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INVARIANT FAMILIES IN THE HELE-SHAW PROBLEM

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ABSTRACT. This paper contains the recent results on invariant families were previously announced in the author's PhD thesis [PhD]. Some of the results were published in [Ku98, KuP] in Russian and the part concerning the polynomial solutions in [Ku01]. The results concerning the invariant families with bounded distortion of the logarithmic derivative (see Section 5 below) were never published before by the author. The discussion of isoperimetric defect estimates and related questions were firstly announced jointly with V. Tkachev paper [KuT].

We give here only a brief outline of preliminaries concerning the Hele-Shaw model and its basic properties. For more detailed discussion of this theme we refer e.g., to [EJ], [G84], [BF], [Rs72] and [VE].

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1. INTRODUCTION

1.1. **Physical model.** Our aim of this section is to give a brief description of the well-known facts about the evolutionary model of Hele-Shaw flow with the discrete point-sources. Here we restricted ourselves by heuristic derivation of the equations which correspond to the Hele-Shaw flow.

Basically, the Hele-Shaw flow arises if newtonian unpressured viscous fluid moves in a narrow gap between two parallel planes sufficiently slowly. The width of the gap is relevantly small in comparison to sizes of initial viscous configuration. Choose a cartesian system $\mathbf{R}^2(x, y)$ in a parallel to the Hele-Shaw cell plane and denote by $\nu = (u, v) = (u(x, y), v(x, y))$ the velocity field of the fluid at the point (x, y) at time t and by $p(x, y, t)$ the viscous pressure.

From the Darcy law follows that the velocity of the fluid is proportional to the pressure gradient

$$\nu = -\kappa \nabla p(x, y, t) = -\kappa \cdot (p'_x, p'_y), \tag{1.1}$$

where $\kappa > 0$ is a coefficient depending on the medium. Incompressibility of the fluid leads us to the continuity equation

$$\operatorname{div} \nu \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.2}$$

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which holds at every point of the domain $\Omega(t)$ except the points of sink or source. Indeed, the last points are singular for the velocity field ν.

If we have free boundaries and don't take in account the surface tension then the pressure function $p(x,y,t)$ is constant on the boundary $\partial\Omega(t)$. Without loss of generality we have

$$p(x,y,t) \equiv 0, \qquad (x,y) \in \partial\Omega(t).$$

From (1.1) and (1.2) follows that the velocity field ν is potential. Moreover, the function $\Phi(x,y,t) = \kappa\, p(x,y,t)$ is harmonic in $\Omega(t)$ minus the set of singularities:

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} \equiv 0, \qquad \nabla\Phi = -\nu.$$

Observe that the total derivative of the function p on time t vanishes on the boundary $\partial\Omega(t)$, which means

$$\frac{\mathrm{d}\,p}{\mathrm{d}\,t} \equiv \frac{\partial p}{\partial t} + \langle \nu, \nabla\, p\rangle = \frac{\partial p}{\partial t} - \kappa|\nabla p|^2 = \frac{1}{\kappa}\left(\frac{\partial\Phi}{\partial t} - |\nabla\Phi|^2\right) \equiv 0,$$

for $(x,y) \in \partial\Omega(t)$.

Using complex coordinates $z = x + iy$ we notice that the set of sinks and sources is described by the finite set of points $z_1, \dots, z_n$ with powers $q_1, \dots, q_n$, where $q_i > 0$ corresponds to the source and $q_i < 0$ — to the sink. So we equivalently rewrite the last equation in terms of our terminology

$$\Phi(z,t) = -\sum_{j=1}^{n} \frac{q_j}{2\pi} \cdot \ln|z - z_j| + \phi(z,t), \tag{1.3}$$

where $\phi(z,t)$ is smooth everywhere in $\Omega(t)$ harmonic function. Actually, (1.3) means that the quantity of liquid flowing via the point z_j in a unit of time (power) of the source q_j is equal to

$$\int_\gamma \langle \nu, N\rangle\, ds = -\int_\gamma \langle \nabla\Phi, N\rangle\, ds = -\int_{\partial B_\varepsilon(z_j)} \langle \nabla\Phi, N\rangle\, ds -$$

$$-\int_{G\setminus B_\varepsilon(z_j)} \Delta\Phi\, dx\, dy = \int_{\partial B_\varepsilon(z_j)} \frac{q_j}{2\pi}\langle \nabla \ln|z - z_j|, N\rangle\, ds = q_j.$$

Here we denote by N the unit external normal to the boundary curve $\gamma = \partial G$ and by $B_\varepsilon(z_j)$ the disk of radius ε with the center z_j such that $\overline{B_\varepsilon(z_j)} \subset G$.

Definition 1.1. A family $\Omega(t)$, $t \in [0,b) \subset \mathbf{R}^1$ such that a C^2-differentiable in $(\overline{\Omega(t)} \setminus \Pi) \times [0;b)$ function $\Phi(z,t)$ does exist we call a *classical solution* to the Hele-Shaw equation with sources set $\Pi \equiv \{z_1, \dots, z_n\}$ if

$$\begin{aligned}
&\text{(i)} && \Delta\Phi \equiv 0, \qquad z \in \Omega(t), \qquad t \in [0,b),\\
&\text{(ii)} && \Phi(z,t) = 0, \qquad z \in \partial\Omega(t), \qquad t \in [0,b),\\
&\text{(iii)} && \Phi(z,t) = \varphi(z,t) - \sum_{j=1}^{n} \frac{q_j}{2\pi} \ln|z - z_j|, \qquad z \in \Omega(t),\\
&\text{(iv)} && \frac{\partial\Phi}{\partial t} = |\nabla\Phi(z,t)|^2, \qquad z \in \partial\Omega(t), \qquad t \in [0,b),
\end{aligned} \tag{1.4}$$

for some continuous in $\Omega(t)$ harmonic function $\varphi(t)$.

Remark 1.1. We notice that (i) and (iii) can be collected as the following single condition

$$\Delta\Phi(z,t) = \sum_{j=1}^{n} q_j \delta_{z_j}(z) \equiv H(z),$$

where $\delta_a(z)$ is the δ-Dirac function with respect to $a \in \mathbb{C}$ and should be (1.4) considered in distributional sense. In this case the right side of $H(z)$ depends on the initial configuration of sources only.

1.2. **Polubarinova-Kochina equation.** In what follows we shall consider the simplest case of (1.4) with a single source. We can suppose without less of generality that it is situated at the origin: $z_1 = 0$ and its power is normalized by $|q| = 2\pi$. Then we have from (i)–(iii) that $\Phi(z,t) \equiv \hat{q}\, G_{\Omega(t)}(z,0)$ where $G_D(z,\zeta)$ is the Green function of D and $\hat{q} = q/2\pi$.

As the initial data $\Omega(0)$ we consider a simply-connected domain $z \in \mathbb{C}$, $z_1 = 0 \in \Omega(0)$ and assume that $\Omega(0)$ is the image of the unit disk $U = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ by a conformal mapping $w(z) : U \mapsto \Omega(0)$ which is normalized by

$$w(0) = 0, \qquad w'(0) > 0. \tag{1.5}$$

In this case the Green function $G_{\Omega(0)}(z,0)$ with $z = 0$ as a pole has the following representation

$$G_{\Omega(0)}(z;0) = -\ln|f(z)|,$$

where $f(z)$ is the reciprocal to $z = w(\zeta)$ function,

$$w \circ f(z) \equiv z; \qquad f \circ w(\zeta) \equiv \zeta.$$

It follows that

$$\Phi(z;0) = \hat{q} G_{\Omega(0)}(z;0) = -\hat{q} \ln|f(z)|.$$

Let $\{\Omega(t)\}$ be the evolution family associated with a classical solution to (1.4) within the interval $t \in [0,b)$. Then for t, $t > 0$, sufficiently small we conclude that all of $\Omega(t)$ will be simply-connected domains. Thus, by introducing the corresponding conformal mappings $w(\zeta;t) : U \mapsto \Omega(t)$ which satisfy

$$w(0;t) = 0; \qquad\qquad w'_\zeta(0;t) > 0,$$

we arrive at

$$\Phi(z;t) = \hat{q} G_{\Omega(t)}(z;0) = -\hat{q} \ln|f(z;t)|,$$

where $f(w(\zeta;t),t) \equiv \zeta$ and $w(f(z;t);t) \equiv z$ for all $z \in \Omega(t)$, $\zeta \in U$. It follows from the last relations that

$$w'_\zeta(f(z;t);t) \cdot f'_z(z;t) = 1,$$

$$w'_\zeta((f(z;t);t)) \cdot f'_t(z;t) + w'_t(f(z;t);t) = 0 \tag{1.6}$$

whence $z \in \Omega(t)$.

Now we can interpret equation (iv) with our notations. We additionally assume that for every $t \in [0,b)$, $w(\zeta;t)$ is holomorphic at a neighborhood of $\overline{U}$ (which depends on t) and univalent in $\overline{U}$. Then

$$\frac{\partial\Phi}{\partial t} = -\hat{q}\frac{\partial}{\partial t}\ln|f(z;t)| = -\hat{q}\ \mathrm{Re}\left(\frac{\partial}{\partial t}\ln f(z;t)\right) = -\hat{q}\ \mathrm{Re}\left(\frac{f'_t(z;t)}{f(z;t)}\right),$$

and using the fact that $\ln|f(z;t)| = \mathrm{Re}(\ln f(z;t))$ we obtain

$$|\nabla\Phi(z;t)|^2 = \frac{|f'_z(z;t)|^2}{|f(z;t)|^2}.$$

By virtue of our assumptions on $w(\zeta;t)$ we conclude that $f(z;t)$ is univalent holomorphic on $\overline{\Omega(t)}$ as well and it yields by (iv) that for every point $z \in \partial\Omega(t)$

$$\operatorname{Re}\left(\frac{f'_t(z;t)}{f(z;t)}\right) = -\hat{q} \cdot \frac{|f'_z(z;t)|^2}{|f(z;t)|^2}, \tag{1.7}$$

because $\hat{q}^2 = 1$. Taking into account (1.6) and (1.7) we obtain

$$f'_z(z;t) = \frac{1}{w'_\zeta(f(z;t),t)}, \qquad f'_t = -\frac{w'_t(f(z;t);t)}{w'_\zeta(f(z;t);t)},$$

and it follows from (1.7) that

$$\operatorname{Re}\left(\frac{w'_t(f(z;t);t)}{f(z;t)\cdot w'_\zeta(f(z;t);t)}\right) = \hat{q}\frac{1}{|w'_\zeta(f(z;t);t)|^2}\cdot\frac{1}{|f(z;t)|^2}.$$

We use $\zeta = f(z;t)$ so we obtain for every $\zeta \in \partial U$

$$\operatorname{Re}\left(\frac{w'_t(\zeta;t)\,\overline{w'_\zeta(\zeta;t)}\cdot\overline{\zeta}}{|\zeta|^2|w'_\zeta(\zeta;t)|^2}\right) = \hat{q}\frac{1}{|w'_\zeta(\zeta;t)|^2}\cdot\frac{1}{|\zeta)|^2},$$

and after simplification we arrive at

$$\operatorname{Re}\left(\overline{w'_t(\zeta;t)}\cdot w'_\zeta(\zeta,t)\,\zeta\right) \equiv \hat{q}.$$

Definition 1.2. Denote by $O(\overline{U})$ the class of all holomorphic in a neighborhood $\overline{U}$ and univalent in $\overline{U}$ functions $w(z)$ satisfying the normalization (1.5).

Let us give the equivalent formulation of the Hele-Shaw problem by using the previous definition (1.2) (cf. with [G84]).

Problem A. Given a mapping $w_0(z) \in O(\overline{U})$, find $b > 0$ and the family $w(z;t)$, $w(\,\cdot\,;t) \in O(\overline{U})$ for every $t \in [0;b)$ and with initial condition $w(z;0) = w_0(z)$ such that $w(z;t)$ is continuous on t and for all $(z;t) \in \partial U \times [0;b)$ the following relation holds

$$\operatorname{Re}\left(\overline{\frac{\partial w}{\partial t}}(z;t)\cdot\frac{\partial w}{\partial z}(z;t)\cdot z\right) = \hat{q}, \qquad \hat{q}^2 = 1. \tag{1.8}$$

Polubarinova-Kochina [P-K] and Galin [Gl] were the first who derived this form of the Hele-Shaw equation and applied it to constructing of explicit examples .

