Ullemar's formula for Jacobian of complex moments mapping

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Abstract. We establish the explicit formula of the Jacobian of the complex moments mapping that was being previously conjectured by C. Ullemar. We also give alternative representations of the Jacobian in terms of the derivative's roots and resultants. As a corollary we show that the boundary of the class of all locally univalent polynomials in the unit disk is contained in the union of three irreducible algebraic surfaces.

1. Introduction

Let $f(z)$, $f(0)=0$, be a univalent function defined in the unit disk $\mathbb{D}$ and $k \geq 0$ a nonnegative integer. Then the complex moments of $f(z)$ (or the target domain $D = f(\mathbb{D})$) can be defined as

$$M_k(f) = \frac{i}{2\pi} \iint\limits_{\mathbb{D}} f^k(z)|f'(z)|^2 \, dz \wedge d\bar{z} = \frac{i}{2\pi} \iint\limits_{D} \zeta^k \, d\zeta \wedge d\bar{\zeta}, \qquad k \geq 0. \tag{1}$$

The complex moments can be regarded as a natural extension of classical theory of the Stieltjes moments on the real line [**2**]. By analogy with the last case important issues are the determinacy and uniqueness of the corresponding inverse problem. Nevertheless, it follows form one result due to Sakai [**15**] that in general the analytic function $f(z)$ (or the target domain D) can not be uniquely defined by its moments (for further discussion we refer to [**1**], [**16**]). The recent results concerning the reconstruction of a domain by its complex moments can be found in [**8**], [**9**],[**12**]. We should only mention a deep connection between the complex moments theory with the the two-dimensional potential theory [**3**] and the Hele-Shaw problem [**5**], [**14**], [**11**].

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In what follows we consider the special class of quadrature domains D [**16**] which can be represented as the images of the unit disk $\mathbb{D}$ under a locally univalent polynomial mapping

$$P(z) = a_1 z + \ldots + a_n z^n, \qquad a_1 > 0,$$

with real coefficients a_k. Then the first part of (1) is still sensible and represents the moments of the chain $P(\mathbb{D})$.

By $\mathcal{P}_n(\overline{\mathbb{D}})$ and $\mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$ we denote the subclasses of polynomials $P(z)$ of fixed degree $n \geq 1$ which are univalent and locally univalent in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ respectively. It is clear, that $P(z) \in \mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$ if and only if the derivative $P'(z)$ does not vanish in $\overline{\mathbb{D}}$. On the other hand, $\mathcal{P}_n(\overline{\mathbb{D}})$ is a proper subclass of $\mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$ for $n \geq 3$.

It is well known fact (see [**14**], [**7**] and paragraph 2 below) that P produces a *finite* sequence of the moments:

$$M_k(P) = 0, \qquad k \geq \deg P. \tag{2}$$

Moreover, $M_0(P) = \sum_{j=1}^{n} j a_j^2 > 0$, $M_{n-1}(P) = a_1^n a_n \neq 0$ and it follows from Richardson's formula (9) that the moment mapping is a polymorphism

$$\mu(P) = (M_0(P), \ldots, M_{n-1}(P)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \times \mathbb{R}^n. \tag{3}$$

It worth to say that a direct studying of injectivity of μ seems to be an extremely hard problem because of involved structure of the univalent polynomials. The only lower-degree ($n \leq 3$) univalent polynomials in the unit disk have been completely studied (see [**10**], [**4**], [**17**]). In particular, by using these results C. Ullemar established in [**18**] that μ is *globally* injective on $\mathcal{P}_3(\overline{\mathbb{D}})$. On the other hand, she has also constructed the explicit examples which show that μ fails the injectivity on $\mathcal{P}_{3,\, loc}(\overline{\mathbb{D}})$.

The first general result concerning the injectivity of μ on the *locally* univalent class $\mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$ is due to B. Gustafsson [**7**] and states that the differential $d\mu$ has the maximal rang n at every point $P \in \mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$ (actually, Gustafsson has established this property for polynomials with complex coefficients). This means that μ is a locally injective polymorphism on $\mathcal{P}_{n,\, loc}(\overline{\mathbb{D}})$. We should point out that the question whether μ is globally injective on $\mathcal{P}_n(\overline{\mathbb{D}})$ for $n \geq 4$ is still open.

In the mentioned above paper Ullemar has conjectured (after direct computations for lower degree polynomials) the following formula for the Jacobian

$$J(P) \equiv \det\, d\mu(P) = 2^{-\frac{n(n-3)}{2}} a_1^{\frac{n(n-1)}{2}} P'(1) P'(-1) \Delta_n(\widetilde{P'}(z)), \tag{4}$$

which will be in the focus of the present paper. Here by $\Delta_n(\widetilde{P'}(z))$ we denote the main Hurwitz determinant for the Möbius transformation of the derivative $P'(\zeta)$ (see exact definitions in section 4). We have to notice the involved character of (4) because it uses the characteristics of P which are far from the initial definition of the complex moments.

Nevertheless, an important feature of (4) is that it immediately implies the local injectivity property. Really, it is well known fact that the corresponding Hurwitz determinant of a polynomial is positive when it has no roots in a right half plane.

Our main result concerns the following alternative formula for evaluation of $J(P)$ via the inner characteristics of P.

