently small $|\tau|$. The desired assertion follows. ■

COROLLARY 6.4 *We have*

$$\limsup_{\mathbb{C}\ni\tau\to 0}\frac{B_{\mathcal{S}}(\tau)}{|\tau|^2}\le\frac{1}{2}.$$

Proof. Let $\theta\to 0^+$ in (6.7). ■

The improved estimate of the integral means spectrum near the origin. Below, we obtain a better constant instead of $\frac{1}{2}$ in the estimate of Corollary 6.4.

Naturally, if we take into account more terms of the sum in the left hand side of the inequality in Theorem 4.9, we obtain more precise information. We now analyze the estimate obtained by considering the first two terms. As in the proof of Proposition 6.1, we fix some θ with $0<\theta<1$ and some positive β, and we plug in $\alpha=\beta+2\theta-1$ and $g=g_\tau=[\varphi']^{\tau/2}$ into Theorem 4.9, throwing away all but the first two terms on the left hand side. We use Proposition 5.1 to evaluate $\Phi_{k,\theta}(z)$ for $k=0,1$, and the identity (6.4) to obtain, for $0<\beta<+\infty$,

$$\begin{aligned}(1-\theta)(\beta+1)(\beta+2)\,&\frac{\Gamma(\beta+1+2\theta)\Gamma(\beta+2)}{\Gamma(\beta+1+\theta)\Gamma(\beta+2+\theta)}\left|\frac{1}{\beta+1}-\frac{1}{\tau}\right|^2\left\|\left[\varphi'\right]^{\tau/2}\right\|^2_{\beta-1}\\ &+(2-\theta)\,\frac{\Gamma(\beta+2\theta+1)\Gamma(\beta+4)}{\Gamma(\beta+\theta+2)\Gamma(\beta+\theta+3)}\times\clubsuit\\ &\qquad\le K(\beta,\theta)\left\|\left[\varphi'\right]^{\tau/2}\right\|^2_{\beta-1}+O\Big(\left\|\left[\varphi'\right]^{\tau/2}\right\|^2_{\beta+\theta-1}\Big),\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}\clubsuit=\Bigg\|\frac{1-\theta}{2(\beta+2)(\beta+3)}\,\boldsymbol{\partial}^2\Big\{\left[\varphi'\right]^{\tau/2}\Big\}-\frac{1-\theta}{2(\beta+3)}\,\boldsymbol{\partial}\left\{\frac{\varphi''}{\varphi'}\left[\varphi'\right]^{\tau/2}\right\}&\\ +\left\{\frac{1}{6}\,\frac{\varphi'''}{\varphi'}-\frac{\theta+1}{8}\left(\frac{\varphi''}{\varphi'}\right)^2\right\}\left[\varphi'\right]^{\tau/2}\Bigg\|^2_{\beta+3}&,\end{aligned} \tag{6.9}$$

and $\boldsymbol{\partial} = \mathrm{d}/\mathrm{d}z$ stands for the operator of differentiation. As before, we first apply this inequality to estimate $B_{\mathcal{S}}(\tau)$ near the origin. We consider $\beta = \beta(\tau) = B_0\,|\tau|^2$, where B_0 is some fixed constant with $0 < B_0 < \frac{1}{2}$. We put $\theta = \theta(\tau) = 4|\tau|$, and plug these values into (6.8). By the trivial estimate (6.6), we have

$$\left\| [\varphi']^{\tau/2} \right\|^2_{\beta(\tau)+\theta(\tau)-1} = O(1),$$

uniformly in $\varphi \in \mathcal{S}$ for each fixed $\tau \in \mathbb{C}$. Then (6.8) takes the following form:

$$\begin{aligned} &\frac{2+\epsilon_1(\tau)}{|\tau|^2} \left\| [\varphi']^{\tau/2} \right\|^2_{-1+\beta(\tau)} \\ &\qquad + \big(6+\epsilon_2(\tau)\big) \left\| \left\{ \left(\frac{1}{24}+\epsilon_3(\tau)\right) \left[\frac{\varphi''}{\varphi'}\right]^2 + \epsilon_4(\tau)\,\frac{\varphi'''}{\varphi'} \right\} [\varphi']^{\tau/2} \right\|^2_{3+\beta(\tau)} \\ &\qquad\qquad \le \frac{1+\epsilon_5(\tau)}{B_0\,|\tau|^2} \left\| [\varphi']^{\tau/2} \right\|^2_{-1+\beta(\tau)} + O(1), \end{aligned} \tag{6.10}$$

where the last $O(1)$ is uniform in $\varphi \in \mathcal{S}$ for each fixed τ. For $k = 1,2,3,4,5$, the functions $\epsilon_k(\tau)$ satisfy

$$\lim_{\tau\to 0} \epsilon_k(\tau) = 0;$$

and for $k = 1,2,5$, the functions are in addition real-valued.