1.3. **Kufarev-Vinogradov equation and HS-Problem.** To simplify the previous equation we assume that $w(z;t)$ belongs to $O(\overline{U})$ for $t \in [0;b)$, $b > 0$. Then it follows from (1.8) that

$$\operatorname{Re}\left(\frac{\overline{w'_t(z;t)}}{\overline{z}\,\overline{w'_z(z;t)}}\right) = \frac{\hat{q}}{|z\,w'_z(z;t)|^2} = \frac{\hat{q}}{|w'_z(z;t)|^2}, \qquad z \in \partial U. \tag{1.9}$$

But $\varphi(z;t) = w'_t(z;t)/z\,w'_z(z;t)$ is a regular holomorphic function of z for fixed $t \in [0;b)$ and by virtue of (1.9) we have for the real part

$$\operatorname{Re}\varphi(z;t) = \frac{\hat{q}}{|w'_z(z;t)|^2},$$

which yields by Schwarz representation formula [Ev] that

$$\varphi(z;t) = \frac{\hat{q}}{2\pi}\int_0^{2\pi}\frac{1}{|w'_z(e^{i\theta};t)|^2}\cdot\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta,$$

or what is the same

$$w'_t(z;t) = z\, w'_z(z;t)\frac{\hat{q}}{2\pi}\int_0^{2\pi} \frac{1}{|w'_z(e^{i\theta};t)|^2}\cdot\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta. \tag{1.10}$$

Remark 1.2. The last integro-differential relation (1.10) was firstly obtained by P.P. Kufarev and Yu.P. Vinogradov in [VKf]. They also were settled the problem of existing and uniqueness of solutions to Problem HS on small time intervals and proved that given a univalent in the unit disk initial data $w(z,0) \in O(\overline{U})$ normalized by $w'_z(z,0) > 0$ there exists a small $\varepsilon > 0$ and $w(z,t) \in O(\overline{U})$ for all $t \in [0;\varepsilon)$ such that (1.10) holds (we refer to this assertion as to *Kufarev–Vinogradov theorem*) .

Remark 1.3. Relations (1.8) and (1.10) was studied by B. Gustafsson in [G84] to establish the local and global solvability of Problem HS for rational and polynomial initial data $w_0(z)$.

It was mentioned above that our main goal is the case of (1.10) for a single source normalized by $\hat{q} = +1$.

Definition 1.3 (Problem HS.). Given an initial holomorphic mapping $w_0(z) \in O(\overline{U})$, a family of holomorphic functions $w(z;t)$ such that $w(z;t)$ is of C^1 on t, $t \in [0;b)$ we call an HS-solution to the Hele-Shaw equation with initial data $\Omega_0 \equiv w(U;0) = w_0(U)$ if
(a) $w(z;t) \in O(\overline{U})$ for all $t \in [0;b)$ and $w(z;0) = w_0(z)$;
(b) (1.5) one holds;
(c) for every $z \in U$ and $t \in [0;b)$ relation (1.10) holds.

Let now $\Gamma \subset O(\overline{U})$ be a subclass of holomorphic univalent functions.

Definition 1.4. The class $\Gamma \subset O(\overline{U})$ is said to be *invariant* (for the Problem HS) if for every initial mapping $w_0(z) \in \Gamma$ a HS-solution $w(z;t)$ with $w(z,0) = w_0(z)$ does exist in a small interval $t \in [0,b)$ and belongs to Γ.

The simplest case of the invariant class is just $O(\overline{U})$. On the other hand, we are mostly interested in those invariant classes which can be characterized by geometric properties of $w(z) \in \Gamma$. We give a brief list of known invariant classes.

Let $S^\star = \{w \in O(\overline{U}) : \operatorname{Re}(zw'_z/w) > 0\}$ be the class of star-like functions which map the unit disk on a star-shaped domain. In recent paper [HPV] the invariance property of $S^\star$ has been established. In section 5 we obtain some extensions of this property.

Other examples are

a) a class $\mathcal{P}_n(\overline{U})$ of all univalent in the unit disk polynomials of fixed degree n, [Gl], [Rs72];
b) a subclass $\mathcal{P}_{n,odd}(\overline{U}) \subset \mathcal{P}_n(\overline{U})$ consisting of all odd polynomials;
c) a subclass of $\mathcal{P}_n(\overline{U})$ consisting of polynomials $w(z) = a_1 z + a_n z^n$ [HR], [Rs94].

Another examples will be discussed in Section 5 below.

2. Preliminary assertions

2.1. **$\star$-derivative.** We have mentioned before that a function $w(z)$ is a star-like in the unit disk U if the image $w(U)$ is a star-shaped domain U with respect to the origin. Then the last requirement is equivalent to that for all $z \in \partial\overline{U}$ the following inequality holds [Al]

$$\operatorname{Re}\left(\frac{z\,w'(z)}{w(z)}\right) = \frac{d\arg\, w}{d\arg\, z} > 0.$$

Let us denote the inner term by

$$w^{\star}(z) \equiv \frac{z\, w'(z)}{w(z)}$$

which can be formally written as

$$\frac{d \ln w}{d \ln z} = \frac{dw}{w} \cdot \frac{z}{dz}.$$

We call it the $\star$-derivative of w. This characteristic has a clear geometrical sense (see Proposition 2.2 below) and plays an important role in geometric function and univalent function theories [Go, Du].

The properties of $w^{\star}(z)$ below follow immediately from the definition of $\star$-operator.

Proposition 2.1. *Let $w(z)$ and $u(z)$ are non-vanished holomorphic functions in a unit disk. Then*

1) $(w(z)\, u(z))^{\star} = w^{\star}(z) + u^{\star}(z)$;
2) $\left(\dfrac{w(z)}{u(z)}\right)^{\star} = w^{\star}(z) - u^{\star}(z)$;
3) *for any* $\alpha \in \mathbf{R}$, $(w^{\alpha}(z))^{\star} = \alpha w^{\star}(z)$;
4) $(az)^{\star} = 1$ *for any* $a \in \mathbb{C}$;
5) *if the composition* $w(u(z))$ *is defined then* $(w(u(z)))^{\star} = w^{\star}(u(z)) \cdot u^{\star}(z)$.

Moreover,

$$w'^{\star}(z) = w^{\star\star}(z) + w^{\star}(z) - 1 \tag{2.1}$$

Proposition 2.2. *Let $w(z)$ be an analytic in the unit disk U and $w(z) \neq 0$ in U. Then*

$$\frac{\partial \ln |w(r\, e^{i\theta})|}{\partial r} = \frac{1}{r} \operatorname{Re} w^{\star}(r\, e^{i\theta}), \qquad \frac{\partial \ln |w(r\, e^{i\theta})|}{\partial \theta} = -\operatorname{Im} w^{\star}(r\, e^{i\theta});$$

$$\frac{\partial \arg w(r\, e^{i\theta})}{\partial r} = \frac{1}{r} \operatorname{Im} w^{\star}(r\, e^{i\theta}), \qquad \frac{\partial \arg w(r\, e^{i\theta})}{\partial \theta} = \operatorname{Re} w^{\star}(r\, e^{i\theta}),$$

for any choice of $0 < r < 1$, $\theta \in [0; 2\pi]$.

Proof. It is sufficient to prove the first property only. We have from

$$\ln w(re^{i\theta}) = \ln |w(re^{i\theta})| + i \arg w(re^{i\theta})$$

the following relation

$$\frac{\partial \ln |w(r\, e^{i\theta})|}{\partial r} = \operatorname{Re} \frac{\partial}{\partial r} \left(\ln w(r\, e^{i\theta})\right) = \operatorname{Re} \left(\frac{w'_z(re^{i\theta})}{w(re^{i\theta})} \cdot e^{i\theta} \right) = \frac{1}{r} \operatorname{Re} w^{\star}(r\, e^{i\theta}),$$

and

$$\frac{\partial \ln |w(r\, e^{i\theta})|}{\partial \theta} = \operatorname{Im} \left(\frac{w'_z(re^{i\theta}) \cdot rie^{i\theta}}{w(re^{i\theta})} \right) = -\operatorname{Re} w^{\star}(r\, e^{i\theta}).$$

□

Corollary 2.1. *Let $w(z)$ be an analytic function in the unit disk U such that $w(z) \neq 0$ in U. Then for the Jacobian of the mapping*

$$P : (r; \theta) \mapsto (\ln |w(re^{i\theta})|; \arg w(re^{i\theta}))$$

there holds

$$J_P(r; \theta) \equiv \frac{\partial(\ln |w|; \arg w)}{\partial(r; \theta)} = \frac{1}{r} |w^{\star}(re^{i\theta})|^2.$$

2.2. **Schwarz integral formula.** Here we adopt the method due to [HPV] of transformation of the Schwarz integral formula. Let $u = u(e^{i\theta})$ be some differentiable function defined on the unit circle ∂U. We denote by

$$S_u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta; t)\cdot\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$$

the *Schwarz integral* which represents an analytic function by its real part values on the unit circle. So, for any analytic function $f(z)$ holomorphic in $\overline{U}$ (i.e., in a neighborhood of $\overline{U}$) one holds

$$S_u(z) \equiv f(z) \quad \text{in } \overline{U}, \qquad u = \operatorname{Re} f(e^{i\theta}).$$

Moreover, if $u(e^{i\theta})$ is a real-valued function for $\theta \in [0; 2\pi]$ then $S_u(z)$ is analytic in U and for any $z_0 \in \partial U$ the following limit does exist

$$\lim_{z\to z_0 z\in U} \operatorname{Re}(S_u(z)) = u(z_0).$$

Given a function $u(\zeta) \in C^1(\partial U)$ we denote by

$$u'(\zeta) \equiv \frac{d}{d\theta}u(e^{i\theta}), \qquad \zeta = e^{i\theta} \in \partial U.$$

Then for $z \in U$ we have

$$\frac{d}{dz}S_u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{2e^{i\theta}}{(e^{i\theta}-z)^2}\,d\theta.$$

On the other hand,

$$\frac{d}{d\theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = -\frac{2ie^{i\theta}z}{(e^{i\theta}-z)^2},$$

and by virtue of integrating by parts we obtain

$$\frac{d}{dz}S_u(z) = \frac{i}{2\pi z}\int_0^{2\pi} u(e^{i\theta})\,d\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = -\frac{i}{2\pi z}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\cdot\frac{d}{d\theta}u(e^{i\theta}),$$

which yields

$$S_{u'}(z) = iz\cdot\frac{d}{dz}(S_u(z)). \tag{2.2}$$

Now take $z = re^{i\theta}$. It follows from the fact

$$\frac{\partial f(re^{i\theta})}{\partial r} = \frac{df}{dz}(re^{i\theta})\,\frac{\partial(re^{i\theta})}{\partial r} = e^{i\theta}\frac{df}{dz}(re^{i\theta}),$$

$$\frac{\partial f(re^{i\theta})}{\partial \theta} = \frac{d}{dz}(re^{i\theta})\,\frac{\partial(re^{i\theta})}{\partial \theta} = ire^{i\theta}\frac{df}{dz}(re^{i\theta}),$$

and (2.2) that

$$\frac{\partial}{\partial r}S_u(z) = e^{i\theta}\frac{d}{dz}(S_u(z)) = \frac{e^{i\theta}}{ire^{i\theta}}S_{u'}(z) = -\frac{i}{|z|}S_{u'}(z),$$

$$\frac{\partial}{\partial \theta}S_u(z) = ire^{i\theta}\frac{d}{dz}(S_u(z)) = \frac{iz}{iz}S_{u'}(z) = S_{u'}(z), \tag{2.3}$$

2.3. **Imaginary part of Schwarz operator.** We recall that given a real-valued function $u(e^{it})$ we can associate the so-called conjugate function (see [Gt]) defined as the imaginary part of the Schwarz operator $S_u(z)$. It turns out that this function plays an important role in further considerations.

We need a special inequality (2.7) for the conjugate functions in the case when the kernel of the Schwarz operator $u(e^{it})$ is equal to absolute value of a holomorphic in the unit disk function. This fact is an easy consequence of the maximum principle and the well known normal derivative lemma in PDE's theory [GT].