THEOREM 1 (Derivative Roots Formula). *Let $P(z) = a_1 z + \ldots + a_n z^n$, $a_k \in \mathbb{R}$ and $\zeta_1, \ldots, \zeta_{n-1}$ are all zeroes of the derivative $P'(z)$. Then*

$$J(P) = 2a_1^{\frac{n(n-1)}{2}} (na_n)^n \cdot \prod_{i \le j} (\zeta_i \zeta_j - 1) = \\ = 2a_1^{\frac{n(n-1)}{2}} (na_n)^{n-2}\, P'(1)\, P'(-1) \prod_{i<j} (\zeta_i \zeta_j - 1) \tag{5}$$

Actually, the product in the right side of (5) is a symmetric function of the roots and it follows that the Jacobian can be written as a homogeneous form of the coefficients a_k. More precisely, (see also section 6)

$$J(P) = 2a_1^{\frac{n(n-1)}{2}} V_{n-1}(b_1, \ldots, b_n) \sum_{j=1}^{n} b_j \sum_{k=1}^{n} (-1)^k b_k,$$

where $b_k = ka_k$ are the coefficients of $P'(z)$ and V_{n-1} is a homogeneous irreducible polynomial of degree $(n-1)$.

THEOREM 2 (Resultant Formula). *Let $A^*(z) = z^p A(1/z)$ be the reciprocal polynomial to $A(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_p z^p$. Then in the notations of Theorem 1 we have*

$$J(P)^2 = 4(-1)^{n-1} a_1^{n(n-1)}\, \mathcal{R}(P', P'^*) \cdot P'(-1) P'(1), \tag{6}$$

where $\mathcal{R}(A, B)$ denotes the resultant of the corresponding polynomials.

We obtain the Ullemar's formula (4) as a corollary of Theorem 1 and some auxiliary properties of Hurwitz determinants which has been established in section 4.

Now we can outline an alternative proof of the mentioned above Gustafsson's result.

COROLLARY 1. *The mapping $\mu(P)$ is locally injective on the set $\mathcal{P}_{n,\,loc}(\overline{\mathbb{D}})$, $n \ge 1$.*

PROOF. Indeed, for any polynomial $P(z) \in \mathcal{P}_{n,\,loc}(\overline{\mathbb{D}})$ with real coefficients we have $a_n \neq 0$ and $a_1 = P'(0) \neq 0$. Moreover, $|P'(\zeta)| \neq 0$ in $\overline{\mathbb{D}}$ and it follows that all zeroes of the first derivative $|\zeta_k| > 1$, $k = 1, \ldots, n-1$. It follows from (5) that $J(P) \neq 0$. $\square$

Another interesting consequence of our representation of $J(P)$ is its connection with the structure properties of class $\mathcal{P}_{n,\,loc}(\overline{\mathbb{D}})$. Really, let us identify a polynomial $P(z)=\sum_{j=1}^{n} a_j z^j$ with the point $(a_1,\ldots,a_n)\in\mathbb{R}^n$.

The following assertion describes the structure of the class

$$\mathcal{P}_{loc}^{n}=\cup_{1\le j\le n}\mathcal{P}_{j,\,loc}(\overline{\mathbb{D}}).$$

THEOREM 3. *If $n\ge 3$ then the boundary of the class $\mathcal{P}_{loc}^{n}$ is contained in the following three irreducible algebraic components: the hyperplanes*

$$\begin{aligned}\Pi^{+}: &\quad P'(1)=a_1+2a_2+\ldots+na_n=0,\\ \Pi^{-}: &\quad P'(-1)=a_1-2a_2+\ldots+(-1)^{n-1}na_n=0,\end{aligned}\tag{7}$$

and an algebraic surface of $(n-1)$th order given by

$$\mathcal{A}:\quad V_{n-1}(a_1,2a_2,\ldots,na_n)=0.\tag{8}$$

We emphasize that it follows from the preceding results that $\mathcal{P}_{loc}^{n}$ is exactly an open component of $J(P)\neq 0$.

We should mention that closely related result has been obtained by Quine [**13**] for the univalent classes $\mathcal{P}_n(\overline{\mathbb{D}})$. However, in the last case only upper estimates for the degree of the boundary $\partial\mathcal{P}_n(\overline{\mathbb{D}})$ have been established.

We notice that the previous formulae as well as the suitable modifications of the facts below are still valid for the polynomials with complex coefficients. But in this case we need somewhat another technique which will be accomplished in a forthcoming paper.

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2. Preliminary results

Due to S. Richardson [**14**] one can write the following expressions for $M_k(P)$

$$M_k(P)=\sum i_1\cdot a_{i_1}\cdot\ldots\cdot a_{i_{k+1}}\overline{a}_{i_1+\ldots+i_{k+1}},\tag{9}$$

where the sum is taken over all possible sets of indices $i_1,\ \ldots,\ i_k\ge 1$ and it is assumed that $a_j=0$ for $j\ge n+1$. These formulae are easy to use for straightforward manipulations with the moments but actually they are useless for investigation of their analytic properties.

We use in the sequel the following residues representation

$$M_k(P)=\frac{1}{k+1}\operatorname*{res}_{\zeta=0}\left(P^{k+1}(\zeta)P'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta}\right).\tag{10}$$

Really, it follows from Stokes' formula that

$$\frac{i}{2\pi}\iint\limits_{G} w^k\,dw\wedge d\overline{w}=\frac{i}{2\pi(k+1)}\int\limits_{\partial G} w^{k+1}\,d\overline{w},\tag{11}$$

where G is an arbitrary 2-chain in complex plane. Letting $G = P(\mathbb{D})$ and taking into account that $\bar{\zeta} = \zeta^{-1}$ along $\partial\mathbb{D}$ and the fact that $\overline{P'(z)} = P'(\bar{z})$ for polynomials with real coefficients we have from (11)

$$M_k(P) = \frac{i}{2\pi(k+1)} \int\limits_{\partial\mathbb{D}} P^{k+1}(\zeta)\overline{P'(\zeta)}\, d\bar{\zeta} = \frac{1}{2\pi(k+1)} \int\limits_{\partial\mathbb{D}} P^{k+1}(\zeta) P'\left(\frac{1}{\zeta}\right) \frac{d\zeta}{\zeta^2}$$

which proves (10).