LEMMA 6.5 $(-1 < \alpha < +\infty)$ *There exists a positive constant $C_7(\alpha)$ such that, for any $g \in \mathcal{H}_\alpha(\mathbb{D})$,*

$$\left\| \frac{\varphi'''}{\varphi'}\, g \right\|^2_{\alpha+4} \le C_7(\alpha)\, \|g\|^2_\alpha.$$

Moreover,

$$C_7(\alpha) = O\left(\frac{1}{\alpha+1}\right)$$

as $\alpha \to -1^+$.

Proof. The assertion follows from the identity

$$\frac{\varphi'''}{\varphi'}\, g = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\varphi''}{\varphi'}\, g\right) + \left[\frac{\varphi''}{\varphi'}\right]^2 g - \frac{\varphi''}{\varphi'}\, g'$$

combined with the classical pointwise estimate (1.9) and Proposition 4.7. ■

As we apply the above lemma, we obtain from (6.10) that the inequality

$$\left\| \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2} \right\|^2_{\beta(\tau)+3} \le \frac{96}{|\tau|^2} \left(\frac{1}{B_0} - 2 + \varepsilon\right) \left\| [\varphi']^{\tau/2} \right\|^2_{\beta(\tau)-1} + O(1) \tag{6.11}$$

holds for each fixed positive ε, for sufficiently small values of $|\tau|$.

LEMMA 6.6 $(0 < \beta < +\infty)$ *For each $g \in \mathcal{H}_{\beta-1}$, we have*

$$\left\| \frac{\varphi''}{\varphi'}\, g \right\|^2_{\beta+1} \le \frac{\beta+2}{\sqrt{\beta(\beta+4)}}\, \|g\|_{\beta-1} \left\| \left[\frac{\varphi''}{\varphi'}\right]^2 g \right\|_{\beta+3}. \tag{6.12}$$

Proof. This follows from a standard application of the Cauchy-Schwarz–Bunyakovskiĭ inequality. ∎

By estimate (6.11) and Lemma 6.6, we have the following chain of inequalities (as before, $\beta(\tau) = B_0|\tau|^2$):

$$\begin{aligned}\left\|[\varphi']^{\tau/2}\right\|^2_{\beta(\tau)-1} &= \frac{|\tau|^2}{4\,(\beta(\tau)+1)(\beta(\tau)+2)} \left\|\frac{\varphi''}{\varphi'}\,[\varphi']^{\tau/2}\right\|^2_{1+\beta(\tau)} + O(1) \\ &\le \frac{|\tau|^2}{4\,(\beta(\tau)+1)\sqrt{\beta(\tau)(\beta(\tau)+4)}} \left\|\left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2}\right\|_{\beta+3} \left\|[\varphi']^{\tau/2}\right\|_{\beta(\tau)-1} + O(1) \\ &\le \frac{1+\epsilon_6(\tau)}{8\sqrt{B_0}}\sqrt{96\left(\frac{1}{B_0}-2+\varepsilon\right)}\left\|[\varphi']^{\tau/2}\right\|^2_{\beta(\tau)-1} + O\Big(\left\|[\varphi']^{\tau/2}\right\|_{\beta(\tau)-1}\Big) + O(1),\end{aligned}$$

where the function $\epsilon_6(\tau)$ is real-valued with limit $\epsilon_6(\tau) \to 0$ as $\tau \to 0$. This inequality implies that

$$\left\|[\varphi']^{\tau/2}\right\|_{\beta(\tau)-1} = O(1)$$

uniformly in $\varphi \in \mathcal{S}$, provided that

$$\frac{1+\epsilon_6(\tau)}{8\sqrt{B_0}}\sqrt{96\left(\frac{1}{B_0}-2+\varepsilon\right)} < 1.$$

We conclude that

$$\limsup_{\mathbb{C}\ni\tau\to 0} \frac{B_{\mathcal{S}}(\tau)}{|\tau|^2} \le B_0$$

holds for each real constant B_0, $0 < B_0 < \frac{1}{2}$, for which

$$96\left(\frac{1}{B_0}-2\right) < 64\,B_0.$$

By solving this last inequality for B_0, we obtain the following estimate.

