Isolated Singularities of Harmonic Functions

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Abstract

The main result of this paper gives a sufficient condition for removability of an isolated singularity of a harmonic function. The condition is given in terms of Newtonian capacity. In addition, an application to an approximation problem is presented.

Introduction

This note deals with a problem of removable isolated singularities of harmonic functions. Suppose u is harmonic and of at most polynomial growth on a punctured neighborhood of a point x_0 and its gradient ∇u satisfies $|\nabla u(x)| \leq C|x - x_0|^{-1}$ on a set K, where x_0 is an accumulation point of K. Under these hypotheses, what size restrictions on K ensure that x_0 is a removable singularity of u? This type of growth constraint is rather natural in the case of harmonic functions, while for analytic functions, a bound on the function is often used, and the answer in this case is very well known (see e.g. [1], Proposition 2.4.4):

Theorem A. *Assume f is holomorphic and single valued on the punctured disk $\{z : \; 0 < |z - z_0| < r\}$ and bounded on a sequence of points clustering at z_0. Then f cannot have a polar singularity at z_0.*

The two dimensional case of the removable singularities problem for harmonic functions mentioned above, and with a bounded gradient on the set K, is settled by Theorem A. However, in higher dimensions the situation is more complicated. We present here a generalization of the above for harmonic functions in $\mathbb{R}^n, n \geq 3$. The Newtonian capacity will measure the *"size"* of K.

In order to state our result, we introduce some notations. We let $B_r(x_0) = \{x : \; |x - x_0| < r\}$ be the ball of radius r centered at x_0, $B_r = B_r(0)$, $S(a, b, x_0) = \{x : a \leq |x - x_0| \leq b\}$ and $S(a, b)$ shall denote $S(a, b, 0)$. The gradient of a function is denoted by ∇, Δ is the Laplace operator and cap (K) denotes the Newtonian capacity of the set K (see [4]).

Theorem 0.1. *Let m be a positive integer and assume that a function u satisfies*

(a) $\Delta u = 0$ *in* $B_1(x_0) \setminus \{x_0\}$,

(b) $u(x) = O(|x - x_0|^{-m})$ *as* $x \to x_0$, *and*

(c) $|\nabla u(x)| \leq \frac{C}{|x-x_0|}$ *on* $K \cap (B_1(x_0) \setminus \{x_0\})$.

If

$$\limsup_{r\to 0} \frac{\text{cap}\,\Big(K \cap S(\frac{1}{2}r, r, x_0)\Big)}{\text{cap}\,\Big(S(\frac{1}{2}r, r)\Big)} > 0, \tag{0.1}$$

then u extends to a harmonic function on all of $B_1(x_0)$.

Remark. The condition (0.1) is called a capacity density condition. It has appeared in various papers on potential theory, see [2] and the references there.

The plan of the paper is as follows: In Section 1 we prove Theorem 0.1 and we will also provide an example showing that the result is not true, if the capacity density condition (0.1) is abandoned. In Section 2 we will apply Theorem 0.1 to an approximation problem.

1 Proof of the Main Result

The proof relies on certain results from [2]. Let F be a closed bounded set of $\mathbb{R}^n$, and $C^1(F)$ the set of restriction to F of continuously differentiable functions on $\mathbb{R}^n$.

Definition 1.1 (Λ-stable). *A family $\mathcal{E}$ of subsets of F is Λ-stable, if for every sequence $\{E_j\} \subset \mathcal{E}$, the set*

$$\bigcap_{m=1}^{\infty} \left(\overline{\bigcup_{j=m}^{\infty} E_j} \right)$$

belongs to the family $\mathcal{E}$. Here $\overline{(\ \cdot\ )}$ denotes the closure.

Definition 1.2 (Sets of uniqueness). *Let P be a closed subspace of $C^1(F)$. A closed set K of F is a set of uniqueness for P, if the only p in P with gradient vanishing on all of K are constants.*

Lemma 1.3 ([2]). *Suppose that P is a finite-dimensional subspace of $C^1(F)$ and that $\mathcal{E}$ is a Λ-stable family of subsets of F, each of which is a set of uniqueness for P. Then there is a constant $C_{\mathcal{E}}$ such that for each K in $\mathcal{E}$*

$$\max_{x\in F}|p(x)| \le C_{\mathcal{E}} \max_{x\in K}|\nabla p(x)|$$

for every p in P with $p(x_0) = 0$, where x_0 is a fixed point in F.