Moreover, we notice that in our assumptions $P(0) = 0$ and it follows that $P(\zeta) = zP_1(z)$ where P_1 is a polynomial. Thus, the expression

$$\zeta^{k+1} P_1^{k+1}(\zeta) P'\left(\frac{1}{\zeta}\right) = \zeta^{k-n}(a_1 + \ldots + a_n\zeta^{n-1})(a_1\zeta^{n-1} + \ldots a_n)$$

is also a polynomial for all $k \geq n$ and it follows from (10) that for all $k \geq n$ we have

$$M_k(P) = \frac{1}{k+1} \operatorname*{res}_{\zeta=0} \zeta^{k+1} P_1^{k+1}(\zeta) P'\left(\frac{1}{\zeta}\right) = 0.$$

As a consequence, we have (2) and, therefore, the mapping μ in (3) is well-defined.

Given meromorphic functions H_1 and H_2 we write

$$H_1(z) \equiv H_2(z) \mod [m_1; m_2]$$

if the Laurent series of $H_2 - H_1$ does not contain z^m with $m \in [m_1; m_2]$. Then we have the following representation for the Jacobian of μ

Lemma 1. *For any k, $0 \leq k \leq n-1$,*

$$P'(z)\left(P^k(z) + P^k\left(\frac{1}{z}\right)\right) \equiv \sum_{\nu=1}^{n} \frac{\partial M_k(P)}{\partial a_\nu} \cdot z^{\nu-1} \mod [0; n-1]. \tag{12}$$

Proof. We denote by $\lambda_m(f(z)) = \operatorname{res}_{z=0}(f(z)z^{-1-m})$. Then it follows from relations

$$\frac{\partial P(1/z)}{\partial a_\nu} = \frac{1}{z^\nu}, \qquad \frac{\partial P'(z)}{\partial a_\nu} = \nu z^{\nu-1},$$

and (10) that

$$\frac{\partial M_k(P)}{\partial a_\nu} = \lambda_0\left(P^k(1/z)P'(z)z^{1-\nu}\right) + \frac{\nu}{k+1}\lambda_0\left(P^{k+1}(1/z)z^\nu\right). \tag{13}$$

On the other hand, integrating by parts yields

$$\lambda_0\left(z^\nu P^{k+1}(1/z)\right) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} P^{k+1}\left(1/z\right) z^{\nu-1}\, dz = \frac{1}{2\pi i\nu} \int\limits_{\partial\mathbb{D}} d\left(z^\nu P^{k+1}\left(1/z\right)\right) +$$

$$+\frac{k+1}{2\pi i\nu} \int\limits_{\partial\mathbb{D}} P^{k+1}\left(1/z\right) P'\left(1/z\right) z^{\nu-2}\, dz = \frac{k+1}{\nu}\lambda_0\left(P^k\left(1/z\right) P'\left(1/z\right) z^{\nu-1}\right),$$

and by using $\lambda_0(f(1/z)) = \lambda_0(f(z))$ we arrive at

$$\lambda_0\left(P^{k+1}\left(z^{-1}\right)z^{\nu}\right) = \frac{k+1}{\nu}\lambda_0\left(P^k(z)P'(z)z^{1-\nu}\right). \tag{14}$$

Substituting of (14) to (13) we come to

$$\begin{aligned}\frac{\partial M_k(P)}{\partial a_\nu} &= \lambda_0\left[P'(z)z^{1-\nu}\Big(P^k(z)+P^k\left(z^{-1}\right)\Big)\right] = \\ &= \lambda_{\nu-1}\left[P'(z)\Big(P^k(z)+P^k\left(z^{-1}\right)\Big)\right]\end{aligned}$$

that is equivalent to the required formula (12). □

We notice that for an arbitrary index $k \in \{0,\dots,n-1\}$ the decomposition

$$P^k(z) + P^k\left(z^{-1}\right) = \sum_{m=-nk}^{nk} h_m^{(k)} z^m, \tag{15}$$

yields the symmetricity condition $h_m^{(k)} = h_{-m}^{(k)}$.

In order to study (12) we to consider a more general case. Really, given an arbitrary vector $x = (x_0, x_1, \dots, x_{n-1})$ we associate with the following Toeplitz matrix

$$\mathcal{T}(x) = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & \cdots & \cdots & x_0 \end{pmatrix}.$$

Then for $y = (y_0, y_1, \dots, y_{n-1})$ we can introduce the *dual* matrix $\mathcal{B}(y)$ by

$$\mathcal{T}(x)\cdot y^\top = \mathcal{B}(y)\cdot x^\top, \qquad \forall x \in \mathbb{R}^n. \tag{16}$$

Unlike $\mathcal{T}(x)$, the matrix $\mathcal{B}(y)$ is not symmetric yet and it has some more complicated structure. We postpone the discussion of the properties of $B(y)$ for the next section.

Let now $H_k(z)$ be rational functions which Laurent series have the form

$$H_k(z) = \sum_{m=-N}^{N} h_{|m|}^{(k)}\, z^m,$$

and let $B(z) = b_0 + b_1 z + \ldots + b_{n-1}z^{n-1}$ be an arbitrary polynomial such that $b_{n-1} \neq 0$. We keep in mind that $H_k(z) = P^k(z) + P^k\left(z^{-1}\right)$ and $B(z) = P'(z)$ in the sequel. Then we can find the polynomials $\Phi_k(z) = \sum_{\nu=0}^{n-1}\varphi_\nu^{(k)} z^\nu$, $0 \le k \le n-1$, such that

$$B(z)\cdot H_k(z) \equiv \Phi_k(z) \mod [0; n-1]. \tag{17}$$

To proceed we consider the vectors $h^{(k)} = (h_0^{(k)}, \ldots, h_{n-1}^{(k)})$ and $b = (b_0, \ldots, b_{n-1})$. It follows from (17) that the matrix identity holds