THEOREM 6.7 *We have that*

$$\limsup_{\mathbb{C}\ni\tau\to 0} \frac{B_{\mathcal{S}}(\tau)}{|\tau|^2} \le \frac{\sqrt{15}-3}{2} = 0.43649\ldots \tag{6.13}$$

REMARK 6.8 The best previous estimate of this type was $B_{\mathcal{S}}(t) \le (3+\varepsilon)\,t^2$ for real t near the origin (see [15]).

An optimization method to estimate $\mathrm{B}_{\mathcal{S}}$ using two terms. Our next goal is to estimate the function $B_{\mathcal{S}}(\tau)$ using the the inequality (6.8), which employs the first two terms on the left hand side of the inequality in Theorem 4.9. This time we intend to take into account somehow all possible values of θ at the same time, rather than considering a single value at a time. This of course requires that the estimates we have obtained so far are sufficiently uniform in θ, if θ is confined to some compact interval $[\theta_0, 1]$, which is true and possible to verify without too much effort. We fix $\tau \in \mathbb{C}$ and β with $0 < \beta < +\infty$, and rewrite (6.8) as follows, using (5.2):

$$\begin{aligned}&\left\|A_1(\theta)\,\boldsymbol{\partial}^2\Big\{[\varphi']^{\tau/2}\Big\} + A_2(\theta)\left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{\tau/2}\right\|_{\beta+3} \\ &\qquad\qquad \le \left\|[\varphi']^{\tau/2}\right\|_{-1+\beta} + O\Big(\left\|[\varphi']^{\tau/2}\right\|_{-1+\beta+\theta}\Big),\end{aligned} \tag{6.14}$$

where

$$A_1(\theta) = \left[\frac{1-\theta}{2(\beta+2)(\beta+3)} - \frac{1-\theta}{\tau\,(\beta+3)} + \frac{1}{3\tau}\right] \left\{(2-\theta)\,\frac{\Gamma(\beta+2\theta+1)\,\Gamma(\beta+4)}{\Gamma(\beta+\theta+2)\,\Gamma(\beta+\theta+3)}\right\}^{\frac{1}{2}}$$
$$\times \left\{K(\beta,\theta) - (1-\theta)(\beta+1)(\beta+2)\frac{\Gamma(\beta+2\theta+1)\,\Gamma(\beta+2)}{\Gamma(\beta+\theta+1)\,\Gamma(\beta+\theta+2)}\left|\frac{1}{\beta+1} - \frac{1}{\tau}\right|^2\right\}^{-1/2}, \tag{6.15}$$

and

$$A_2(\theta) = \left[\frac{1}{6}\left(1-\frac{\tau}{2}\right) - \frac{\theta+1}{8}\right] \left\{(2-\theta)\,\frac{\Gamma(\beta+2\theta+1)\Gamma(\beta+4)}{\Gamma(\beta+\theta+2)\Gamma(\beta+\theta+3)}\right\}^{\frac{1}{2}}$$
$$\times \left\{K(\beta,\theta) - (1-\theta)(\beta+1)(\beta+2)\frac{\Gamma(\beta+2\theta+1)\Gamma(\beta+2)}{\Gamma(\beta+\theta+1)\Gamma(\beta+\theta+2)}\left|\frac{1}{\beta+1} - \frac{1}{\tau}\right|^2\right\}^{-\frac{1}{2}}; \tag{6.16}$$

we recall the definition of the function $K(\beta,\theta)$ in (6.1). Without loss of generality, we may assume that

$$(1-\theta)(\beta+1)(\beta+2)\frac{\Gamma(\beta+2\theta+1)\Gamma(\beta+2)}{\Gamma(\beta+\theta+1)\Gamma(\beta+\theta+2)}\left|\frac{1}{\beta+1} - \frac{1}{\tau}\right|^2 < K(\beta,\theta) \tag{6.17}$$

holds for all θ, $0<\theta\le 1$; for otherwise, we may apply Proposition 6.1 in conjunction with Remark 6.2 to get the desired inequality $\mathrm{B}_{\mathcal{S}}(\tau) \le \beta$. This means that the square roots which are used to define the functions A_1 and A_2 produce real-valued functions on the whole interval $0<\theta\le 1$. For each θ, $0<\theta\le 1$, we consider the disk

$$\mathcal{D}_\theta = \left\{ w\in\mathbb{C}:\ \left|A_1(\theta) - w\,A_2(\theta)\right| \le \frac{1}{\sqrt{(\beta+1)_4}}\right\}. \tag{6.18}$$

Here, of course, $(\beta+1)_4 = (\beta+1)(\beta+2)(\beta+3)(\beta+4)$.

We have the following result.