Let us consider a subharmonic in the closed unit disk function $v = v(z)$ such that

$$v(e^{i\theta}) = 0, \qquad 0 \le \theta \le 2\pi. \tag{2.4}$$

We assume that $v(z)$ is of C^2 class in $\overline{U}$. Then by the maximum principle we have

$$\frac{\partial}{\partial r} v(re^{i\theta})\bigg|_{r=1} \equiv \lim_{z \to 1-0} \frac{v(e^{i\theta}) - v(re^{i\theta})}{1-r} \ge 0. \tag{2.5}$$

Lemma 2.1. *Let $u(z)$ be a harmonic in $\overline{U}$ function and $\varphi(z)$ be an analytic in $\overline{U}$ function. Let the following equality hold*

$$u(e^{i\theta}) = |\varphi(e^{i\theta})|, \qquad \forall \theta \in [0; 2\pi].$$

Then

$$\frac{\partial u}{\partial r}(re^{i\theta})\bigg|_{r=1} \le \frac{\partial}{\partial r}|\varphi(re^{i\theta})|\bigg|_{r=1}, \qquad \forall \theta \in [0; 2\pi]. \tag{2.6}$$

Proof. Because $|\varphi(z)|$ is subharmonic in $\overline{U}$, the difference $|\varphi(z)| - u(z)$ is too. Applying (2.5) to the latter we arrive at (2.6). ☐

Corollary 2.2. *Let $\varphi(z)$ be an analytic in $\overline{U}$ function such that $\varphi(z) \ne 0$ in $\overline{U}$. Moreover, let $u(e^{i\theta}) = |\varphi(e^{i\theta})|^2$. Then*

$$\operatorname{Im} S_{u'}(e^{i\theta}) \le 2|\varphi(e^{i\theta})|^2 \cdot \operatorname{Re} \varphi^\star(e^{i\theta}). \tag{2.7}$$

Proof. By virtue of (2.3) we obtain

$$\operatorname{Im} S_{u'}(re^{i\theta}) = \operatorname{Im}\left(ir\frac{\partial}{\partial r} S_u(re^{i\theta})\right) = r \operatorname{Re} \frac{\partial}{\partial r} S_u(re^{i\theta}) =$$

$$= r\frac{\partial}{\partial r} \operatorname{Re} S_u(re^{i\theta}). \tag{2.8}$$

Now, let $v(z) = \operatorname{Re} S_u(z)$. Then $v(z)$ is a harmonic in the unit disk U function which is continuous up to the boundary ∂U with its consequent derivatives. It follows by (2.8)

$$S_{u'}(e^{i\theta}) = \lim_{r\to 1-0} \operatorname{Im} S_{u'}(re^{i\theta}) = \lim_{r \to 1-0} r\frac{\partial}{\partial r} \operatorname{Re} S_u(re^{i\theta}) = \lim_{r\to 1-0} r\frac{\partial}{\partial r} v(re^{i\theta}) = \frac{\partial}{\partial r} v(re^{i\theta})\bigg|_{r=1}.$$

On the other hand,

$$v(e^{i\theta}) = \operatorname{Re} S_u(e^{i\theta}) = u(e^{i\theta}) = |\varphi(e^{i\theta})|^2.$$

Using Lemma 2.1 we conclude that

$$a \equiv \frac{\partial}{\partial r}|\varphi(re^{i\theta})|^2\bigg|_{r=1} \ge \frac{\partial}{\partial r} v(re^{i\theta})\bigg|_{r=1} = \operatorname{Im} S_{u'}(e^{i\theta}).$$

To find a we use analytic behavior of $\varphi(z)$ and applying Proposition 2.3 we obtain

$$\frac{\partial}{\partial r}|\varphi(re^{i\theta})|^2 = 2|\varphi(re^{i\theta})|^2 \cdot \frac{\partial}{\partial r}\ln|\varphi(re^{i\theta})| = 2|\varphi(re^{i\theta})|^2 \cdot \frac{1}{r}\operatorname{Re}\varphi^\star(re^{i\theta}),$$

whence

$$a = 2|\varphi(e^{i\theta})|^2 \cdot \operatorname{Re}\varphi^\star(e^{i\theta})$$

which completes the proof. □

Remark 2.1. In what follows we will apply Corollary 2.2 in the special case where $u(e^{it})$ coincides with $|w'_z(z,t)|^{-2}$. In our notations it corresponds to $\varphi(z) = w'_z(z,t)^{-1}$. Clearly, by univalence of $w(z,t)$ the function $\varphi(z)$ takes no zeroes in the unit disk and it follows that

$$\operatorname{Im} S_{u'}(e^{i\theta}) \le \frac{2}{|w'_z(z,t)|}\operatorname{Re}\left(\frac{1}{w'_z(z,t)}\right)^\star = -\frac{2\operatorname{Re} w'^\star_z(z,t)}{|w'_z(z,t)|}.$$

2.4. **Area-preserving homeomorphism.** Here we recall some basic facts related to geometric properties of Hele-Shaw cell properties.

Let $w(z;t)$ be a solution of Problem HS for $t \in [0;b)$. We introduce the characteristics

$$\varrho(z;t) = \ln|w(z;t)|, \qquad \varphi(z;t) = \arg w(z;t).$$

From the univalence property of $w(z;t)$ and the initial condition $w(0;t) = 0$ it follows that functions $\varrho(z;t)$ is defined everywhere in the unit disk punctured at the origin, $\overline{U} \setminus \{0\}$. We can regard there the function $\varphi(z;t)$ to be a smooth on both variables multi-valued branch of $\arg w(z,t)$ (on z) such that the function $\varphi(e^{i\theta},t)$ is single-valued on θ.

Denote by $u(\theta;t) = |w'_z(e^{i\theta};t)|^{-2}$ and take

$$S_u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta;t)\cdot\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta.$$

From (1.10) we have

$$\frac{\partial}{\partial t}\ln\frac{w(z;t)}{z} = \frac{1}{w(z;t)}\cdot\frac{\partial}{\partial t}w(z;t) = \frac{w'_z(z;t)\cdot z}{w(z;t)}\cdot S_u(z),$$

whence

$$\frac{\partial}{\partial t}\ln\frac{w(z;t)}{z} = w^\star(z;t)\cdot S_u(z). \tag{2.9}$$

So, we arrive at the following differential relation

$$\frac{\partial}{\partial t}\left(\varrho(z;t) - \ln|z| + i\varphi(z;t)\right) = w^\star(z;t)\cdot S_u(z).$$

By separating the real and imaginary parts we obtain from the last equality

$$\frac{\partial\varrho(z;t)}{\partial t} = \operatorname{Re} w^\star(z;t)\cdot S_u(z), \qquad \frac{\partial\varphi(z;t)}{\partial t} = \operatorname{Im}(w^\star(z;t)\cdot S_u(z)).$$

Using formulae from the proposition 2.2 gives us further

$$\begin{aligned}\frac{\partial\varrho}{\partial t}\cdot\frac{\partial\varphi}{\partial\theta} - \frac{\partial\varrho}{\partial\theta}\cdot\frac{\partial\varphi}{\partial t} &= \operatorname{Re}(w^\star S_u)\cdot\operatorname{Re}(w^\star) + \operatorname{Im} w^\star\cdot\operatorname{Im}(w^\star S_u) =\\ &= \operatorname{Re}(w^\star S_u)\cdot\overline{w^\star} = |w^\star|^2\operatorname{Re} S_u.\end{aligned}$$

From the other hand, if $z = e^{i\theta}$ then from the characteristic property of Schwarz integral it follows

$$\operatorname{Re} S_u(e^{i\theta}) = \frac{1}{|w'_z(e^{i\theta};t)|^2},$$

and, henceforth,

$$\frac{\partial(\varrho;\varphi)}{\partial(t;\theta)} \equiv \det\begin{pmatrix} \varrho'_t & \varrho'_\theta \\ \varphi'_t & \varphi'_\theta \end{pmatrix} = \frac{w^\star(e^{i\theta};t)|^2}{|w'_z(e^{i\theta};t)|^2} = \frac{1}{|w(e^{i\theta};t)|^2} = e^{-2\varrho(e^{i\theta};t)}.$$

By changing the variables we obtain

$$\frac{\partial\left(e^{2\varrho(e^{i\theta};t)},\varphi(e^{i\theta};t)\right)}{\partial(t,\theta)} = 2. \tag{2.10}$$

and as a consequence of (2.10) we have

Proposition 2.3. *The mapping*

$$f(t;\theta) = \frac{e^{2\varrho(e^{i\theta};t)} + i\varphi(e^{i\theta};t)}{\sqrt{2}} = \frac{|w(e^{i\theta};t)|^2 + i\arg w(e^{i\theta};t)}{\sqrt{2}}$$

is local homeomorphism which preserves the area.

2.5. **Richardson-type theorem for sector.** Let $w(z;t)$ be a HS-solution with the interval of existence $[0,b)$ and $\zeta_k \equiv w(e^{i\theta_k};0)$, $k = 1,2$ be two points on the initial domain's boundary $\partial\Omega(0)$. Then

$$\zeta(t) \equiv w(e^{i\theta_k};t), \qquad t \in [0;b),$$

defines trajectories ejected from ζ_k after the Hele-Shaw flow. Denote by $\Omega_t(\zeta_1,\zeta_2)$ the sector which is cut out from the ring $\Omega(t)\setminus\overline{\Omega(0)}$ by these trajectories (from the point ζ_1 to ζ_2 in the positive direction of the boundary $\partial\Omega(0)$).

Then the following result is a slight generalization of one theorem due to Richardson [Rs72] on the linear dependence of the area of the Hele-Shaw cell on the time parameter (the so called "Area Theorem").

Theorem 2.1. *The area of the sector $\Omega_t(\zeta_1;\zeta_2)$ has linear growth on t*

$$|\Omega_t(\zeta_1;\zeta_2)| = (\theta_2-\theta_1)\cdot t = c(\zeta_1;\zeta_2)\cdot t.$$

Proof. We notice that $\Omega_t(\zeta_1;\zeta_2)$ admits the following representation

$$\Omega(t)\setminus\overline{\Omega(0)} = \{\zeta\in\mathbb{C}: \quad \zeta = w(e^{i\theta};\tau); \quad \theta\in[\theta_1;\theta_2], \quad \tau\in(0;t)\}.$$

Then the area element is

$$dx\wedge dy = \left(\frac{d\zeta + d\overline{\zeta}}{2}\right)\wedge\left(\frac{d\zeta - d\overline{\zeta}}{2i}\right) = \frac{i}{2}d\zeta\wedge d\overline{\zeta}$$

and by the decomposition

$$\zeta = w(e^{i\theta};\tau) = |w(e^{i\theta};\tau)|\cdot e^{i\arg w(e^{i\theta};\tau)} = e^{\varrho(e^{i\theta};\tau) + i\varphi(e^{i\theta};\tau)},$$

we arrive at

$$d\zeta = e^{\varrho+i\varphi}(d\varrho + id\varphi),$$
$$d\overline{\zeta} = e^{\varrho-i\varphi}(d\varrho - id\varphi),$$

which yields

$$dx\wedge dy = \frac{i}{2}e^{2\varrho}(-2id\varrho\wedge d\varphi) = e^{2\varrho}d\varrho\wedge d\varphi =$$
$$= e^{2\varrho}\frac{\partial(\varrho,\varphi)}{\partial(\tau,\theta)}\cdot d\tau\wedge d\theta = \frac{1}{2}\frac{\partial(e^{2\varrho},\varphi)}{\partial(\tau,\theta)}\cdot d\tau\wedge d\theta = d\tau\wedge d\theta.$$

Hence,

$$|\Omega_t(\zeta_1;\zeta_2)| = \iint\limits_{\Omega_t(\zeta_1;\zeta_2)} dx\wedge dy = \int\limits_{\theta_1}^{\theta_2} d\,\theta \int\limits_0^t d\,\tau = (\theta_2-\theta_1)t,$$

□

Substituting $\theta_1 = 0$, $\theta_2 = 2\pi$ we obtain the result due to Richardson

Corollary 2.3 (Area Theorem, [Rs72])**.** *Let $|\Omega(t)|$ be the area of $\Omega(t)$. Then*

$$|\Omega(t)| = |\Omega(0)| + 2\pi t. \tag{2.11}$$

The last fact is a part of theorem due to Richardson [Rs72] concerning the conservative law for complex moments. Actually it states that for every positive integer n

$$\iint_{\Omega(t)} (x+iy)^n\, dx\wedge dy = \iint_{\Omega(0)} (x+iy)^n\, dx\wedge dy, \qquad t\geq 0,$$

while for $n = 0$ one holds linear law (2.11).

From this view point the Proposition 2.3 could be regarded as a local version of Corollary 2.3.

3. Envelope functions

3.1. **Envelopes.** Let $H(\theta;t) \in C^1(\mathbf{R}\times(a,b))$, $H(\theta+2\pi;t) = H(\theta;t)$ be an arbitrary function of two real variables. It follows from continuity of $H(\theta;t)$ that the following function is well-defined

$$h(t) = \max_{\theta\in[0;2\pi]} H(\theta;t),$$

and we call it the *upper envelope* to $H(t)$. Moreover, by periodicity of H the following set is a non-empty compact

$$E_t \equiv E_t(H) = \{\theta\in[0;2\pi] : H(\theta;t) = h(t)\}.$$

Lemma 3.1. *The upper envelope function $h(t)$ is a locally Lipschitz function on $(a;b)$.*

Proof. Really, let the segment $[a_1;b_1]\subset(a;b)$ be chosen arbitrary. Let

$$M_1 = \sup\{|H'_t(\theta;t)| : 0\leq\theta\leq 2\pi, t\in[a_1;b_1]\}.$$

Then continuity of $H'_t(\theta;t)$ implies $M_1 < +\infty$. Let t_1, t_2, $t_1 < t_2$, an arbitrary pair of points from $[a_1;b_1]$ and $\theta_i \in E_{t_i}$, $i = 1,2$, the corresponding extremal points. By the mean value theorem we have

$$\begin{aligned} h(t_2)-h(t_1) = H(\theta_2;t_2) - H(\theta_1;t_1) \leq H(\theta_2;t_2) - H(\theta_2;t_1) = \\ = H'_t(\theta_2;\xi')(t_2-t_1) \leq M_1(t_2-t_1), \end{aligned}$$

where $\xi' \in (t_1;t_2)\subset[a_1;b_1]$.