$$(\varphi_0^{(k)}, \ldots, \varphi_{n-1}^{(k)})^\top \equiv {\varphi^{(k)}}^\top = \mathcal{T}(h^{(k)}) \cdot b^\top,$$

which by the definition of dual matrix (16) implies ${\varphi^{(k)}}^\top = \mathcal{B}(b) \cdot {h^{(k)}}^\top$, $0 \leq k \leq n-1$. Therefore, denoting by Φ and H the matrices which are formed by combination of the columns $\varphi^{(k)}$ and ${h^{(k)}}^\top$ respectively we get $\Phi = \mathcal{B}(b)H$ and it follows that

$$\det\, \Phi = \det\, \mathcal{B}(b) \cdot \det\, H. \tag{18}$$

In our previous notations $B(z) = P'(z)$, $H_k(z) = P^k(z) + P^k(1/z)$ and we obtain from (12)

$$\varphi_\nu^{(k)} = \frac{\partial M_k(P)}{\partial a_\nu}, \qquad d\mu(P) = \Phi.$$

Thus, the problem of evaluating of the complex moments mapping Jacobian $J(P)$ can be reduced by (18) to the general problem of evaluating of determinants of the corresponding matrices $\mathcal{B}(b)$ and H (here $b_{j-1} = ja_j$ are the coefficients of $P'(z)$).

The last determinant can be found as follows. We notice that matrix $\|h_i^{(k)}\|$ has the low-triangular shape in our case. Indeed, $P(z) = zP_1(z)$, where $P_1(z)$ is a polynomial, and

$$P^k(z) + P^k\left(z^{-1}\right) = z^k P_1(z) + \frac{1}{z^k} P_1^k\left(z^{-1}\right) = \sum_{m=k}^{kn} (z^m + z^{-m}) h_m^{(k)},$$

which easily implies $h_m^{(k)} = 0$, $0 \leq m \leq k-1$. Moreover, we have for the diagonal elements $h_0^{(0)} = 2$ and $h_k^{(k)} = a_1^k$. This yields the desirable identity

$$\det\, H = \det\, \|h_i^{(k)}\| = 2 \cdot a_1 \cdot a_1^2 \cdot \ldots \cdot a_1^{n-1} = 2a_1^{\frac{n(n-1)}{2}}. \tag{19}$$

3. Toeplitz determinants

The explicit form of $\det\, \mathcal{B}(y)$ in terms of the coefficients $y_0, \ldots, y_m$ is messy and useless for the further analysis. However, it turns out, that this determinant can be briefly written in terms of some intrinsic characteristics of y. Namely, let us associate with any vector y the polynomial

$$B_y(z) = y_0 + y_1 z + \ldots + y_m z^m.$$

We also assume that $y_m \neq 0$.

THEOREM 4. *Let ζ_1, …, ζ_m are the roots of $B_y(\zeta)$. Then*

$$\det\, \mathcal{B}(y) = y_m^{m+1} \prod_{i \geq j} (\zeta_i \zeta_j - 1), \tag{20}$$

Proof. First we notice that left-hand side of (20) is an algebraic function of $y_0, \ldots, y_{n-2}$ and it is sufficient to prove that (20) is valid for every $\zeta = (\zeta_1, \ldots, \zeta_m)$ outside of some proper algebraic submanifold of $\mathbb{C}^m$. Namely, we will suppose that $\zeta_i \neq \zeta_j$ for $i \neq j$ and $\zeta_i\zeta_j \neq 1$ for all i, j.

Then given a nonnegative integer k and $\zeta \in \mathbb{C}$ we consider the following vector

$$\{\zeta\}_k = (0, \ldots, 0, 1, \zeta, \zeta^2, \ldots, \zeta^{m-k})^\top \in \mathbb{C}^{m+1}, \qquad \{\zeta\} \equiv \{\zeta\}_0.$$

Then substitution of $x = \{\zeta\}^\top$ in (16) gives

$$\mathcal{T}(\{\zeta\}) \cdot y^\top = B_y(\zeta) \cdot \{\zeta^{-1}\} + \sum_{i=0}^{i=m-1} y_i \left(\{\zeta\}_i - \{\zeta^{-1}\}_i\right)$$

and taking ζ^{-1} instead of ζ in the previous formula we arrive at

$$\mathcal{T}(\{\zeta\} + \{\zeta^{-1}\}) \cdot y^\top = B_y(\zeta) \cdot \{\zeta^{-1}\} + B_y(\zeta^{-1}) \cdot \{\zeta\}. \tag{21}$$

Let $\zeta = \zeta_i$ be a root of the polynomial $B_y(\zeta)$. Then it follows from (21) that

$$\mathcal{T}(\{\zeta_i\} + \{\zeta_i^{-1}\}) \cdot y^\top = B_y(\zeta_i^{-1}) \cdot \{\zeta_i\} \tag{22}$$

and

$$\mathcal{T}(e) \cdot y^\top = B_y(1) \cdot e^\top, \tag{23}$$

where $e = (2, \ldots, 2) \in \mathbb{C}^{m+1}$. Applying (16) to (22) and (23) we obtain for all $i = 1, \ldots, m$

$$\mathcal{B}(y)(\{\zeta_i\} + \{\zeta_i^{-1}\}) = B_y(\zeta_i^{-1}) \cdot \{\zeta_i\},$$

and

$$\mathcal{B}(y) \cdot e^\top = B_y(1) \cdot e^\top,$$

Combining the preceding expressions to the matrix identity we attain the relation for determinants

$$\begin{aligned} &\det \mathcal{B}(y) \det \mathcal{W}(1, \zeta_1, \ldots, \zeta_m) = \\ &= 2B_y(1) \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \zeta_1 & \ldots & \zeta_1^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_m & \ldots & \zeta_m^m \end{pmatrix} \prod_{j=1}^m B_y(\zeta_j^{-1}) = \\ &= 2B_y(1) \prod_{k=1}^m B_y(\zeta_k^{-1}) \cdot \prod_{i<j} (\zeta_j - \zeta_i) \cdot \prod_{i=1}^m (1 - \zeta_i), \end{aligned} \tag{24}$$

where by $\mathcal{W}(\alpha_0, \alpha_1, \ldots, \alpha_m)$ we denote $\mathcal{W}_{ij} = \|\alpha_j^i + \alpha_j^{-i}\|_{i,j=0}^m$.