PROPOSITION 6.9 *Suppose there exists a certain* θ_0*, with* $0<\theta_0\le 1$*, such that*
(a) the intersection $\bigcap_{\theta_0\le\theta\le 1}\mathcal{D}_\theta$ *is empty, and*
(b) the estimate $\left\|[\varphi']^{\tau/2}\right\|_{-1+\beta+\theta_0} = O(1)$ *holds uniformly in* $\varphi\in\mathcal{S}$*.*
Then

$$\left\|[\varphi']^{\tau/2}\right\|_{-1+\beta} = O(1)$$

holds uniformly in $\varphi\in\mathcal{S}$*, so that in particular,* $\mathrm{B}_{\mathcal{S}}(\tau)\le\beta$*.*

Proof. A standard compactness argument shows that the assumption (a) remains valid if we replace the disks $\mathcal{D}_\theta$ by the slightly bigger disks

$$\mathcal{D}_\theta^\varepsilon = \left\{ w\in\mathbb{C}:\ \left|A_1(\theta) - w\,A_2(\theta)\right| \le \frac{1+\varepsilon}{\sqrt{(\beta+1)_4}}\right\},$$

for a small enough positive ε. This means that

$$\inf_{w\in\mathbb{C}} \left\|A_1 - w\,A_2\right\|_{C[\theta_0,1]} \ge \frac{1+\varepsilon}{\sqrt{(\beta+1)_4}}$$

holds, if, as is standard, $C[\theta_0, 1]$ is the Banach space of complex-valued functions continuous in $[\theta_0, 1]$, supplied with the uniform norm. By standard duality, this entails that there exists a complex Borel measure μ on the interval $[\theta_0, 1]$ such that the total variation of μ is 1, and, in addition,

$$\frac{1+\varepsilon}{\sqrt{(\beta+1)_4}} \le \left| \int_{\theta_0}^{1} A_1(\theta)\, \mathrm{d}\mu(\theta) \right| \qquad \text{while} \qquad \int_{\theta_0}^{1} A_2(\theta)\, \mathrm{d}\mu(\theta) = 0.$$

We find that an application of (6.14) leads to

$$\begin{aligned}
\frac{1+\varepsilon}{\sqrt{(\beta+1)_4}} \left\| \boldsymbol{\partial}^2 \left\{ [\varphi']^{\tau/2} \right\} \right\|_{\beta+3} &\le \left\| \left\{ \int_{\theta_0}^{1} A_1(\theta)\, \mathrm{d}\mu(\theta) \right\} \boldsymbol{\partial}^2 \left\{ [\varphi']^{\tau/2} \right\} \right\|_{\beta+3} \\
&= \left\| \int_{\theta_0}^{1} \left\{ A_1(\theta)\, \boldsymbol{\partial}^2 \left\{ [\varphi']^{\tau/2} \right\} + A_2(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 (\varphi')^{t/2} \right\} \mathrm{d}\mu(\theta) \right\|_{\beta+3} \\
&\le \int_{\theta_0}^{1} \left\| A_1(\theta)\, \boldsymbol{\partial}^2 \left\{ [\varphi']^{\tau/2} \right\} + A_2(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 (\varphi')^{\tau/2} \right\|_{\beta+3} |\mathrm{d}\mu(\theta)| \\
&\le \left\| [\varphi']^{\tau/2} \right\|_{-1+\beta} + O\Big(\left\| [\varphi']^{\tau/2} \right\|_{-1+\beta+\theta_0} \Big).
\end{aligned}$$

In view of Proposition 4.7 and the assumption (b), the desired conclusion follows. ■

REMARK 6.10 For $\tau = t$ real, it suffices to verify the assumption (a) of Proposition 6.9 along the real line only, as can be seen from the observation that the functions $A_1(\theta)$ and $A_2(\theta)$ are real-valued then. This means that if we put

$$\mathcal{I}_\theta = \left\{ x \in \mathbb{R} : \ \big| A_1(\theta) - x\, A_2(\theta) \big| \le \frac{1}{\sqrt{(\beta+1)_4}} \right\}, \tag{6.19}$$

which constitutes a closed interval, it is enough to check that

$$\bigcap_{\theta_0 \le \theta \le 1} \mathcal{I}_\theta = \emptyset. \tag{6.20}$$

This criterion can be easily checked by computer calculations. Indeed, if we denote the left and right end points of I_θ by $\alpha_1(\theta)$ and $\alpha_2(\theta)$, so that

$$I_\theta = [\alpha_1(\theta), \alpha_2(\theta)],$$

then the criterion (6.20) is equivalent to

$$\inf_{\theta \in [\theta_0, 1]} \alpha_2(\theta) < \sup_{\theta \in [\theta_0, 1]} \alpha_1(\theta),$$

which is easily treated numerically.