Similarly, we obtain

$$h(t_2)-h(t_1) \geq H(\theta_1;t_2) - H(\theta_1;t_1) = H'_t(\theta_1;\xi'')(t_2-t_1) \geq -M_1(t_2-t_1).$$

Combing the inequalities obtained we conclude that h satisfies the Lipschitz condition on $[a_1;b_1]$ with constant M_1. Hence, $h\in \mathrm{Lip}_{loc}(a;b)$ and the required assertion is proved.

□

Corollary 3.1. *The function $h(t)$ is an absolutely continuous function on $(a;b)$ and has almost everywhere in $(a;b)$ the first derivative.*

Lemma 3.2. *Let for any $t \in (a;b)$ and $\theta_0 \in E_t(H)$ the inequality $H'_t(\theta_0;t) > 0$ (or $H'_t(\theta_0;t) < 0$) hold. Then $h(t)$ is strictly increasing (or strictly decreasing) Lipschitz function on $(a;b)$ (in particular, it is absolutely continuous).*

Proof. Firstly, we notice that it is sufficient to treat the positive derivative situation only. Really, if it is the case, we can introduce an auxiliary function $\tilde{H}(\theta;t) = H(\theta;-t)$. Then we have $\tilde{h}(t) = h(-t)$ and it follows from $\tilde{H}'_t(\theta;t) = -H'_t(\theta;-t) > 0$ that $\tilde{h}(t)$ is increasing while $h(t)$ is decreasing function.

Now, return to the positive derivative case: $H'_t(\theta_0;t) > 0$ for every $\theta_0 \in E_t(H)$. Then we claim that $h(t)$ has no local maximum points. To prove it we show that given an arbitrary $\tau \in (a;b)$ we can find $\varepsilon_\tau > 0$ such that

$$h(t) > h(\tau), \qquad \forall t \in (\tau; \tau + \varepsilon_\tau). \tag{3.1}$$

Really, otherwise, a point $\tau_0 \in (a;b)$ and a sequence of $\tau_k > \tau_0$, $\tau_k \to \tau_0$ as $k \to \infty$ do exist such that $h(\tau_k) \le h(\tau_0)$. Choose $\theta_0 \in E_{\tau_0}(H)$. Then by the definition of h we have

$$H(\theta_0;\tau_0) = h(\tau_0) \ge h(\tau_k) \ge H(\theta_0;\tau_k)$$

and by the mean value theorem we obtain

$$0 \ge H(\theta_0;\tau_k) - H(\theta_0;\tau_0) = H'_t(\theta_0;\xi_k)(\tau_k - \tau_0),$$

for some $\xi_k \in (\tau_0;\tau_k)$. The last implies that $H'_t(\theta_0;\xi_k) \le 0$ and it follows from continuity of the derivative H'_t and the convergency of $\xi_k \to \tau_0$ that

$$H'_t(\theta_0;\tau_0) = \lim_{k\to+\infty} H'_t(\theta_0;\xi_k) \le 0.$$

But the last inequality contradicts to the fact that H'_t is being positive at $(\theta_0;\tau_0)$. Thus, we have established the existence of ε_τ. Clearly, it implies that $h(t)$ has no local maximum points.

Finally, to show that $h(t)$ is strictly increasing we assume the opposite and find a pair $t_1 < t_2$ form (a,b) such that $h(t_1) \ge h(t_2)$. Then by (3.1) and by continuity of $h(t)$ we have that there exists a global maximum point of $h(t)$ on $[t_1,t_2]$ which contradicts to the last claim and proves the lemma. □

Lemma 3.3. *If we assume that semi-definite inequalities hold in Lemma 3.2 then $h(t)$ will be still monotone but non strictly monotone in general.*

Proof. Really, let the following inequality holds

$$H'_t(\theta_0;t) \ge 0, \qquad \forall \theta_0 \in E_t(H).$$

We can involve an auxiliary function $F(\theta;t) = \varepsilon t + H(\theta,t)$ where $\varepsilon > 0$ is chosen arbitrary. Clearly, the derivative $F'_t(\theta_0;t) = \varepsilon + H'_t(\theta_0;t) > 0$ is positive for any $\theta_0 \in E_t(H)$. Hence, applying Lemma 3.2 we conclude that the function $f(t) \equiv \max_{\theta\in[0;2\pi]} F(\theta,t)$ is strictly increasing on $(a;b)$. But $f(t) = h(t) + \varepsilon t$, whence for any two values $t_2 > t_1$ from $(a;b)$ we obtain

$$0 < f(t_2) - f(t_1) = h(t_2) - h(t_1) + \varepsilon(t_2 - t_1),$$

and taking $\varepsilon \to 0$ we arrive at $h(t_2) \ge h(t_1)$. It follows that h increases. □

3.2. **Barriers.** Here we involve the auxiliary notion of the barriers for envelope functions. This simple but useful technique allows us to establish more delicate properties of the further objects.

Definition 3.1. A function is called $\lambda(t)$ a *lower* (resp. *upper*) *barrier* for $h(t)$ for $t \in (a; b)$ if the inequality

$$H_t'(\theta_0; t) \geq \lambda(t)$$

(resp. $H_t'(\theta_0; t) \leq \lambda(t)$) holds for any $\theta_0 \in E_t(H)$.

Lemma 3.4. *Let $\lambda(t)$ be a lower (resp. upper) barrier for $h(t)$ for $t \in (a; b)$. Then almost everywhere in $(a; b)$ the inequality holds $h'(t) \geq \lambda(t)$ (resp. $h'(t) \leq \lambda(t)$).*

Proof. Let F be a subset of $(a; b)$ where the derivative $h'(t)$ does exist (by Corollary 3.1 $h'(t)$ does exist almost everywhere in $(a; b)$). We claim that the assertion of the lemma is valid everywhere on F. Arguing similarly to that in Lemma 3.2 it is sufficient to treat the case of the lower barrier only.

We fix an arbitrary point $t_0 \in F$ and assume that $t_1 > t_0$. Then for $\theta_i \in E_{t_i}(H)$, $i = 1, 2$:

$$h(t_1) - h(t_0) = H(\theta_1; t_1) - H(\theta_0; t_0) \geq H(\theta_0; t_1) - H(\theta_0; t_0).$$

By the mean value theorem we obtain for some $\xi_1 \in (t_0; t_1)$

$$\frac{h(t_1) - h(t_0)}{t_1 - t_0} \geq H_t'(\theta_0; \xi_1).$$

Taking the limit as $t_1 \to t_0$ we arrive at $h'(t_0) \geq H_t'(\theta_0; t_0) \geq \lambda(t_0)$ and the lemma is proved. □

Corollary 3.2. *Let $\lambda(t)$ be a locally integrable function which is a lower barrier for $h(t)$. Then for all $a_1 < b_1$ from $[a; b]$*

$$h(b_1) - h(a_1) \geq \int_{a_1}^{b_1} \lambda(t)\, dt.$$

Proof. It is just a consequence of absolute continuity of the envelope function $h(t)$:

$$h(b_1) - h(a_1) = \int_{a_1}^{b_1} h'(t)\, dt \geq \int_{a_1}^{b_1} \lambda(t)\, dt.$$

□

Remark 3.1. Certainly, all the assertions were formulated above are still valid for the upper envelope $g(t) = \min_{\theta \in [0; 2\pi]} H(\theta; t)$ which one is an lower envelope for the function $\tilde{H} = -H(\theta; t)$.

3.3. **Inner and outer radii.** Let $w(z; t)$, $t \in [0; b)$ be a HS-solution. As before we denote by $\Omega(t) = w(U; t)$ the image of the unit disk by the conformal mapping representing the solution. By the definition we have $\zeta = 0 \in \Omega(t)$ and it allows us to introduce the inner and outer radii of $\Omega(t)$ with respect to the origin by letting

$$R_i(t) = \min_{\theta} |w(e^{i\theta}; t)|,$$

$$R_e(t) = \max_{z \in \overline{U}} |w(z; t)| = \max_{\theta} |w(e^{i\theta}; t)|.$$

Clearly, $R_i(t) = \text{dist}\,(0, \partial\Omega(t))$.

Definition 3.2. Given a real $R \geq 0$, we call the function

$$w(z;t) = z\sqrt{2t + R^2} \tag{3.2}$$

to be a *trivial* HS-solution.

Obviously, (3.2) produces an evolution family (a solution to the Hele-Shaw equation) with initial domain is being the disk of radius R with the origin as its center. Moreover, it immediately follows from Kufarev–Vinogradov local existence theorem that the trivial solutions (3.2) are the only solutions to Problem HS which have an initial data $w(z,t_0) = c_0 z$, $c_0 = const$.

In further considerations we need also the following characteristics of $w(z;t)$ which are well-known as the distortions characteristics in the geometric function theory

$$M(t) = \max_{|z|=1} |w'_z(z;t)|^2, \qquad m(t) = \min_{|z|=1} |w'_z(z;t)|^2. \tag{3.3}$$

As a consequence of the maximum principle for subharmonic functions we also have

$$M(t) = \max_{|z|\leq 1} |w'_z(z;t)|^2.$$

Theorem 3.1. *Let $w(z;t)$ be a non-trivial solution to Problem HS for all $t \in [0;b)$. Then $R_i(t)$ and $R_e(t)$ are strictly increasing locally-Lipschitz functions in $[0;b)$ and almost everywhere in $[0;b)$ there holds*

$$R'_e(t) \geq R_e(t) \cdot \frac{1}{M(t)}, \tag{3.4}$$

$$R'_i(t) \leq R_i(t) \cdot \frac{1}{m(t)}. \tag{3.5}$$

Proof. We consider $t_0 \in [0;b)$ and let $\theta_0 \in [0;2\pi]$ be a value such that

$$R_e(t_0) = |w(e^{i\theta_0};t_0)|. \tag{3.6}$$

Then

$$v(z;t) = \ln\left|\frac{w(z;t)}{z}\right|$$

is an analytic function which is well-defined everywhere in $\overline{U}$ for all $t_0 \in [0;b)$ by virtue of $w(z;t)/z \neq 0$ (i.e. $w(z,t)$ is univalent in $\overline{U}$, $w(0,t) = 0$ and $w'_z(0;t) \neq 0$). Because of non-triviality of w the fraction $w(z;t_0)/z$ is not identically constant, we can apply Schwarz lemma:

$$|w(z;t)| < R_e(t) \cdot |z|$$

for all $z \in U$. Hence,

$$\ln R_e(t) = \max_{|z|\leq 1} v(z;t) = \max_{|z|=1} v(z;t), \tag{3.7}$$

and what is more, for $t = t_0$ the maximum of the right hand of (3.7) attains in the same points $z = e^{i\theta}$ that (3.6) does.