The determinant of $\mathcal{W}(\alpha_0, \ldots, \alpha_m)$ can be easily evaluated similar to that for the Vandermonians (see also [**19**, Part 4]):

$$\det\ \mathcal{W}(\alpha_0, \ldots, \alpha_0) = \frac{2}{(\alpha_0 \ldots \alpha_m)^m} \prod_{i<j} (\alpha_j - \alpha_i) \cdot \prod_{i<j} (\alpha_i\alpha_j - 1). \tag{25}$$

Thus, we have from (24), (25) and by virtue of

$$B_y(1)=y_m\cdot\prod_{i=1}^{m}(1-\zeta_i),\qquad \zeta_1\cdots\zeta_m=(-1)^m\frac{y_1}{y_m},$$

that

$$\det\mathcal{B}(y)\cdot\frac{(-1)^m}{(\zeta_1\ldots\zeta_m)^m}\prod_{i<j}(\zeta_j-\zeta_i)\cdot\prod_{i<j}(\zeta_i\zeta_j-1)\prod_{i=1}^{m}(1-\zeta_i)^2=$$

$$=\frac{1}{y_m}\cdot\prod_{k=1}^{m}B_y(\zeta_k^{-1})\cdot B_y^2(1)\prod_{i<j}(\zeta_j-\zeta_i),$$

and after substitution

$$B_y(\zeta_k^{-1})=\frac{y_m}{\zeta_k^m}\prod_{i=1}^{m}(1-\zeta_i\zeta_k),$$

we finally come to required relation (20). □

Now we are ready to establish our main result.

Proof of Theorem 1. We recall that in the previous notations $d\mu(P)=\Phi$. Then applying (19) and Theorem 4 to (18) we obtain

$$J(P)\equiv\det\left[\frac{\partial M_k(P)}{\partial a_i}\right]=2a_n^{\frac{n(n-1)}{2}}\cdot b_{n-1}^n\prod_{i\le j}(\zeta_i\zeta_j-1),$$

where $b_{n-1}=na_n$ is the leading coefficient of $B(z)\equiv P'(z)$. Thus,

$$J(P)=2a_n^{\frac{n(n-1)}{2}}\cdot a_n^n n^n\prod_{i\le j}(\zeta_i\zeta_j-1)$$

which proves the theorem. □

4. Hurwitz determinants and Ullemar's formula

Let us consider an arbitrary polynomial $R(z)=r_0+r_1z+\ldots+r_mz^m$ of the degree $m\ge 1$. The $m\times m$-matrix

$$\mathcal{G}(R)\equiv\begin{pmatrix} r_{m-1} & r_{m-3} & \ldots & r_{1-m}\\ r_m & r_{m-2} & \ldots & r_{2-m}\\ \vdots & \vdots & \ddots & \vdots\\ r_{2m-2} & r_{2m-4} & \ldots & r_0\end{pmatrix}$$

is said to be the Hurwitzian matrix of the polynomial $R(z)$ [**6**]. One can easily see that

$$\mathcal{G}_{ij}(R)=r_{m+i-2j}\tag{26}$$

where $r_p=0$ if p is a negative integer or $p>\deg R$.

We denote by the Hurwitz determinant of R, $\Delta(R)$, the main diagonal minor of $(m-1)$th order of the matrix $\mathcal{G}(R)$. It is an easy consequence of the definition of $\mathcal{G}(R)$ that

$$\det \mathcal{G}(R) = r_0\, \Delta(R). \tag{27}$$

THEOREM 5. *The Hurwitz determinant of $R(z)$, $\deg R = m$, has the following form*

$$\Delta(R) = (-1)^{\frac{m^2-m}{2}} r_m^{m-1} \prod_{1\le i<j\le m} (z_i + z_j), \tag{28}$$

where z_i are all the roots of $R(z)$.

Before we give the proof of the theorem let us formulate some of its corollaries.

Let us consider the Möbius transformation of the polynomial $R(z)$ given by

$$\widetilde{R}(z) = (z+1)^m R\left(\frac{z-1}{z+1}\right) \equiv \widetilde{r}_0 + \widetilde{r}_1 z + \ldots + \widetilde{r}_m z^m.$$

It is clear that the polynomial $\widetilde{R}(z)$ have its roots as $\zeta_k = \frac{1+z_k}{1-z_k}$ where z_1, $\ldots z_m$ are the roots of $R(z)$. In particular, all roots of $R(z)$ are contained in the unit disk if and only if the roots of $\widetilde{R}(z)$ lie in the right half-plane.

Using the previous relations between the roots we get

$$\prod_{1\le i<j\le m} (\zeta_i + \zeta_j) = 2^{\frac{m(m-1)}{2}} \prod_{1\le i<j\le m} (1 - z_i z_j) \left(\prod_{i=1}^{m} (1 - z_i)\right)^{1-m}.$$

Then the following identities

$$\prod_{i=1}^{m} (1 - z_i) = \frac{R(1)}{r_m},$$

$$\widetilde{r}_m = \lim_{z\to\infty} z^{-m} \widetilde{R}(z) = R(1),$$

together with (28) yield

COROLLARY 2. *In previous notations*

$$\Delta(\widetilde{R}) = 2^{\frac{m^2-m}{2}} r_m^{m-1} \prod_{1\le i<j\le m} (z_i z_j - 1), \tag{29}$$

where r_m is the main coefficient of polynomial $R(z)$ with z_i to be its roots.