Remark. The requirement $p(x_0) = 0$ is not necessary in case the subspace P does not contain a nonzero constant function.

Set $F = S(\frac{1}{2}, 1) = \{\frac{1}{2} \le |x| \le 1\}$ and let $\mathcal{E}$ be the family of all Borel subsets E of F satisfying

$$\text{cap}\,(E) \ge \gamma, \tag{1.1}$$

where γ is a positive constant.

To verify that $\mathcal{E}$ is Λ-stable one has to check the following three conditions:

(i) $\mathcal{E}$ is closed under formation of countable unions.

(ii) If $E \in \mathcal{E}$, then $\overline{E} \in \mathcal{E}$.

(iii) The intersection of a countable, monotone decreasing sequence of closed nonempty sets from $\mathcal{E}$ is again in $\mathcal{E}$.

It is known that the family defined by (1.1) has these properties. We refer to [4] and [2] for further details. We now introduce the finite dimensional space $\mathcal{H}_m$, consisting of harmonic functions of the following form:

$$h \in \mathcal{H}_m \Leftrightarrow h(x) = \sum_{j=0}^{m} \frac{p_j(x)}{|x|^{2j+n-2}},$$

where p_j is a homogeneous harmonic polynomial of degree j.

The set $\{x: \ |\nabla u(x)| = 0\}$ is called the critical set of the function u. Kuran proved that the critical set of a non-constant harmonic function has zero Newtonian capacity (see [3] Lemma 1). That is, a compact subset K of $S(\frac{1}{2}, 1)$ of positive capacity is a set of uniqueness for the space $\mathcal{H}_m$.

Applying Lemma 1.3 with $\mathcal{E}$ as above and with $P = \mathcal{H}_m$ we get

Corollary 1.4. *Given a positive constant γ and a positive integer m, there is a positive constant $c(\gamma, m)$ such that*

$$\max_{x\in S(\frac{1}{2},1)} |h(x)| \leq c(\gamma, m) \max_{x\in K} |\nabla h(x)| \tag{1.2}$$

holds for every h in $\mathcal{H}_m$ and for every compact set $K \subset S(\frac{1}{2}, 1)$ with $\mathrm{cap}\,(K) \geq \gamma$.

Proof of Theorem 0.1. First, we may assume that $x_0 = 0$. Conditions (a) and (b) of Theorem 0.1 imply that $u = u_0 + h$, where u_0 is harmonic in B_1 and h in $\mathcal{H}_m$. By the capacity density condition (0.1), there is a sequence of positive numbers $\{r_j\}$, $r_j \searrow 0$ and there is a constant $\gamma > 0$ such that

$$\frac{\mathrm{cap}\,\Big(K \cap S(\frac{1}{2}r_j, r_j)\Big)}{\mathrm{cap}\,\Big(S(\frac{1}{2}r_j, r_j)\Big)} \geq \gamma, \qquad j = 1, 2, ... \tag{1.3}$$

Let

$$h_j(x) = \frac{h(r_j x)}{r_j^2}$$

and

$$K_j = \{x : \ r_j x \in K\}.$$

Then, (1.3) becomes

$$\frac{\mathrm{cap}\,\Big(K_j \cap S(\frac{1}{2}, 1)\Big)}{\mathrm{cap}\,\Big(S(\frac{1}{2}, 1)\Big)} \geq \gamma, \qquad j = 1, 2, ...$$

and by condition (c),

$$|\nabla h_j(x)| = \frac{|\nabla h_j(r_j x)|}{r_j} \leq \frac{C}{r_j^2 |x|} \leq \frac{2C}{r_j^2} \qquad \text{for } x \in K_j \cap S(\frac{1}{2}, 1).$$

Since $h_j \in \mathcal{H}_m$, we may apply Corollary 1.4 and (1.2) and obtain

$$\max_{S(\frac{1}{2},1)} |h_j(x)| \leq \frac{2Cc(\gamma, m)}{r_j^2},$$

or

$$\max_{S(\frac{1}{2}r_j,r_j)} |h(x)| \leq 2Cc(\gamma, m),$$

where $r_j \to 0$. This, together with h belonging to $\mathcal{H}_m$, implies $h \equiv 0$. Hence $u = u_0$ is harmonic in B_1. □