REMARK 6.11 It would be desirable to change the implementation of the optimization method so that we may incorporate the information supplied by Lemma 6.6, so as to obtain a more optimal estimate based on the first two terms. If we do this in a straightforward manner, focussing on the term containing $A_2(\theta)$ instead of $A_1(\theta)$, we are to replace the intervals $\mathcal{I}_\theta = \mathcal{D}_\theta \cap \mathbb{R}$ by

$$\mathcal{J}_\theta = \left\{ x \in \mathbb{R} : \ \big| A_2(\theta) - x\, A_1(\theta) \big| \le \frac{t^2}{4(\beta+1)\sqrt{\beta(\beta+4)}} \right\},$$

and the criterion $\bigcap_{\theta_0 \le \theta \le 1} \mathcal{J}_\theta = \emptyset$ then permits us to conclude that $\mathrm{B}_{\mathcal{S}}(t) \le \beta$. Numerical simulation shows that this criterion is more powerful for (real) t near the origin than the criterion (a) of Proposition 6.9.

Numerical implementation. By successive application of Proposition 6.9 for real $\tau = t$, taking into account Remark 6.10, we obtain the estimate $\mathrm{B}_{\mathcal{S}}(t) \le \mathrm{B}_*(t)$, where the function $\mathrm{B}_*(t)$ is tabulated below. We use suitably small values of θ_0. The function $\mathrm{B}_*(t)$ is also graphed. For some values of t, the method outlined in Remark 6.11 is used in place of Proposition 6.9; this is then indicated with an asterisk (*).

The tabulated bounds for $\mathrm{B}_{\mathcal{S}}(-1)$ and $\mathrm{B}_{\mathcal{S}}(-2)$ are to be compared with the bounds that were found recently by the second-named author in [17]; there, it was shown that $\mathrm{B}_{\mathcal{S}}(-1) \le 0.420$ and $\mathrm{B}_{\mathcal{S}}(-2) \le 1.246$. It should be noted that the inequality of Theorem 1 in [17] leading to these bounds is a particular case of our main inequality – the inequality of Theorem 4.9 – if we put $\theta = 1$ and, like in (6.14), take into account only the first two terms in the sum on the left hand side. In this particular case, the first term vanishes and the constant $C_6(\alpha, \theta)$ which appears in (4.9) vanishes as well, because $L_\theta = 0$ for $\theta = 1$.

t	$\mathrm{B}_*(t)$	$\max\{-t-1, 0\}$
−20.000	19.028	19.000
−10.000	9.040	9.000
−8.000	7.049	7.000
−6.000	5.067	5.000
−5.000	4.082	4.000
−4.000	3.105	3.000
−3.000	2.144	2.000
−2.500	1.674	1.500
−2.400	1.582	1.400
−2.300	1.490	1.300
−2.200	1.398	1.200
−2.100	1.308	1.100
−2.000	1.218	1.000
−1.900	1.130	0.900
−1.800	1.042	0.800
−1.752	1.001	0.752
−1.700	0.956	0.700
−1.600	0.871	0.600
−1.500	0.787	0.500
−1.400	0.706	0.400
−1.300	0.626	0.300
−1.200	0.549	0.200
−1.100	0.474	0.100
−1.000	0.403	0.000
−0.900	0.336	0.000
−0.800	0.272	0.000
−0.700	0.213*	0.000
−0.600	0.159*	0.000
−0.500	0.112*	0.000
−0.400	0.072*	0.000
−0.300	0.0404*	0.000
−0.200	0.0179*	0.000
−0.150	0.0100*	0.000
−0.100	0.00443*	0.000
−0.050	0.00110*	0.000

t	$\mathrm{B}_*(t)$	$\max\{3t-1, 0\}$
0.000	0.00000	0.000
0.050	0.00141*	0.000
0.100	0.0065	0.000
0.150	0.0157	0.000
0.200	0.031	0.000
0.250	0.056	0.000
0.300	0.101	0.000
0.350	0.190	0.050
0.400	0.314	0.200
0.450	0.447	0.350
0.500	0.585	0.500
0.600	0.870	0.800
0.700	1.159	1.100
0.800	1.452	1.400
0.900	1.746	1.700
1.000	2.041	2.000
1.100	2.337	2.300
1.200	2.634	2.600
1.300	2.932	2.900
1.400	3.230	3.200
1.500	3.528	3.500
1.600	3.826	3.800
1.700	4.124	4.100
1.800	4.423	4.400
1.900	4.722	4.700
2.000	5.021	5.000
2.100	5.320	5.300
2.200	5.619	5.600
2.300	5.918	5.900
2.400	6.217	6.200
2.500	6.517	6.500
3.000	8.014	8.000
4.000	11.011	11.000
5.000	14.010	14.000
6.000	17.010	17.000

TABLES 1 AND 2.