Now (29) and Theorem 1 immediately imply the Ullemar's conjectured formula (4)

COROLLARY 3 (Ullemar Formula). *The Jacobian of the complex moment mapping μ has the following representation*

$$J(P) \equiv \det\, d\mu(P) = 2^{-\frac{n(n-3)}{2}} a_1^{\frac{n(n-1)}{2}} P'(1) P(-1) \Delta_n(\widetilde{P'}(z))$$

where $n = \deg P$.

Proof of Theorem 5. Similar to the proof of Theorem 1 we can also assume that $R(z)$ has no multiple roots.

Then we have from (26) for an arbitrary $\zeta \in \mathbb{C}$ and integer i that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R)\zeta^{2m-2j} = \sum_{j=1}^{m} r_{m+i-2j}\zeta^{2m-2j} = \zeta^{m-i}\sum\nolimits_{k}^{\star}\zeta^{k} r_{k},$$

where the kth index in the last sum has the form $k(i,j) = m+i-2j$ and $j = 1, \dots, m$.

Now we consider a fixed integer $i \in [1;m]$ and let $\bar{i} = (m-i)$. It is clear that k takes the only even (or only odd) values ranging between $(-\bar{i})$ and $(2m-2-\bar{i})$ with changing j in $[1;m]$. Moreover, we notice that both k and $\bar{i}$ have the same evenness and

$$-\bar{i} \le k(i,j) \le 2m-2-\bar{i}, \qquad \bar{i} = 0, 1, \dots, m-1.$$

Hence, for every fixed i the indices $k(i,j)$ take all the values of $\bar{i}$ from interval $\{0, 1, \dots, n\}$ when $j \in [0;m]$. By virtue of this property we conclude that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R)\zeta^{2m-2j} = \zeta^{m-i} R^{[\bar{i}]}(\zeta), \tag{30}$$

where we denote by $R^{[p]}(\zeta)$ the even (or odd) part of $R(\zeta)$

$$R^{[p]}(\zeta) = \frac{1}{2}\bigg(R(\zeta) + (-1)^p R(-\zeta)\bigg).$$

Let now $\zeta = \zeta_k$ be kth root of the polynomial $R(z)$. Taking into account that

$$R^{[p]}(\zeta_k) = R(\zeta_k) - R^{[p+1]}(\zeta_k) = -R^{[p+1]}(\zeta_k),$$

we have

$$R^{[p]}(\zeta_k) = (-1)^p R^{\mathrm{ev}}(\zeta_k).$$

Here $R^{\mathrm{ev}} = R^{[0]}$ is the even part of $R(z)$ and we see from (30) that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R)\zeta_k^{2m-2j} = (-\zeta_k)^{m-i} R^{\mathrm{ev}}(\zeta_k).$$

Combining the last identities for $k = 1, 2, \dots, m$ into the matrix form we have for the determinants

$$\det \mathcal{G}(R) \det \mathcal{V}(\zeta_1^2, \dots, \zeta_m^2) = (-1)^{\frac{m(m-1)}{2}} \det \mathcal{V}(\zeta_1, \dots, \zeta_m) \prod_{k=1}^{m} R^{ev}(\zeta_k). \tag{31}$$

where $\mathcal{V}(a_1, \dots, a_m) = \|a_j^{k-1}\|_{j,k=1}^{m}$ is the Vandermonian matrix.

On the other hand, we have for the even part

$$R^{ev}(\zeta_k) = \frac{1}{2}R(\zeta_k) = \frac{r_m}{2}\prod_{i=1}^{m}(\zeta_i + \zeta_k),$$

and it follows from (31)

$$\det \mathcal{G}(R)\cdot \prod_{1\le i<j\le m}(\zeta_j^2-\zeta_i^2)=\frac{(-1)^m r_m^m}{2^m}\prod_{i,j=1}^{m}(\zeta_i+\zeta_j)\prod_{1\le i<j\le m}(\zeta_i-\zeta_j).$$

Thus, using (27) we arrive at

$$\Delta(R)=\frac{(-1)^{\frac{m^2+m}{2}}r_m^m}{2^m r_0}\prod_{1\le i\le j\le m}(\zeta_i+\zeta_j). \tag{32}$$

Finally, writing the last product as

$$\prod_{1\le i\le j\le m}(\zeta_i+\zeta_j)=\prod_{i=1}(2\zeta_i)\prod_{1\le i<j\le m}(\zeta_i+\zeta_j)=\frac{(-2)^m r_0}{r_m}\prod_{1\le i<j\le m}(\zeta_i+\zeta_j),$$

we obtain the required identity (28) and the theorem is proved. □

5. Representations via resultants

We recall that given polynomials $A(z)=A_n(z-\alpha_1)\dots(z-\alpha_n)$, $B(z)=B_n(z-\beta_1)\dots(z-\beta_n)$ the expression

$$\mathcal{R}(A,B)=A_n^nB_n^n\prod_{i,j=1}^{n}(\alpha_i-\beta_j)$$

is called the *resultant* of A and B.