Example. We give an example showing that if (0.1) is not assumed, then the conclusion of Theorem 0.1 no longer holds. Let $n = 3$, and take

$$K = \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq x_3^4\}.$$

We verify that (0.1) does not hold. For each $0 < r < 1$, the set $S(\frac{1}{2}r, r) \cap K = \{\frac{1}{2}r \leq |x| \leq r\} \cap K$ is included in a cylinder of height r and base radius r^4, and hence it is contained in an ellipsoid with semi-axes r^4, r^4 and r. The Newtonian capacity of an elongated ellipsoid

$$\{x \in \mathbb{R}^3 : \frac{x_1^2 + x_2^2}{a^2} + \frac{x_3^2}{b^2}, 0 < a < b\}$$

is equal to

$$\frac{2\sqrt{b^2 - a^2}}{\pi \log\left(\frac{b+\sqrt{b^2-a^2}}{b-\sqrt{b^2-a^2}}\right)},$$

see [4] page 165. Setting $a = r^4$ and $b = r$, we have

$$\text{cap}\left(K \cap S(\frac{1}{2}r, r)\right) \leq \frac{2r\sqrt{1-r^6}}{\pi\left(\log\left(\frac{1+\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)\right)} = \frac{2r\sqrt{1-r^6}}{\pi\left(\log\left(1 + \frac{2\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)\right)}.$$

Since the shell $S(\frac{1}{2}r, r)$ contains a ball of radius $\frac{r}{2}$,

$$\text{cap}\,(S(\frac{1}{2}r, r)) \geq \text{cap}\,(B_{\frac{r}{2}}) = \frac{r}{\pi 2}.$$

Having these two estimates, we see that,

$$\frac{\text{cap}\left(S(\frac{1}{2}r, r) \cap K\right)}{\text{cap}\left(S(\frac{1}{2}r, r)\right)} \leq \frac{4\sqrt{1-r^6}}{\log\left(1 + \frac{2\sqrt{1-r^6}}{1-\sqrt{1-r^6}}\right)} \to 0 \quad \text{as } r \to 0.$$

Thus (0.1) fails.

Let

$$u(x) = \frac{x_1 x_2}{|x|^5}.$$

It is easy to check that $|\nabla u(x)| \leq \frac{C}{|x|}$ on K, for a positive constant C. Thus u satisfies the conditions (a), (b) and (c) of Theorem 0.1, and the origin is not a removable singularity of u.

2 An approximation problem

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$. We denote by $L^p(\Omega)$ the usual Lebesgue space, and by $HL^p(\Omega)$ its subspace of harmonic functions in Ω. We consider the subspace $\Phi_{x_0}(\Omega)$, of $HL^1(\Omega)$ consisting of harmonic functions whose partial derivatives of order less than or equal m vanish at the point x_0. We address the question: Under which conditions, is the subspace $\Phi_{x_0}(\Omega)$ dense in $HL^1(\Omega)$ in the L^1-norm. One should note that if x_0 is an interior point of Ω, then it is impossible that $\Phi_{x_0}(\Omega)$ will be dense in $HL^1(\Omega)$, because convergence in L^1-norm for harmonic functions implies the convergence in the supremum-norm in a neighborhood of x_0, and hence, for example, the function $h \equiv 1$ cannot be approximated by functions from $\Phi_{x_0}(\Omega)$. However, in case x_0 is a boundary point of Ω, we will show that the capacity density condition (2.2) suffices for such approximation.