On the other hand, we have

$$\frac{\partial}{\partial t} v(z;t) = \frac{\partial}{\partial t} \ln\left|\frac{w(z;t)}{z}\right|,$$

and by virtue of (2.9),

$$\frac{\partial}{\partial t} v(z;t) = \operatorname{Re} \frac{\partial}{\partial t} \ln\left(\frac{w(z;t)}{z}\right) = \operatorname{Re}(w^{\star}(z;t) \cdot S_u(z)). \tag{3.8}$$

Now, using the extremal properties of θ_0 and relation (3.7) we find that

$$\frac{\partial}{\partial\theta} v(e^{i\theta}; t_0)\bigg|_{\theta=\theta_0} = 0,$$

which yields from Proposition 2.2 that

$$-\operatorname{Im}\left(\frac{w(z;t)}{z}\right)^{\star} = 0, \qquad \text{for} \quad z = e^{i\theta_0}, \quad t = t_0.$$

Simplifying the left hand of the previous equality (see Proposition 2.1) we arrive at

$$\operatorname{Im} w^{\star}(e^{i\theta_0}; t_0) = 0. \tag{3.9}$$

and substituting of (3.9) in (3.8) implies

$$\frac{\partial}{\partial t} v(e^{i\theta_0}; t)\bigg|_{t=t_0} = \operatorname{Re} w^{\star}(e^{i\theta_0}; t_0) \cdot \operatorname{Re} S_u(e^{i\theta_0}) - \operatorname{Im} w^{\star}(e^{i\theta_0}; t_0) \cdot \operatorname{Im} S_u(e^{i\theta_0}) =$$

$$= u(e^{i\theta_0}; t_0) \cdot \operatorname{Re} w^{\star}(e^{i\theta_0}; t_0) = \frac{1}{|w_z'(e^{i\theta_0}; t_0)|^2} \operatorname{Re} w^{\star}(e^{i\theta_0}; t_0). \tag{3.10}$$

We notice now that $v(z; t_0)$ is actually a harmonic in $\overline{U}$ function which attains its maximum at $z = e^{i\theta_0}$. Hence, it follows from the normal derivative lemma [GT] that the derivative of $v(z; t_0)$ along the outward normal to the unit circle at the point $z = e^{i\theta_0}$ is positive:

$$\frac{\partial}{\partial r} v(re^{i\theta_0}; t_0) > 0, \qquad \text{for} \qquad r = 1. \tag{3.11}$$

The direct computations show that

$$\frac{\partial}{\partial r} v(re^{i\theta_0}; t_0) = \frac{\partial}{\partial r} \ln\left|\frac{w(re^{i\theta_0}; t_0)}{r}\right| = \frac{1}{r} \operatorname{Re} w^{\star}(re^{i\theta_0}; t_0) - \frac{1}{r},$$

which implies after the substitution of $r = 1$ and (3.11) that

$$\operatorname{Re} w^{\star}(e^{i\theta_0}; t_0) > 1.$$

We notice also that it follows from (3.9) that $w^{\star}(e^{i\theta_0}; t_0)$ takes a real value. Thus, we have from (3.10)

$$v_t'(e^{i\theta_0}; t_0) = \frac{1}{|w_z'(e^{i\theta_0}; t_0)|^2} \cdot \operatorname{Re} w^{\star}(e^{i\theta_0}; t_0) > \frac{1}{|w_z'(e^{i\theta_0}; t_0)|^2} \geq \frac{1}{M(t_0)}. \tag{3.12}$$

On the other hand, we notice that $\ln R_e(t)$ is actually an upper envelope to $v(e^{i\theta}; t)$ with the lower barrier is being equal to $1/M(t)$. Hence, from (3.12) and Lemmas 3.1, 3.4 we conclude that the function $\ln R_e(t)$ is strictly increasing and locally-Lipschitz in $[0; b)$. Moreover, almost everywhere in $[0; b)$ one holds

$$\frac{d}{dt} \ln R_e(t) \geq \frac{1}{M(t)}.$$

Thus, we have established the required property (3.4).

To prove (3.5) we notice that by virtue of harmonicity of $v(z; t)$ there exists a point $\theta_1(t)$ such that

$$\ln R_i(t) = \min_{|z|=1} v(z, t) = \min_{|z|\leq 1} v(z, t) \equiv v(e^{i\theta_1(t)}; t).$$

Arguing similar to that above we can obtain that

$$\operatorname{Im} w^{\star}(e^{i\theta_1(t_0)}; t_0) = 0$$

and by the definition of the inner radius and by the normal derivative lemma we have

$$\frac{\partial}{\partial r} v(re^{i\theta_1}; t_0)\Big|_{r=1} < 0, \qquad \theta_1 = \theta_1(t_0),$$

whence

$$\operatorname{Re} w^\star(e^{i\theta_1}; t_0) < 1.$$

Thus,

$$v'_t(e^{i\theta_1}; t_0) < \frac{1}{|w'_z(e^{i\theta_1}; t_0)|^2} \leq \frac{1}{m(t_0)}.$$

Since the function $\ln R_i(t)$ is a lower envelope of $v(e^{i\theta}; t)$ with an upper barrier $1/m(t)$ and we similarly arrive at (3.5).

We must only to show that $R_i(t)$ does strictly increase. By analogy with the previous case, to prove the required property it is sufficient to establish the inequality $\operatorname{Re} w^\star(e^{i\theta_1}; t_0) > 0$. With this aim we recall that $\operatorname{Im} w^\star(e^{i\theta_1}; t_0) = 0$. Moreover, it follows from the property of $w(z; t)$ being univalent in $\overline{U}$ that $w'_z(z; t) \neq 0$ in $\overline{U}$ and we have as a consequence that $w^\star(e^{i\theta_1}; t_0) \neq 0$. Thus, $\beta = \operatorname{Re} w^\star(e^{i\theta_1}; t_0) \neq 0$.

On the other hand, applying Proposition 2.2 we see that

$$\beta = \operatorname{Re} w^\star(e^{i\theta_1}; t_0) = \frac{\partial}{\partial \theta} \arg \left(w(e^{i\theta}; t_0)\right)\Big|_{\theta=\theta_0}.$$

We notice that by the argument principle, $w(e^{i\theta}; t)$ gives a parametrization of the corresponding plane Jordan curve in the right direction with respect to the origin (in other words, it agrees with the orientation of the boundary of $\Omega(t) = w(U; t_0)$).

To finish the proof we notice that the segment between $\zeta = 0$ and $\zeta = w(e^{i\theta_1}; t_0)$ is entirely containing in $\overline{\Omega(t_0)}$ (by the definition of $R_i(t)$). It follows that near the point θ_1 the function $\arg w(e^{i\theta}; t_0)$ is actually strictly increasing (e.g., see [Ax, n. 67]). Thus, $\beta \geq 0$ and because $\beta \neq 0$ the latter is a strictly positive quantity which completes the proof. □

Remark 3.2. Some related results to that ones in Theorem 3.1 have been obtained in [Kh] and [HKh]. We notice that our estimates (4.17) and (4.18) following from (3.4) for the upper radius are more sharp than those given in [HKh].

4. *A priori* ESTIMATES

4.1. **Maximal distortion function.** Further we need the modification of (2.9). We have from (2.9) and (2.2)

$$\frac{\partial}{\partial t}\left(\frac{w'_z}{w}\right) = (w^\star)'_z \cdot S_u + \frac{w^\star}{iz} \cdot S_{u'}.$$

and by further multiplication on $z/w^\star$ we arrive at

$$\frac{1}{w^\star}\frac{\partial}{\partial t}(w^\star) = w^{\star\star} \cdot S_u - iS_{u'}.$$

We have shown above (see proof of Theorem 3.1) that $w(z,t)/z$ as well as $w'_z(z,t)$ are non-vanishing in $\overline{U}$ functions provided that $w \in O(\overline{U})$. It follows that $w^\star \neq 0$ for a HS-solution and due to simply-connectedness of U, one can well define there the logarithm branches of $\ln w^\star(z; t)$ and $\ln w'_z(z; t)$. Moreover, an easy computation shows that for all $w \in O(\overline{U})$:

$$w^\star(0, t) = 1.$$

Thus,

$$\frac{\partial \ln w^\star}{\partial t} = w^{\star\star} S_u - iS_{u'}. \tag{4.1}$$

From (4.1) and (2.9) we obtain

$$\frac{\partial}{\partial t}\ln(w'_z) = (zw'_z)^\star S_u - iS_{u'} = [1 + (w'_z)^\star] S_u - iS_{u'}. \tag{4.2}$$

The following property characterizes the asymptotic behavior of the maximal distortion function $M(t)$ (3.3).

Theorem 4.1. *Let $w(z;t)$ be a non-trivial HS-solution in $[0;b)$. Then the function $M(t)-2t$ is strictly decreasing and locally-Lipschitz in $[0;b)$.*

Proof. To prove this fact we fix $t_0 \in [0;b)$ and choose $\theta_0 \in [0;2\pi]$ such that $|w'_z(e^{i\theta_0};t)|^2 = M(t_0)$. Similarly to that above we have from extremal property of θ_0 and Proposition 2.2 that

$$0 = \frac{\partial}{\partial \theta}\ln|w'_z(e^{i\theta};t_0)|\Big|_{\theta=\theta_0} = -\operatorname{Im} w'^{\star}_z(e^{i\theta_0};t_0). \tag{4.3}$$

On the other hand, we notice that $v(z;t) \equiv \ln|w'_z(z;t)|$ is a harmonic function of z which is continuous in $\overline{U}$. Moreover, again by extremality of θ_0 and non-triviality of the solution we can apply the normal derivative lemma which implies

$$v'_r(re^{i\theta_0};t_0)\big|_{r=1} > 0,$$

whence taking into account the fact

$$v'_r(re^{i\theta_0};t_0) = \frac{\partial}{\partial r}\ln|w'_z(re^{i\theta_0};t_0)| = \frac{1}{r}\operatorname{Re} w'^{\star}_z(re^{i\theta_0};t_0),$$

we obtain for $r = 1$

$$\operatorname{Re} w'^{\star}_z(e^{i\theta_0};t_0) > 0. \tag{4.4}$$

From (4.2) we see that

$$\begin{aligned}\frac{\partial}{\partial t}\ln|w'_z(e^{i\theta_0};t_0)| &= \operatorname{Re}(1 + w'^{\star}_z(e^{i\theta_0};t_0)) \cdot \operatorname{Re} S_u(e^{i\theta_0};t_0) - \\ &\quad - \operatorname{Im}(1 + w'^{\star}_z(e^{i\theta_0};t_0)) \cdot \operatorname{Im} S_u(e^{i\theta_0};t_0) + \operatorname{Im} S_{u'}(e^{i\theta_0};t_0).\end{aligned} \tag{4.5}$$

The middle term in the right hand of (4.5) is equal to zero since (4.3).

To estimate the last term we can apply Corollary 2.2. In our notations $\varphi(z) = 1/w'_z(z;t)$ which yields

$$\operatorname{Im} S_{u'}(e^{i\theta};t) \le \frac{2}{|w'_z(e^{i\theta};t)|^2} \cdot \operatorname{Re}\left(\frac{1}{w'_z(z;t)}\right)^\star\Bigg|_{z=e^{i\theta}} = -\frac{2\operatorname{Re} w'^{\star}_z(e^{i\theta};t)}{|w(e^{i\theta};t)|^2}. \tag{4.6}$$

Consequently, we have from (4.5)

$$\frac{\partial}{\partial t}\ln|w'_z(e^{i\theta_0};t)| = \frac{1 - \operatorname{Re} w'^{\star}_z(e^{i\theta_0};t)}{|w'_z(e^{i\theta_0};t)|^2},$$

that is just the same as the following

$$\frac{1}{2}\cdot\frac{\partial}{\partial t}|w'_z(e^{i\theta_0};t)|^2 \le 1 - \operatorname{Re} w'^{\star}_z(e^{i\theta_0};t).$$

Simplifying the last inequality yields

$$\frac{\partial}{\partial t}(|w'_z(e^{i\theta_0};t)|^2 - 2t) \le -2\operatorname{Re} w'^{\star}_z(e^{i\theta_0};t).$$

But the function $M(t)-2t$ is an upper envelope to $|w'_z(e^{i\theta_0};t)|^2-2t$. Thus, using (4.4) and barriers lemmas we obtain that $(M(t)-2t)$ is a locally Lipschitz strictly decreasing function which completes the proof. □

4.2. **Estimates for $M(t)$.** Now we study the function $H(t)=M(t)-2t$ in more detail. Firstly we notice that the following is an immediate consequence of decreasing property of H:

$$H(t)=M(t)-2t\le H(0)=M(0).$$

Now we show that $H(t)$ is bounded from below. To do it we observe that by the Area Theorem (Corollary 2.3) the following relation holds

$$|\Omega(t)|\equiv\iint\limits_U |w'_z(z;t)|^2\,dxdy=|\Omega(0)|+2\pi t. \tag{4.7}$$

Taking into account the definition of $M(t)$ we can derive from (4.7) that

$$M(t)\ge\frac{1}{\pi}|\Omega(0)|+2t,$$

i.e. $H(t)\ge\frac{1}{\pi}|\Omega(0)|$. We arrive at the following result

Corollary 4.1. *Let $w(z;t)$ be a HS-solution. Then the function*

$$H(t)\equiv\max_{|z|\le 1}|w'_z(z;t)|^2-2t$$

strictly decreasing and locally-Lipschitz in $[0;b)$ and it is bounded from below by the quantity which depends only on the initial data $\Omega(0)$:

$$H(t)\ge\frac{1}{\pi}|\Omega(0)|. \tag{4.8}$$

In particular, the following limit does exist $\lim_{t\to b-0}H(t)$.

Remark 4.1. We emphasize that the last limit is of most interest when $b=+\infty$. Moreover, it follows from direct computations (valid e.g. for the trivial or polynomial solutions) estimate (4.8) is asymptotically sharp.