If $A(z)$ and $B(z)$ are mutually reciprocal polynomials

$$B(z)=z^nA(1/z)\equiv A^*(z),$$

then $B_{n-j}=A_j$, $j=0,\dots n$ and the we have for their roots: $\beta_j=\frac{1}{\alpha_j}$. Then the corresponding resultant can be rewritten in the matrix form

$$\mathcal{R}(A,A^*)=\det\begin{pmatrix} A_0 & A_1 & \dots & \dots & A_n & & & \\ & A_0 & A_1 & \dots & \dots & A_n & & \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & A_0 & A_1 & \dots & \dots & A_n \\ A_n & A_{n-1} & \dots & \dots & A_0 & & & \\ & A_n & A_{n-1} & \dots & \dots & A_0 & & \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & A_n & A_{n-1} & \dots & \dots & A_0 \end{pmatrix} \tag{33}$$

It is easy to see that the last matrix is of $2n$th order and has as diagonal elements A_0. On the other hand,

$$\mathcal{R}(A,A^*)=A_n^nA_0^n\prod_{i,j=1}^{n}\left(\alpha_i-\frac{1}{\alpha_j}\right)=\frac{A_n^nA_0^n}{(\alpha_1\dots\alpha_n)^n}\prod_{i>j}^{n}(\alpha_i\alpha_j-1)^2\prod_{i=1}^{n}(\alpha_i^2-1)$$

and by Viète formulae

$$\alpha_1 \cdots \alpha_n = (-1)^n \frac{A_0}{A_n}, \qquad A(1)A(-1) = A_n^2 \prod_{i=1}^{n} (\alpha_i^2 - 1),$$

we conclude that

$$\begin{aligned} \mathcal{R}(A, A^*) &= (-1)^n A(-1)A(1)A_n^{2n-2} \prod_{i>j}^{n} (\alpha_i\alpha_j - 1)^2 = \\ &= \frac{(-1)^n A_n^{2n+2}}{A(1)A(-1)} \left[\prod_{i\geq j}^{n} (\alpha_i\alpha_j - 1) \right]^2 . \end{aligned} \tag{34}$$

By introducing the following expression

$$W_n(A) = A_n^{n+1} \prod_{i\leq j} (\alpha_i\alpha_j - 1), \tag{35}$$

we have from (34)

$$W_n^2(A) = (-1)^n \mathcal{R}(A, A^*)A(-1)A(1). \tag{36}$$

As a consequence, we see that $W_n(A) \equiv W_n(A_0, A_1, \ldots, A_n)$ is a homogeneous form of degree $\deg A = n$. Moreover, it follows from simplicity of linear multipliers $A(\pm 1)$ that actually $W_n(A)$ has the following representation

$$W_n(A) = A(-1)A(1)V_n(A), \qquad V_n(A) = A_n^{n-1} \prod_{i<j} (\alpha_i\alpha_j - 1) \tag{37}$$

where $V_n(A)$ is a homogeneous form of A_k of degree $(\deg A - 2)$.

It easily follows from the representation

$$V_n(A) = A_0^{n-1} \prod_{i<j} \left(1 - \frac{1}{\alpha_i\alpha_j}\right), \tag{38}$$

that the recursion hold

$$V_n(A_0, A_1, \ldots, A_k, 0, \ldots, 0) = A_0^{n-k} V_k(A_0, A_1, \ldots, A_k). \tag{39}$$

To demonstrate the explicit form of V_k we list it for $n = 3$ and $n = 4$:

$$\begin{aligned} V_3(A) &= A_0^2 - A_0A_2 + A_1A_3 - A_3^2 \\ V_4(A) &= A_4(-A_1^2 + A_3A_1 + A_4^2 - A_4A_2 - A_0A_4 + 2A_0A_2 - A_0^2) + \\ &\quad + A_0(A_0^2 - A_0A_2 + A_1A_3 - A_3^2). \end{aligned}$$

THEOREM 6. *The form $V_n(A) \equiv V_n(A_0, A_1, \ldots, A_n) \in \mathbb{C}[A_0, A_1, \ldots, A_n]$ is irreducible polynomial over $\mathbb{C}$.*

Proof. A simple analysis of the denominator of the right-hand side of (38) shows that A_n can not be a divisor of $V_n(A)$. On the other hand, we notice that $V_n(A)$ is symmetric function of the roots α_k of $A(z) = 0$.

Let $H_1(A)$ and $H_2(A)$ be two nontrivial divisors of $V_n(A)$. It is consequence of homogeneity of $V_n(A)$ that both of $H_k(A)$ are homogeneous too. Moreover, in our assumption $h_k = \deg H_k \geq 1$.

Let σ_k is kth symmetric function of $(\alpha_1, \ldots, \alpha_n)$. Then substituting Viéte formulae

$$A_k = A_n \sigma_k(\alpha_1, \ldots, \alpha_n)$$

in $H_k(A)$ implies by virtue of homogeneity of H_k that

$$H_k(A) = A_n^{h_k} Y_k(\alpha_1, \ldots, \alpha_n),$$

and it follows from (37) that $h_1 + h_2 = n - 1$ and Y_k must be a divisor of

$$\prod_{i<j} (\alpha_i \alpha_j - 1).$$

But the last product consists of irreducible multipliers $(\alpha_i \alpha_j - 1)$ only. Moreover, if one $(\alpha_i \alpha_j - 1)$ come in Y_1 as a divisor then by symmetry the others have to be the divisors too.