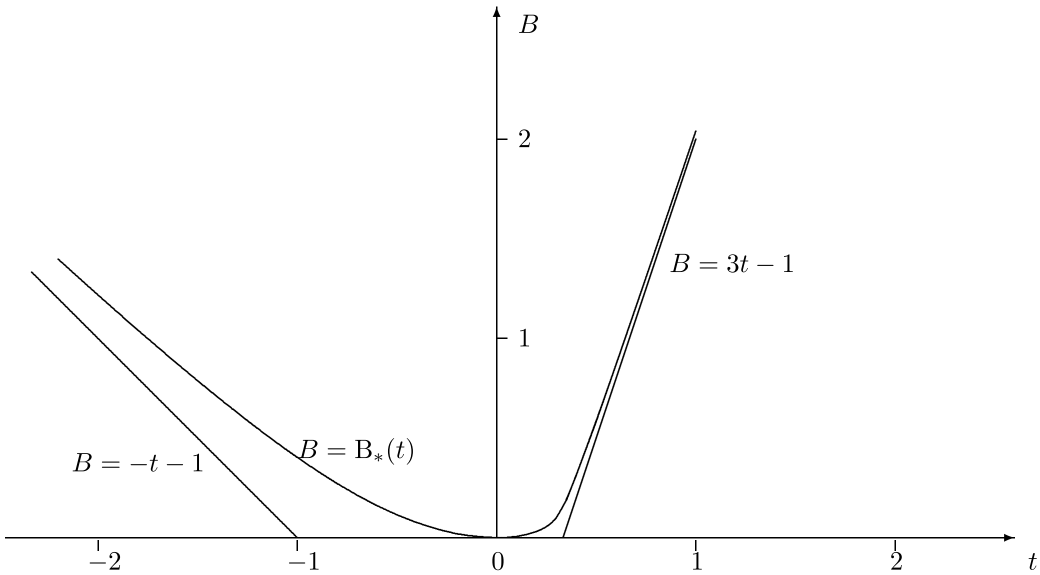

FIGURE 1. Graph of $B = \mathrm{B}_*(t)$, the estimated universal spectral function; support lines included.

REMARK 6.12 By taking advantage of the fact that the function $\mathrm{B}_\Sigma(t)$ is convex, with $\mathrm{B}_\Sigma(t) \le \mathrm{B}_\mathcal{S}(t)$ and $\mathrm{B}_\Sigma(2) = 1$, we derive from a somewhat larger supply of sample values of the graphed function $\mathrm{B}_*(t)$ that $\mathrm{B}_\Sigma(1) \le 0.4600$, improving the best earlier known estimate, due to Makarov and Pommerenke [12], which was $\mathrm{B}_\Sigma(1) \le 0.4886$. The value of $\mathrm{B}_\Sigma(1)$ describes the growth of the length of Green lines (the level curves of the Green function) as they approach the boundary of an arbitrary simply connected bounded planar domain. It also determines the rate of decay of the Laurent series coefficients of functions in the class Σ (see [3]).

The optimization method to estimate $\mathrm{B}_\mathcal{S}$ using three or more terms. How do we implement the optimization method if we take into account more than two terms on the left hand side of the inequality of Theorem 4.9? We outline here briefly an extension of the method which applies to the case of three terms. The method may of course be extended to include more than three terms as well.

For simplicity, we consider real $\tau = t$ only. As we take the first three terms on the left hand side of the inequality of Theorem 4.9 into account, putting, as before, $\alpha = \beta + 2\theta - 1$ and $g = [\varphi']^{t/2}$, we obtain an inequality of the form

$$\left\| A_1(\theta)\, \boldsymbol{\partial}^2\Big\{[\varphi']^{t/2}\Big\} + A_2(\theta) \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{t/2} \right\|^2_{\beta+3} + \frac{1}{(\beta+5)(\beta+6)}$$
$$\times \left\| A_3(\theta)\, \boldsymbol{\partial}^3 \left\{[\varphi']^{t/2}\right\} + A_4(\theta)\, \boldsymbol{\partial} \left\{ \left[\frac{\varphi''}{\varphi'}\right]^2 [\varphi']^{t/2} \right\} + A_5(\theta) \left[\frac{\varphi''}{\varphi'}\right]^3 [\varphi']^{t/2} \right\|^2_{\beta+5} \le$$
$$\le \left\| [\varphi']^{t/2} \right\|^2_{\beta-1} + O\Big(\left\| [\varphi']^{t/2} \right\|^2_{\beta-1+\theta} \Big), \quad (6.21)$$

where the functions A_1 and A_2 are given by (6.15) and (6.16), and the functions A_3, A_4, A_5 are continuous on $]0,1]$, and given by certain explicit expressions. As before, we assume that condition (6.17) is fulfilled for all θ, $0 < \theta \le 1$. The process of deriving equation