Let $a_1(t)=w'_z(0;t)$ be the leading coefficient of the Taylor expansion of $w(z;t)$

$$w(z;t)=a_1(t)z+a_2(t)z^2+a_3(t)z^3\dots. \tag{4.9}$$

By the definition of a HS-solution, $a_1(t)$ is a positive real. Moreover, it easily follows from the definition of $M(t)$ that

$$a_1^2(t)=|w'_z(0;t)|^2\le\max_{|z|\le 1}|w'_z(0;t)|^2=M(t).$$

On the other hand, we have

$$\frac{w'_t(z;t)}{z}=w'_z(z;t)\cdot\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|w'_z(e^{i\theta};t)|^2}\cdot\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta,$$

and substituting $z=0$ we see that

$$\frac{da_1(t)}{dt}=a_1(t)\cdot\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1}{|w'_z(e^{i\theta};t)|^2}\,d\theta,$$

whence

$$\frac{d}{dt}\ln a_1(t) \geq \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1}{M(t)}\,d\theta = \frac{1}{M(t)} \geq \frac{1}{M(0)+2t} = \frac{1}{2}\frac{d}{dt}\ln(M(0)+2t).$$

Integration of the last inequality yields that

$$\frac{a_1^2(t)}{M(0)+2t}$$

is an increasing function.

Theorem 4.2. *The function $a_1^2(t)-2t$ is strictly increasing and of Lip_{loc} provided that $w(z;t)$ is a non-trivial HS-solution. Moreover,*

$$a_1^2(t) - 2t \leq \frac{1}{\pi}|\Omega(0)|. \tag{4.10}$$

Proof. The function $v(z;t) = \ln|w'_z(z;t)|$ is harmonic and continuous in $\overline{U}$. By the mean value theorem for harmonic functions we have

$$\frac{1}{2\pi}\int\limits_0^{2\pi} v(e^{i\theta};t)\,d\theta = v(0;t) = \ln a_1(t). \tag{4.11}$$

On the other hand, for any continuous in $[0;2\pi]$ function $\lambda(\theta)$ the Jensen inequality [HLP]

$$\frac{1}{2\pi}\int\limits_0^{2\pi} e^{\lambda(\theta)}\,d\theta \geq \exp\frac{1}{2\pi}\int\limits_0^{2\pi}\lambda(\theta)\,d\theta. \tag{4.12}$$

Using (4.11) and (4.12) we get from (4.10)

$$\frac{a_1'(t)}{a_1(t)} = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{-2v(e^{i\theta};t)}\,d\theta \geq \exp\left[\frac{1}{2\pi}\int\limits_0^{2\pi} -2v(e^{i\theta};t)\,d\theta\right] = \frac{1}{a_1^2(t)},$$

whence $2a_1'a_1 = (a_1^2)' \geq 2$ and as a consequence, the function $a_1^2(t)-2t$ is decreasing. To establish (4.10), it remains to notice that

$$\pi a_1^2(t) \leq \iint\limits_{\overline{U}} |w'_z(z;t)|^2\,dxdy = |\Omega(t)| = |\Omega(0)| + 2\pi t.$$

□

4.3. **Estimates for $\sigma(t)$.** Another helpful consequence of the results formulated above is the following uniform estimate for the Taylor coefficients of a HS-solution.

Theorem 4.3. *Let $w(z;t)$ be a HS-solution in $[0;b)$ with expansion (4.9). Then for any $t \in [0;b)$*

$$g(t) \equiv \sum_{k=2}^{\infty} k|a_k^2(t)| \leq \frac{1}{\pi}|\Omega(0)| - a_1^2(0).$$

and $g(t)$ is a decreasing function of t.

Proof. Indeed, because the mapping $w(z,t)$ is univalent we have

$$\pi \sum_{k=1}^{\infty} k|a_k(t)|^2 = \iint_{\overline{U}} |w'_z(z;t)|^2\, dxdy = |\Omega(t)| = |\Omega(0)| + 2\pi t.$$

It follows that

$$\sum_{k=2}^{\infty} k|a_k(t)|^2 = \frac{1}{\pi}|\Omega(0)| + 2t - a_1^2(t) \equiv g(t).$$

From the Theorem 4.2 we conclude that the function $g(t)$ is strictly decreasing for $t \in [0;b)$. Moreover, $g(0) = \frac{1}{\pi}|\Omega(0)| - a_1^2(0)$ gives us the desired inequality. □

Theorem 4.4. *Let $w(z;t)$ be a nontrivial HS-solution and*

$$\sigma(t) = \max_{z\in\overline{U}} |w(z;t)\cdot w'_z(z;t)|.$$

Then $\sigma(t) - 2t$ is strictly decreasing and locally-Lipschitz and

$$a_1(0)^2 \le a_1(t)^2 - 2t \le \sigma(t) - 2t \le \sigma(0). \tag{4.13}$$

Proof. Multiplying (2.9) by 2 and adding with (4.1) we have

$$\frac{\partial}{\partial t} \ln\left(\frac{w^2}{z^2}\cdot\frac{zw'_z}{w}\right) = (w^{\star\star} + 2w^\star)S_u - iS_{u'},$$

which implies by virtue of $w^{\star\star} + 2w^\star = (w^\star w^2)^\star = (w'_z zw)^\star$ that

$$\frac{\partial}{\partial t} \ln\left(\frac{w}{z}\cdot w'_z\right) = (w_z'^{\star} + w^\star + 1)S_u - iS_{u'}. \tag{4.14}$$

We consider as a test harmonic function $v(z;t) = \ln\left|\frac{w}{z}\cdot w'_z\right|$. Then $v(z;t)$ is well-defined in $\overline{U}$ by univalency of $w(z;t)$ and $w(z;t) \in O(U)$. Thus,

$$\ln\sigma(t) = \max_{|z|=1} \ln|w(z;t)\cdot w'_z| = \max_{|z|\le 1} v(z;t) = \max_{|z|=1} v(z;t).$$

Let $z = e^{i\theta_0}$ be a maximum point of $v(z;t)$ for $t = t_0 \le 0$. Arguing similar to that above we arrive at

$$\frac{\partial}{\partial\theta}v(e^{i\theta};t_0) = \frac{\partial}{\partial\theta}\ln\left|\frac{w}{z}\cdot w'_z\right| = -\operatorname{Im}\left(\frac{w}{z}\cdot w'_z\right)^\star,$$

whence after $\theta = \theta_0$ we obtain for $t = t_0$ that

$$\operatorname{Im}(w_z'^{\star} + w^\star - 1) = 0.$$

The last can be rewritten as

$$\operatorname{Im}(w_z'^{\star}(e^{i\theta_0};t_0) + w^\star(e^{i\theta_0};t_0)) = 0. \tag{4.15}$$

Now we suppose that $\ln\left|\frac{w}{z}\cdot w'_z\right|$ is identically constant for some $t = t_0 \ge 0$. It follows then that $ww'_z = cz$, or $w^2 = cz^2 + c_1$. Taking into account that $w(0,t_0) = 0$ we obtain $c_1 = 0$ which implies that $w(z;t)$ is trivial and leads to a contradiction. Thus, given an arbitrary t the function $\ln\left|\frac{w}{z}\cdot w'_z\right|$ is different from a constant. Consequently, due to extremality of $z = e^{i\theta_0}$ we have for a normal derivative

$$\left.\frac{\partial}{\partial\varrho}v(\varrho e^{i\theta};t_0)\right|_{\varrho=1} > 0.$$

By simplification we obtain

$$\operatorname{Re}(w_z'^{\star} + w^\star - 1) > 0, \qquad z = e^{i\theta_0}, \qquad t = t_0 \tag{4.16}$$

and going to the real part of (4.14) we arrive at

$$\frac{\partial}{\partial t}\ln v(z;t) = \operatorname{Re}(w_z'^{\star} + w^{\star} + 1)\operatorname{Re} S_u - \operatorname{Im}(w_z'^{\star} + w^{\star} + 1)\operatorname{Im} S_u + \operatorname{Im} S_{u'}.$$

We notice that at $z = e^{i\theta_0}$ and $t = t_0$ relations (4.15) and (4.16) does hold. Therefore, by virtue of (4.6) we find from the last inequality that

$$\begin{aligned}\frac{\partial}{\partial t}\ln v(e^{i\theta_0};t)\Big|_{t=t_0} &\le \frac{1}{|w_z'|^2}\operatorname{Re}(w_z'^{\star} + w^{\star} + 1) - \frac{2\operatorname{Re} w_z'^{\star}}{|w_z'|^2} = \\ &= -\frac{1}{|w_z'|^2}\operatorname{Re}(w_z'^{\star} + w^{\star} - 1) + \frac{2\operatorname{Re} w^{\star}}{|w_z'|^2} < \\ &< \frac{2\operatorname{Re} w^{\star}}{|w_z'|^2} \le \frac{2}{\left|w_z' \cdot \dfrac{w}{z}\right|} = \frac{2}{v(e^{i\theta_0};t_0)}.\end{aligned}$$

It follows that

$$\frac{\partial}{\partial t}(v(\varrho e^{i\theta};t_0) - 2t)\Big|_{t=t_0} > 0$$

and by Lemma 3.2 we obtain that

$$\sigma(t) - 2t = \max_{|z|=1} v(z;t) - 2t$$

is strictly decreasing.

To complete the proof we need to settle (4.13). The right side of the inequality follows from monotonicity of $\sigma(t) - 2t$. To prove the left hand we observe that

$$\max_{z=1} v(z;t) - 2t \ge v(0;t) - 2t = \lim_{z\to 0}\left|\frac{w(z;t)}{z}\cdot w_z'(z;t)\right| - 2t =$$

$$= |w_z'(0;t)|^2 - 2t = a_1(t)^2 - 2t.$$

By Theorem 4.2 the function $a_1(t)^2 - 2t$ is strictly increasing and

$$a_1(t)^2 - 2t \ge a_1(0)^2,$$

which completes the proof. ☐

4.4. **Collective estimates.** The next assertion establishes the connection of the inner radius $R_e(t)$ with characteristics $\sigma(t)$ and $M(t)$.

Theorem 4.5. *Let $w(z;t)$ be a HS-solution. Then*

$$\frac{1}{2}R_e^2(t) \le \sigma(t) \le R_e(t)\sqrt{M(t)} \le M(t) \tag{4.17}$$

for any $t \in [0;b)$. In particular,

$$R_e(t) \ge a_1(t) \ge \sqrt{2t + a_1(0)}. \tag{4.18}$$

Proof. It is a direct consequence of the definitions that

$$\sigma(t) = \max_{|z|=1}|w(z;t)w_z'(z;t)| \le \max_{|z|=1}|w(z;t)|\max_{|z|=1}|w_z'(z;t)| = R_e(t)\sqrt{M(t)}.$$

On the other hand, let ζ be a point where the value $R_e(t) = \max_{z=1} |w(z;t)| = |w(\zeta;t)|$ is attained. Then by harmonicity of $|w(z;t)|$, ζ belongs to the unit circle ∂U, i.e. $|\zeta| = 1$. We have

$$R_e(t) = |w(\zeta;t) - w(0;t)| = \left| \int_0^1 w'_z(\varrho\zeta;t)\, d\varrho \right| \le \int_0^1 \sqrt{M(t)}\, d\varrho = \sqrt{M(t)},$$

whence

$$R_e(t) \le \sqrt{M(t)},$$

and the right hand of (4.17) is proved.

Similarly, by the definition of outer radius we have

$$R_e^2(t) = |w(\zeta;t)|^2 = \left| \int_0^1 (w^2(\varrho\zeta;t))'_\varrho\, d\varrho \right| = 2\left| \int_0^1 w'_z(\varrho\zeta;t)\cdot w(\varrho\zeta,t)\,\zeta\, d\varrho \right| \le 2\sigma(t),$$

which proves (4.17). Finally, applying Schwarz lemma for $w(z,t)/R_e(t)$ and using (4.13) we arrive at (4.18). □

5. Main results

5.1. **Functions with bounded angle variation.** Let us consider the following function

$$\Lambda(z;t) = \arg w^\star(z;t) = \arg \frac{zw'_z(z,t)}{w(z,t)}$$

where $w(z;t)$ be a HS-solution. Without loss of generality, we can assume that $\Lambda(z;t)$ is defined as

$$\Lambda(0;t) = \arg w^\star(0;t) = \arg 1 = 0$$

because $w^\star(z;t) \ne 0$ in the unit disk and $w^\star(0,t)$ is a positive real. Denote by

$$\Lambda^+(t) = \max_{\theta\in[0;2\pi]} \Lambda(e^{i\theta};t) = \max_{|z|\le 1} \Lambda(z;t),$$

$$\Lambda^-(t) = \min_{\theta\in[0;2\pi]} \Lambda(e^{i\theta};t) = \min_{|z|\le 1} \Lambda(z;t),$$

where we use harmonicity of $\Lambda(z;t)$ for the equality of boundary and inner extremal values.