We recall first a result of Sakai, [5]. Let $N(x) = c_n|x|^{2-n}$ be the Newtonian kernel, and $U^\mu = N * \mu$ the Newtonian potential of a distribution μ with a compact support. The constant c_n is chosen so that

$$\Delta U^\mu = \mu. \tag{2.1}$$

We set

$$\Phi(\Omega) = \text{span}\,\{N(y-x),\ \partial_i N(y-x),\ x \in \mathbb{R}^n \setminus \Omega, i = 1, ..., n\}.$$

Theorem B. (Sakai [5]). *For any bounded set Ω of $\mathbb{R}^n$, $\Phi(\Omega)$ is dense in $HL^1(\Omega)$.*

For a point $x_0 \in \partial\Omega$ and a positive integer m, we set

$$\begin{aligned}\Phi_{x_0,m}(\Omega) = \text{span}\,\{&N(y-x) - \textstyle\sum_{|\alpha|\leq m} \frac{(y-x_0)^\alpha}{\alpha!}(\partial^\alpha N)(x_0 - x),\\ &\partial_i N(y-x) - \textstyle\sum_{|\alpha|\leq m} \frac{(y-x_0)^\alpha}{\alpha!}(\partial^\alpha \partial_i N)(x_0 - x),\\ &x \in \mathbb{R}^n \setminus \Omega, x \neq x_0, i = 1, ..., n\}.\end{aligned}$$

Theorem 2.1. *Let Ω be a bounded domain, m a positive integer and $x_0 \in \partial\Omega$. If*

$$\limsup_{r\to 0} \frac{\text{cap}\,\left((\mathbb{R}^n \setminus \Omega) \cap S(\frac{1}{2}r, r, x_0)\right)}{\text{cap}\,\left(S(\frac{1}{2}r, r)\right)} > 0, \tag{2.2}$$

then $\Phi_{x_0,m}(\Omega)$ is dense in $HL^1(\Omega)$ in the topology of the L^1-norm.

Corollary 2.2. *Let Ω be a bounded domain such that Ω is the interior of its closure, m a positive integer and $x_0 \in \partial\Omega$. If (2.2) holds, then the harmonic functions on a neighborhood of the closure of Ω such that their partial derivatives of order less or equal to m vanish at the point x_0, are dense in $HL^1(\Omega)$.*

Proof of Theorem 2.1. By the the Hahn-Banach theorem, we have to show that if $f \in L^\infty(\Omega)$ annihilates $\Phi_{x_0,m}(\Omega)$, then f annihilates $HL^1(\Omega)$ too. Let

$$v(x) = \int_\Omega \left(N(y-x) - \sum_{|\alpha|\le m} \frac{(y-x_0)^\alpha}{\alpha!} (\partial^\alpha N)(x_0 - x) \right) f(y)dy.$$

Then

$$\partial_i v(x) = \int_\Omega \left(\partial_i N(y-x) - \sum_{|\alpha|\le m} \frac{(y-x_0)^\alpha}{\alpha!} (\partial^\alpha \partial_i N)(x_0 - x) \right) f(y)dy$$

and by (2.1),

$$\Delta v = \chi_\Omega f \qquad \text{in } \mathbb{R}^n \setminus \{x_0\},$$

where χ_Ω denotes the characteristic function of Ω. Also, $v(x)$ and $\nabla v(x)$ vanish on $\mathbb{R}^n \setminus (\Omega \cup \{x_0\})$ since f annihilates $\Phi_{x_0,m}(\Omega)$. Setting

$$u(x) = U^{(\chi_\Omega f)}(x) - v(x),$$

we have that u satisfies:

(a) $\Delta u = 0$ in $\mathbb{R}^n \setminus \{x_0\}$,

(b) $u(x) = O(|x - x_0|^{(-n+2-m)})$ as $x \to x_0$,
and since $U^{(\chi_\Omega f)}$ is C^1 and $\nabla v(x)$ vanishes on $\mathbb{R}^n \setminus \Omega$,

(c) $|\nabla u(x)| \le C$ on $\mathbb{R}^n \setminus \Omega$.

So we are now in the situation where Theorem 0.1 can be applied. By the hypothesis (2.2), we conclude that u is harmonic in a neighborhood of x_0. This means that

$$\int_\Omega \left(\sum_{|\alpha|\le m} \frac{(y-x_0)^\alpha}{\alpha!} (\partial^\alpha N)(x_0 - x) \right) f(y)dy = 0$$

for all x. Therefore, $U^{(\chi_\Omega f)}$ and $\nabla U^{(\chi_\Omega f)}$ vanish on $\mathbb{R}^n \setminus \Omega$. That is, f annihilates the class $\Phi(\Omega)$ and By Sakai's theorem, f annihilates $HL^1(\Omega)$. $\square$

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