It follows that one of Y_k contains none α_i, i.e. it has a form A_n^p. Thus, applying the remark in the beginning of the proof we see that $p = 0$. But it means that Y_k must be a constant multiplier that contradicts to our assumption and proves the theorem. $\square$

Proof of Theorem 2. Substituting the derivative

$$P'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1} \equiv b_1 + b_2 z + \ldots + b_n z^{n-1},$$

instead of $A(z)$ we have from (34) and (36) that

$$\left[b_n^n \prod_{i \geq j}^{n-1} (\zeta_i \zeta_j - 1) \right]^2 = (-1)^{n-1} \mathcal{R}(P', P'^*) P'(-1) P'(1). \tag{40}$$

By comparison of the relations obtained with the definition (35) we arrive at the following formula

$$W_{n-1}(P')^2 = (-1)^{n-1} \mathcal{R}(P', P'^*) P'(-1) P'(1). \tag{41}$$

The last identity with property (5) yields the required representation of $J(P)$

$$J^2(P) = 4 b_1^{n^2 - n} W_{n-1}^2(P') = (-1)^{n-1} \mathcal{R}(P', P'^*) P'(-1) P'(1) \tag{42}$$

which completes the proof. $\square$

6. Proof of Theorem 3

Let $P(z) = a_1z + \dots a_nz^n$, $P \in \mathcal{P}^n_{loc}$. As above we identify $P'(z)$ and the vector of its coefficients $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$, where $b_k = ka_k$. We write also $\mathcal{R}(p, q) = \mathcal{R}(P', Q')$ for the corresponding vectors p and q. Moreover, in what follows by S we denote the linear transform $S(a) = b : \mathbb{R}^n \to \mathbb{R}^n$ of the form $S(P) = P'(z)$.

Then the following equivalence is a simple consequence of (36) and (37)

$$b \in \ker W_{n-1} \Leftrightarrow \mathcal{R}(b, b^*) = 0, \tag{43}$$

where $b^* = (b_n, \dots, b_1)$ corresponds to P'^*.

Lemma 2. *The set $\mathcal{P}^n_{loc}$ is an open subset of $\mathbb{R}^n$ which contains two components. A polynomial $P(z)$ is an element of the boundary $\partial\mathcal{P}^n_{loc}$ if and only if*

1) $P'(z)$ contains no zeroes in $\mathbb{D}$;

2) $\mathcal{R}(P', P'^*) = 0$.

Proof. The first statement obviously follows form the fact

$$\inf_{z\in\overline{\mathbb{D}}} |P'(z)| > 0, \qquad \forall P \in \mathcal{P}^n_{loc} \tag{44}$$

Moreover, let $P \in \mathcal{P}^n_{loc}$. Then the homotopy

$$a_\lambda = (a_1, a_2t, \dots, a_nt^{n-1}), \qquad t \in [0; 1],$$

corresponds to the homothety $P_t(z) = \frac{1}{t}P(tz)$ and connects $P(z)$ and $Q(z) = a_1z$ inside of $\mathcal{P}^n_{loc}$ because $P'_t(z) = P'(tz) \neq 0$ in $\overline{\mathbb{D}}$. This shows that all polynomials $P(z)$ with $a_1 > 0$ are in the same open component of $\mathcal{P}^n_{loc}$.

On the other hand, the function $a_1(P) = a_1$ is continuous on $\mathcal{P}^n_{loc}$ and it follows from $a_1 \neq 0$ on $\mathcal{P}^n_{loc}$ that $\mathcal{P}^n_{loc}$ consists of two components exactly and the involution $P \to -P$ is a bijection between these components.

Property 1) easily follows form the continuity arguments and (44).

To prove the last assertion we assume that $P \in \partial\mathcal{P}^n_{loc}$. Then we obviously have that

$$\inf_{z\in\overline{\mathbb{D}}} |P'(z)| = 0$$

and it follows that there exist a root ζ_k of $P'(z)$ such that $|\zeta_k| = 1$. It follows from the reality of P that $\overline{\zeta_k} = 1/\zeta_k$ is the root of P' as well. But it means that $P'(z)$ and $P'^*(z)$ has common roots and by the characteristic property of resultant it yields that $\mathcal{R}(P', P'^*) = 0$. □

Proof of Theorem 3. Now, let $\mathcal{P}^n_{loc} = \mathcal{P}^+ \cup \mathcal{P}^-$ be the decomposition in Lemma 2. Let us consider a real-valued continuous function

$$f(a) = \mathcal{R}(S(a), S(a)^*) : \mathcal{P}^n_{loc} \to \mathbb{R}.$$

Then it follows from (40) that f does not changes its sign on each component $\mathcal{P}^\pm$. Really, given an arbitrary $P(z) \in \mathcal{P}^n_{loc}$ we have that all roots ζ_k of $P'(z)$ are

outside of $\overline{\mathbb{D}}$. Thus,
$$|\zeta_i\zeta_j| > 1, \quad \forall\, i, j.$$
and $f(a) \neq 0$. The last inequality together with (43) implies the claimed property.

Hence, $\mathcal{P}^{\pm} \subset \Lambda^{\pm}$ for some open components $\Lambda^{\pm}$ of $f \neq 0$. On the other hand, by property 2) in Lemma 2 we have $f(a) = 0$ for all $a \in \partial\mathcal{P}^n_{loc}$ and it implies by (43) that actually $\Lambda^{\pm} = \mathcal{P}^{n,\pm}_{loc}$.

Hence, we have obtained that $\mathcal{P}^{\pm}$ coincide with open components of
$$\mathbb{R}^n \setminus \ker W_{n-1} = \mathbb{R}^n \setminus S^{-1}(\ker f).$$

To finish the proof we need to establish that all three algebraic surfaces in the statement of Theorem 3 are actually realized as boundary components of $\mathcal{P}^n_{loc}$ for $n \geq 3$. To see it we notice that hyperplanes $\Pi^{\pm}$ in (7) correspond to the polynomials $P \in \partial\mathcal{P}^n_{loc}$ which have the critical points on the real axe: $P'(\pm 1) = 0$. On the other hand, $\mathcal{A}$ in (8) represents the component of $\partial\mathcal{P}^n_{loc}$ consisting of the polynomials with complex roots $\zeta \notin \mathbb{R}$, $|\zeta| = 1$, $P'(\zeta) = 0$. □

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