We notice that provided that $w(z;t)$ is non-trivial the function $\Lambda(z,t)$ is always different from a constant. Moreover, because $\Lambda(0,t) = 0$ the following inequality holds

$$\Lambda^-(t) < 0 < \Lambda^+(t).$$

Theorem 5.1. *Let $w(z;t)$ be a HS-solution in $[0;b)$ such that*

$$\Lambda^+(0) \in [0;\pi), \qquad (or \quad \Lambda^-(0) \in (-\pi;0]).$$

Then Λ^+ is strictly decreasing (or Λ^- strictly increasing) and locally-Lipschitz in $[0;b)$.

Proof. Firstly, we treat the decreasing of Λ^+. Really, consider $t_0 \in [0;b)$ and $\theta_0 = \theta(t_0)$ the values such that

$$\Lambda^+(t_0) = \Lambda(e^{i\theta_0};t_0) \equiv \arg w^\star(e^{i\theta_0};t_0).$$

By extremal property of θ_0 we have (see Proposition 2.2)

$$0 = \frac{\partial}{\partial\theta}\Lambda(e^{i\theta};t_0) = \operatorname{Re} w^{\star\star}(e^{i\theta_0};t_0),$$

i.e. $w^{\star\star}(e^{i\theta_0};t_0)$ is a purely imaginary value. On the other hand, θ_0 is a global maximum point of harmonic function $\Lambda(z;t_0)$ in the closed unit disk and by the normal derivative lemma we have

$$\frac{\partial}{\partial r}\Lambda(re^{i\theta_0};t_0)\bigg|_{r=1} > 0,$$

whence

$$\operatorname{Im} w^{\star\star}(e^{i\theta_0};t_0) \equiv \lambda > 0, \tag{5.1}$$

where $w^{\star\star}(e^{i\theta_0};t_0) = i\lambda$. It follows from (4.1) after taking the imaginary part at the point $z = e^{i\theta_0}$, $t = t_0$, that

$$\frac{\partial\Lambda}{\partial t} = \operatorname{Re} S_u \cdot \operatorname{Im} w^{\star\star} + \operatorname{Im} S_u \cdot \operatorname{Re} w^{\star\star} - \operatorname{Re} S_{u'} = \frac{\operatorname{Im} w^{\star\star}(e^{i\theta_0};t_0)}{|w'_z(e^{i\theta_0};t_0)|^2} - u'_\theta(e^{i\theta_0};t_0),$$

where $u(\theta;t) = |w'_z(e^{i\theta};t_0)|^{-2}$. By Proposition 2.2 we obtain

$$(\ln u)'_\theta = -2(\ln|w'_z(e^{i\theta};t_0)|)'_\theta = 2\operatorname{Im} w'^{\star}_z(e^{i\theta_0};t_0).$$

Now, taking into account (5.1) and (2.1) we have

$$\frac{\partial\Lambda}{\partial t}(e^{i\theta_0};t_0) = \frac{1}{|w'_z(e^{i\theta_0};t)|^2}\left(\operatorname{Im} w^{\star\star}(e^{i\theta_0};t_0) - 2\operatorname{Im} w'^{\star}_z(e^{i\theta_0};t_0)\right) =$$

$$= \frac{1}{|w'_z(e^{i\theta_0};t)|^2}\left(-\operatorname{Im} w^{\star\star}(e^{i\theta_0};t_0) - 2\operatorname{Im} w^{\star}(e^{i\theta_0};t_0)\right). \tag{5.2}$$

We see from (5.1) and the fact $\arg w^\star(e^{i\theta_0};t_0) \in [0;\pi)$ that the last term in (5.2) is negative. Using Lemma 3.2 we conclude that the upper envelope $\Lambda^+(t)$ is strictly decreasing in $[0;b)$.

Treatment of $\Lambda^-(t)$ is similar and theorem is proved. □

5.2. **Invariant classes and asymptotic behavior.** We notice that the previous characteristics Λ are vastly known in the univalent functions theory. In particularly, (see [Bz], [Zm]) that if a function $w(z)$ defined in the unit disk and has there $\operatorname{Re} aw^\star$ to be positive for a number $a \in \mathbb{C}$ then $w(z)$ is actually univalent in U.

Definition 5.1. We say that $f(z)$, $f(0) = 0$, $f'_z(0) > 0$, is of class $T(\alpha;\beta)$ with $-\pi < \alpha \le \beta < \pi$ if

$$\arg f^\star(e^{i\theta}) \in [\alpha;\beta]$$

for all $\theta \in [0;2\pi]$.

The class $T(\alpha,\beta)$ contains also non-univalent functions but in what follows we dealing with the HS-solutions (and it follows with univalent subclass in $T(\alpha,\beta)$) only. In particular, we list below some well-known subclasses .

- Star-like functions from S^* which are after a suitable normalization by the leading Taylor coefficient ($a_1 = 1$) can be identified with $T(-\frac{\pi}{2},\frac{\pi}{2})$;
- δ-spiral-like functions S_δ which have the following characterizations in our notations: $S_\delta = T(-\frac{\pi}{2}-\delta,\frac{\pi}{2}-\delta)$. Actually, the functions $f(z)$ of this class can be defined as $f(0) = 0$, $f'(0) > 0$ and for some $\delta \in (-\pi/2;\pi/2)$ one holds

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\cdot e^{i\delta}\right) \ge 0, \qquad \forall z : |z| = 1.$$

- α-Star-like functions (S^*_α) which naturally can be included in the star-like class: a function $f(z)$ is of S^*_α iff $f(0)=0$, $f'(0)>0$ and

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \le \alpha < \frac{\pi}{2}, \qquad \forall z : |z|=1.$$

  One easily sees that $S^\star_\alpha = T(-\alpha;\alpha)$.

It follows from Theorem 5.1

Corollary 5.1. *For any admissible α and β the class $T(\alpha;\beta)$ is HS-invariant (i.e. if $w(z;t)$ is a HS-solution with $w(0;t)\in T(\alpha;\beta)$ then $w(z;t)\in T(\alpha;\beta)$ for all $t\in[0;b)$).*

Corollary 5.2. *The classes $S^\star$, S_δ and $S^\star_\alpha$ are HS-invariant.*

First results in this direction were obtained by D. Prokhorov, A. Vasiliev and Yu. Khokhlov in [HPV]. They proved that star-like functions are forming an invariant class while the convex functions (i.e. those one which map the unit disk onto a convex domain) are not. We use here some modification of their method and develop it to establish more detail properties like monotonicity theorems for invariant classes.

Here we establish more delicate property concerning the behavior of characteristic $\Lambda^\pm(t)$. It is clear that the following assertion is of particular interest in the case where the time-interval is infinite.

Theorem 5.2. *Let $w(z;t)$ be a non-trivial HS-solution in $[0;b)$ such that $\Lambda^+(0)\in[0;\pi)$. Then*

$$y^+(t) = \tan\left(\frac{\Lambda^+(t)}{2}\right)\cdot\exp\left(\int_0^t \frac{2d\tau}{\sigma(\tau)}\right)$$

is strictly decreasing in $[0;b)$. Here

$$\sigma(t) = \max_{|z|=1} |w(z,t)w'_z(z,t)|.$$

Proof. Let $\zeta(t)=e^{i\varphi(t)}$, $\varphi(t)\in\mathbb{R}$, such that

$$\Lambda^+(t) \equiv \max_{|z|=1}\arg w^\star(z;t) = \arg w^\star(\zeta(t);t).$$

Obviously, $\Lambda^+(t)>0$ because the solution w is non-trivial and $w^\star(0,t)=1$ and $\Lambda^+(t)<\pi$ as a consequence of Theorem 5.1.

Then we find from (5.2) and (5.1) that for any $t_0\in[0;b)$

$$\frac{\partial\arg w^\star}{\partial t}(\zeta(t_0);t_0) = -\frac{\operatorname{Im} w^{\star\star}(\zeta(t_0);t_0) + 2\operatorname{Im} w^\star(\zeta(t_0);t_0)}{|w'_z(\zeta(t_0);t_0)|^2} <$$

$$< -\frac{2\operatorname{Im} w^\star(\zeta(t_0);t_0)}{|w'_z(\zeta(t_0);t_0)|^2}. \tag{5.3}$$

To simplify the previous inequality we notice that

$$\frac{\operatorname{Im} w^\star(\zeta(t_0);t_0)}{|w'_z(\zeta(t_0);t_0)|^2} = \frac{|w^\star(\zeta(t_0);t_0)|\cdot\sin\Lambda(t_0)}{|w'_z(\zeta(t_0);t_0)|^2} = \frac{\sin\Lambda(t_0)}{|w(\zeta(t_0);t_0)\cdot w'_z(\zeta(t_0);t_0)|}. \tag{5.4}$$

Then, using the following relation

$$|w(\zeta(t_0);t_0)\cdot w'_z(\zeta(t_0);t_0)| \le \sigma(t_0),$$

we obtain from (5.3) and (5.4),

$$\frac{\partial \arg w^\star}{\partial t}(\zeta(t_0); t_0) < -\frac{2\sin\Lambda(t_0)}{\sigma(t_0)}. \tag{5.5}$$

Now denote by

$$H(\theta; t) = \ln\tan\left(\frac{1}{2}\arg w^\star(e^{i\theta}; t)\right).$$

Here and in what follows we consider a neighborhood of the contact point $e^{i\varphi(t)}$ to apply inequality (5.5). It follows that the function $H(\theta, t)$ will be well defined because $\arg w^\star \in (0, \pi)$ there. Then

$$\frac{\partial H(\theta; t)}{\partial t} = \frac{1}{\sin(\arg w^\star(e^{i\theta}; t))}\frac{\partial \arg w^\star}{\partial t}(\theta; t),$$

which yields (by positivity of sin-function since $\Lambda^+(t) \in (0, \pi)$) that

$$\left.\frac{\partial H(\theta, t)}{\partial t}\right|_{(\varphi(t);t)} + \frac{2}{\sigma(t)} < 0 \tag{5.6}$$

for all admissible $t \in [0; b)$. By Theorem 4.4 the function $\sigma(t)$ is positive and absolutely continuous. This yields

$$\frac{d}{dt}\int_0^t \frac{d\tau}{\sigma(\tau)} = \frac{1}{\sigma(t)}.$$

Thus,

$$\left.\frac{\partial}{\partial t}\left(H(\theta; t) + 2\int_0^t \frac{d\tau}{\sigma(\tau)}\right)\right|_{(\varphi(t);t)} < 0.$$

Taking into account that $\theta = \varphi(t_0)$ is the maximum point of $\arg w^\star(e^{i\theta}; t_0)$ for a fixed t_0 in the closed disk $\overline{U}$, it follows that the maximum of $H(\theta; t_0)$ also is also attained at $\varphi(t_0)$.

By Lemma 3.2 we have that the function

$$\max_{\theta\in[0;2\pi]} H(\theta; t) + 2\int_0^t \frac{d\tau}{\sigma(\tau)}$$

is strictly decreasing. From

$$\max_{\theta\in[0;2\pi]} H(\theta; t) = \ln\tan\frac{\Lambda^+(t)}{2},$$

we arrive at the required property and the theorem is proved. □

Corollary 5.3. *Under the hypotheses of Theorem 5.2 the function*

$$(2t + \sigma(0)) \cdot \tan\frac{\Lambda^+(t)}{2}$$

is strictly decreasing and locally-Lipschitz in $[0; b)$.

Proof. By Theorem 4.4 we have for $\sigma(t)$ that

$$\sigma(t) \le \sigma(0) + 2t.$$

Substituting of this inequality in (5.6) and arguing similar to what in the proof of Theorem 5.2 we obtain that the function

$$\ln\tan\frac{\Lambda(t)}{2}+\ln\frac{2t+\sigma(0)}{\sigma(0)},$$

is strictly decreasing and locally-Lipschitz which completes the proof. □

Remark 5.1. The both previous assertion (Theorem 5.2 and Corollary 5.3) are obviously still valid when we consider $\Lambda^-(t)$ with hypothesis that $\Lambda^-(0)\in(-\pi;0]$ and change $\Lambda^+(t)$ by $|\Lambda^-(t)|$.