(6.21) involves not only Theorem 4.9, but also some of the algebraic results of Section 5. A counterpart to Proposition 6.9 is the following.

PROPOSITION 6.13 *Let $\mathcal{E}_\theta$ denote the ellipse in (x,y)-plane defined by the condition*

$$\big|A_1(\theta)-x\,A_2(\theta)\big|^2+\big|A_3(\theta)-x\,A_4(\theta)-y\,A_5(\theta)\big|^2\le\frac{1}{(\beta+1)_4}.$$

Suppose that there exists a certain θ_0, with $0<\theta_0\le 1$, such that
(a) the intersection $\bigcap_{\theta\in[\theta_0,1]}\mathcal{E}_\theta$ is empty;
(b) $\|(\varphi')^{t/2}\|^2_{-1+\beta+\theta_0}=O(1)$ uniformly in $\varphi\in\mathcal{S}$.
Then

$$\Big\|\big[\varphi'\big]^{\tau/2}\Big\|_{-1+\beta}=O(1)$$

holds uniformly in $\varphi\in\mathcal{S}$ and, in particular, $\mathrm{B}_{\mathcal{S}}(t)\le\beta$.

Proof. First, we introduce the operator of integration $\mathbf{I}_0$,

$$\mathbf{I}_0 f(z)=\int_0^z f(w)\,dw,\qquad z\in\mathbb{D}.$$

Then we apply Proposition 4.7 to the second term on the left-hand side of (6.21), which allows us to rewrite (6.21) in the form

$$\begin{aligned}&\left\|A_1(\theta)\,\boldsymbol{\partial}^2\Big\{(\varphi')^{t/2}\Big\}+A_2(\theta)\left[\frac{\varphi''}{\varphi'}\right]^2\big[\varphi'\big]^{t/2}\right\|^2_{\beta+3}\\&\quad+\left\|A_3(\theta)\,\boldsymbol{\partial}^2\left\{\big[\varphi'\big]^{t/2}\right\}+A_4(\theta)\left[\frac{\varphi''}{\varphi'}\right]^2\big[\varphi'\big]^{t/2}+A_5(\theta)\,\mathbf{I}_0\left[\left[\frac{\varphi''}{\varphi'}\right]^3\big[\varphi'\big]^{t/2}\right]\right\|^2_{\beta+3}\\&\qquad\le\Big\|\big[\varphi'\big]^{t/2}\Big\|^2_{-1+\beta}+O\bigg(\Big\|\big[\varphi'\big]^{t/2}\Big\|^2_{-1+\beta+\theta}\bigg).\end{aligned}\tag{6.22}$$

A standard compactness argument shows that the assumption (a) remains valid if the ellipses $\mathcal{E}_\theta$ are replaced by slightly larger ellipses $\mathcal{E}_\theta^\varepsilon$, defined by

$$\big|A_1(\theta)-xA_2(\theta)\big|^2+\big|A_3(\theta)-xA_4(\theta)-yA_5(\theta)\big|^2\le\frac{1+\varepsilon}{(\beta+1)_4},$$

provided that the positive number ε is small enough. Moreover, a similar argument shows that we may assume that a finite intersection of $\mathcal{E}_\theta^\varepsilon$ is empty:

$$\bigcap_{\theta\in\mathfrak{F}}\mathcal{E}_\theta^\varepsilon=\emptyset,$$

for some finite subset $\mathfrak{F}$ of the interval $[\theta_0,1]$. This condition is equivalent to having

$$\max_{\theta\in\mathfrak{F}}\Big\{\big|A_1(\theta)-xA_2(\theta)\big|^2+\big|A_3(\theta)-xA_4(\theta)-yA_5(\theta)\big|^2\Big\}>\frac{1+\varepsilon}{(\beta+1)_4}$$