Corollary 5.4. *Let $-\pi<-\alpha<0<\beta<\pi$ and $\beta+\alpha<\pi$. Then for any initial data $w(z;0)$ from $T(\alpha;\beta)$ we have for a nontrivial HS-solution $w(z,t)$, $t\in[0;b)$, that*

$$w(z;t)\in T(-\alpha(t);\beta(t)),$$

where

$$\alpha(t)=2\mathrm{arctan}\left(\frac{\sigma(0)}{2t+\sigma(0)}\cdot\tan\frac{\alpha}{2}\right)<\alpha,$$
$$\beta(t)=2\mathrm{arctan}\left(\frac{\sigma(0)}{2t+\sigma(0)}\cdot\tan\frac{\beta}{2}\right)<\beta.$$

In particular, if $b=+\infty$,

$$\lim_{t\to+\infty}\alpha(t)=\lim_{t\to+\infty}\beta(t)=0.$$

5.3. **Isoperimetric defect.** Let Ω be a Jordan domain with C^1-regular boundary $\partial\Omega$. The value

$$\delta(\Omega)=P^2(\Omega)-4\pi|\Omega|,$$

is usually called the *isoperimetric defect* of Ω, where $P(\Omega)$ is the perimeter of the domain Ω (which can be defined as one-dimensional Hausdorff measure because of regularity of $\partial\Omega$) and $|\Omega|$ is the area of Ω.

It follows form the well-known isoperimetric inequality for plane domains that $\delta(\Omega)$ is always non-negative and it vanishes if and only if Ω is a round disk (see [BZ] and [Hw]). On the other hand, $\delta(\Omega)$ is an effective upper bound which is vastly used in geometrical problems dealing with distortion of a domain from a disk. Further we basically used the following Annulus Theorem due to Bonnesen [Bs] and in a recent version treated by Fuglede in [Fg].

Definition 5.2. An annulus $K_\zeta(r,R)=\{z\in\mathbb{C}:\ r\le|z-\zeta|\le R\}$ is called the *minimal* for Ω if $\partial\Omega\subset K_\zeta(r,R)$ and the quantity $(R-r)$ can not be decreased. The quantity $\mu(\Omega)=R-r$ we will call the *width* of Ω.

Theorem 5.3 ([BZ], [Fg]). *For any Jordan domain Ω with rectifiable boundary $\partial\Omega$ the minimal annulus $K_\zeta(r,R)$ does exist and*

$$4\pi(R-r)^2\le\delta(\Omega)\le4\pi^2R(R-r).$$

Unlike those considered above characteristics the isoperimetric defect $\delta(\Omega(t))$ (where $\Omega(t)$ is an evolution family of Hele-Shaw flow, $t\in[0;b)$) is not monotonic on t in general case. Really, we can consider an initial data $\Omega(0)=U_a$ to be the unit disk with the center at $a\in(0,1)$ on the real axis. Then the corresponding evolution family $\Omega(t)$ does not contain round disks for $t>0$. The explicit solution in this case was firstly established by P.P. Kufarev in [Kf] and further developed by Richardson in [Rs72]. This family is formed by the domains which are non-convex for small $t > 0$ (see Figure 1).

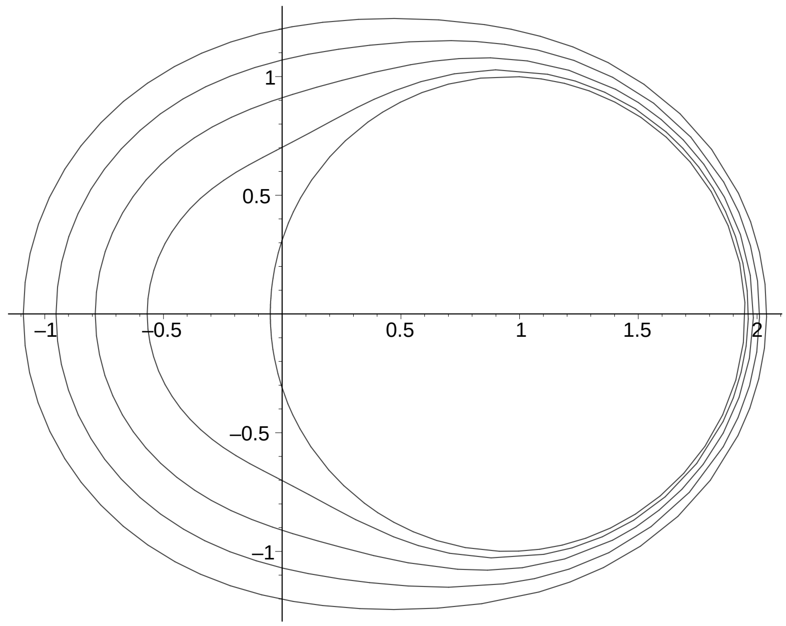

FIGURE 1. Phase portrait of the evolution family for initial domain $\Omega(0) = U_a$, $a = 0.95$, $t \in [0, 1]$

The explicit form of $w(z, t)$ can be written as the following (see [Rs72])

$$w(z,t) = \frac{\beta z}{1-\gamma z} + \lambda z,$$

where β, γ and λ satisfy

$$\left.\begin{array}{rcl} \lambda(\beta+\lambda) & = & 2t \\ (a-\gamma\lambda)(1-\gamma^2) & = & \beta\gamma \\ (1-\beta\lambda)(1-\gamma^2)^2 & = & \beta^2 \end{array}\right\}$$

In particular, in this case $\delta(\Omega(0)) = 0$ while $\delta(\Omega(t)) > 0$ for all $t > 0$. The graph of $\delta(\Omega(t))$-dependence for $a = 0.95$ is shown on the Figure 2 below.

We have the following estimate for the isoperimetric defect and the width of evolution family elements. We also refer to recent paper due to Gustafsson and Sakai [GS] for another approach to the related problem.

Theorem 5.4. *For any admissible $t \geq 0$ the following estimates hold*

$$\delta(\Omega(t)) \leq 4\pi^2 \left(M(t) - 2t - \frac{1}{\pi}|\Omega(0)| \right) \leq 4\pi(\pi M(0) - |\Omega(0)|), \tag{5.7}$$

$$\mu(\Omega(t)) \leq \sqrt{\pi M(0) - |\Omega(0)|}, \tag{5.8}$$

where $M(t) = \max_{\theta\in[0,2\pi]} |w'_z(e^{i\theta};t)|^2$.

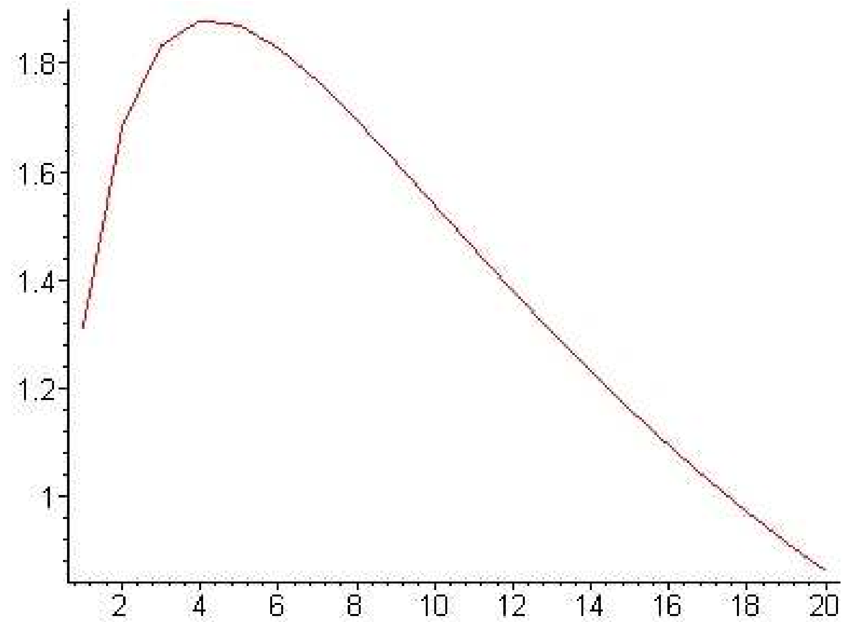

FIGURE 2. Isoperimetric defect for $\Omega(t)$ form Figure 1, $t \in [0,1]$

Proof. By the definition of the perimeter of $\Omega(t)$ and maximal distortion function $M(t)$ we have

$$P(\Omega(t)) = \int_0^{2\pi} |w'_z(e^{i\theta}; t)|\, d\theta \le \int_0^{2\pi} \sqrt{M(t)}\, d\theta = 2\pi\sqrt{M(t)}.$$

Using Theorem 4.1 we conclude that

$$P(\Omega(t)) \le 2\pi\sqrt{M(t)} \le 2\pi\sqrt{M(0) + 2t},$$

whence by (2.11) we obtain

$$\delta(\Omega(t)) \le P(\Omega(t))^2 - 4\pi|\Omega(t)| = P(\Omega(t))^2 - 4\pi(|\Omega(0)| + 2\pi t) \le$$
$$\le 4\pi^2(M(t) - 2t - \frac{1}{\pi}|\Omega(0)|) \le 4\pi(\pi M(0) - |\Omega(0)|).$$

In particular, Theorem 5.3 implies the desired estimates for the width of $\Omega(t)$ and the assertion is proved. □

The following assertion shows that asymptotically the Hele-Shaw evolution family tends to round domain (see also [GS]). We emphasize that the estimate proved above on the isoperimetric defect depends on the initial data only. In particular, the distortion of $\Omega(t)$ from a round domain in the Hausdorff metric is controlled by the initial data via characteristic $\pi M(0) - |\Omega|$ which is zero in the round case.

We recall that given two domains C and D in complex plane the Hausdorff distance is defined as follows

$$d(C, D) = \inf\{\tau : C^\tau \supset D, D^\tau \supset C, \tau \ge 0\},$$

where D^τ is a τ-neighborhood of D, i.e.

$$D^\tau = \{z \in \mathbf{C} : \text{dist }(z, D) \le \tau\}.$$

Theorem 5.5. *Let $\Omega(t)$ is well-defined for all $t \ge 0$. Then the homothetic family*

$$\overline{\Omega(t)} = \frac{1}{\sqrt{2t + M(0)}} \cdot \Omega(t)$$

converges to the unit disk $\overline{U}$ in the Hausdorff metric.

Proof. Consider a scaled domain $\overline{\Omega(t)}$. By virtue of homogeneity property of the isoperimetric defect of order 1 (with respect to homothety) it follows from (5.7) that

$$\delta(\overline{\Omega(t)}) \le \frac{4\pi(\pi M(0) - |\Omega(0)|)}{\sqrt{2t + M(0)}}.$$

In particular, the isoperimetric defect of the scaled domain tends to zero as $t \to +\infty$.

Now we consider the minimal annulus

$$K_{\zeta_t}(r_t; R_t) = \{z \in \mathbf{C} : r_t \le |z - \zeta_t| \le R_t\}$$

for $\overline{\Omega(t)}$. Then by monotonicity and homogeneous of the area we easily arrive at

$$\pi r_t^2 \le |\overline{\Omega(t)}| = \frac{|\Omega(t)|}{2t + M(0)} \le \pi R_t^2,$$

whence

$$\pi r_t^2 \le \frac{2\pi t + |\Omega(0)|}{2t + M(0)} \le \pi R_t^2. \tag{5.9}$$

After rescaling we have from (5.8)

$$r_t = R_t - \mu(\overline{\Omega(t)}) \ge R_t - \sqrt{\frac{\pi M(0) - |\Omega(0)|}{2t + M(0)}}. \tag{5.10}$$

On the other hand, since the boundary of $\overline{\Omega(t)}$ is completely contained in $K_{\zeta_t}(r_t; R_t)$ then by the outer radius definition and by (5.9), (5.10) we get

$$R_e(\overline{\Omega(t)}) \ge |\zeta_t| + r_t \ge$$

$$\ge |\zeta_t| + \sqrt{\frac{2\pi t + |\Omega(0)|}{2\pi t + M(0)}} - \sqrt{\frac{\pi M(0) - |\Omega(0)|}{2t + M(0)}}.$$

Taking into account that $R_e(\Omega(t)) \le \sqrt{M(t)}$ (see (4.17)), we have form the homogeneity of the outer radius that

$$R_e(\overline{\Omega(t)}) \le 1,$$

which yields that

$$|\zeta_t| \le 1 - \sqrt{\frac{2\pi t + |\Omega(0)|}{2\pi t + \pi M(0)}} + \sqrt{\frac{\pi M(0) - |\Omega(0)|}{2t + M(0)}}.$$

Because $\Omega(t)$ is well-defined for all $t \ge 0$ then it follows from (5.9) and (5.10) that the following limits do exist and

$$\lim_{t\to+\infty} r_t = \lim_{t\to+\infty} R_t = 1,$$

where

$$\lim_{t\to+\infty} |\zeta_t| = 0.$$

It finally follows from the definition of the Hausdorff metric and its property for round disks (see e.g., [Hw, Sec. 4]) that

$$\lim_{t\to+\infty} d(\overline{\Omega(t)}; \overline{U}) = 0.$$

□

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