for all $x,y\in\mathbb{R}$, or, expressed differently,

$$\operatorname{dist}_{\mathcal{X}}\left[\begin{pmatrix}A_1\\A_3\end{pmatrix},\ \operatorname{span}\left\{\begin{pmatrix}A_2\\A_4\end{pmatrix},\begin{pmatrix}0\\A_5\end{pmatrix}\right\}\right]>\sqrt{\frac{1+\varepsilon}{(\beta+1)_4}},\tag{6.23}$$

where "span" means the $\mathbb{R}$-linear span, and $\mathrm{dist}_{\mathcal{X}}$ is the distance function on the space $\mathcal{X}$, the $\mathbb{R}$-linear space of vector-valued functions

$$\theta \mapsto \begin{pmatrix} \xi_1(\theta) \\ \xi_2(\theta) \end{pmatrix}, \qquad \theta \in \mathfrak{F},$$

supplied with the norm

$$\left\| \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\|_{\mathcal{X}} = \max_{\theta\in\mathfrak{F}} \sqrt{|\xi_1(\theta)|^2 + |\xi_2(\theta)|^2}.$$

The $\mathbb{R}$-linear space $\mathcal{X}^*$ of vector-valued functions

$$\theta \mapsto \begin{pmatrix} \mu_1(\theta) \\ \mu_2(\theta) \end{pmatrix}, \qquad \theta \in \mathfrak{F},$$

supplied with the norm

$$\left\| \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\|_{\mathcal{X}^*} = \sum_{\theta\in\mathfrak{F}} \sqrt{|\mu_1(\theta)|^2 + |\mu_2(\theta)|^2},$$

is then dual to $\mathcal{X}$ with respect to the natural dual pairing

$$\left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\rangle = \sum_{\theta\in\mathfrak{F}} \Big\{ \xi_1(\theta)\mu_1(\theta) + \xi_2(\theta)\mu_2(\theta) \Big\}.$$

By standard duality theory, the inequality (6.23) means that there exists a vector-valued function

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathcal{X}^*$$

which satisfies

$$\left\| \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\|_{\mathcal{X}^*} = 1; \qquad \sqrt{\frac{1+\varepsilon}{(\beta+1)_4}} < \left| \left\langle \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\rangle \right|, \tag{6.24}$$

while

$$\left\langle \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ A_5 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\rangle = 0.$$

We then have

$$\begin{aligned}
\sqrt{\frac{1+\varepsilon}{(\beta+1)_4}} \left\| \boldsymbol{\partial}^2 [\varphi']^{t/2} \right\|_{\beta+3} &< \left\| \sum_{\theta\in\mathfrak{F}} \Big\{ \mu_1(\theta) A_1(\theta) + \mu_2(\theta) A_3(\theta) \Big\} \boldsymbol{\partial}^2 [\varphi']^{t/2} \right\|_{\beta+3} \\
&= \Bigg\| \sum_{\theta\in\mathfrak{F}} \Bigg\{ \mu_1(\theta) \left[A_1(\theta)\, \boldsymbol{\partial}^2 \Big\{ [\varphi']^{t/2} \Big\} + A_2(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 [\varphi']^{t/2} \right] + \mu_2(\theta) \\
&\times \left[A_3(\theta)\, \boldsymbol{\partial}^2 \left[(\varphi')^{t/2} \right] + A_4(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 [\varphi']^{t/2} + A_5(\theta)\, \mathbf{I}_0 \left[\left[\frac{\varphi''}{\varphi'} \right]^3 [\varphi']^{t/2} \right] \right] \Bigg\} \Bigg\|_{\beta+3} \\
&\le \sum_{\theta\in\mathfrak{F}} \Bigg\{ |\mu_1(\theta)| \left\| A_1(\theta)\, \boldsymbol{\partial}^2 \Big\{ [\varphi']^{t/2} \Big\} + A_2(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 [\varphi']^{t/2} \right\|_{\beta+3} + |\mu_2(\theta)| \\
&\times \left\| A_3(\theta)\, \boldsymbol{\partial}^2 \Big\{ [\varphi']^{t/2} \right] + A_4(\theta) \left[\frac{\varphi''}{\varphi'} \right]^2 [\varphi']^{t/2} + A_5(\theta)\, \mathbf{I}_0 \left[\left[\frac{\varphi''}{\varphi'} \right]^3 [\varphi']^{t/2} \right] \right\|_{\beta+3} \Bigg\} \\
&\le \left\| [\varphi']^{t/2} \right\|_{\beta-1} + O\left(\left\| [\varphi']^{t/2} \right\|_{\beta-1+\theta} \right),
\end{aligned}$$

where in the last step, we appeal to the Minkowski inequality, as well as to (6.24) and (6.22). Since ε is positive, this completes the proof, in view of Proposition 4.7. ■

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