Symmetries for the Euclidean Non-Linear Schrödinger Equation and Related Free Equations

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Abstract

We compare certain infinite dimensional Lie algebras of conserved quantities for the free Newton equation $\ddot{q} = 0$, the free heat system and the euclidean non-linear Schrödinger equation. There is a natural differential operator defined for all polynomials of the conservation laws $I_0, I_1, ...$ in the NLS hierarchy. We discuss the invariant polynomials and point out a connection to the free classical equation. The basic ingredient is the presence of an extra 'Heisenberg' element in addition to $I_0, I_1, ...$

0. Introduction. This work is related to the studies of invariance properties of Schrödinger (Bernstein) and related diffusion processes in [1]-[4], [8], [10], [14] and [15], in particular, the case of Gaussian processes [1]-[2].

Going back to a paper [11] by Sophus Lie from 1881, we know that the Lie algebra for the free heat equation in 1+1 dimensions is, except for a 'trivial' infinite dimensional part stemming from linearity, of dimension six. It is a general fact, see [3], that this Lie algebra has a classical counterpart of constants of motion. In fact, in a certain sense they differ at most by one element which needs a "quantum correction". In particular, they have the same dimension. It is shown in the papers referred to above, how to obtain martingales, or stochastic constants of motion, from the heat Lie algebra.

The classical counterpart to the free heat equation is the free Newton equation $\ddot{q} = 0$, which has a six dimensional *Poisson-Noether Lie algebra* generated by the functions 1, p, $pt - q$, p^2, $p(pt - q)$ and $(pt - q)^2$, where $p := \dot{q}$. It may also be described as all constants of motion in the variables t, q and p which are of order at most two w.r.t. p. With $I_n = p^n$ we get an infinite sequence of constants of motion in involution w.r.t. the usual Poisson bracket in two dimensions.

In the case of the free heat Lie algebra, it is clear that partial derivation w.r.t. the space variable q preserves the heat equation: ∂_q is a *recursion operator* [6],

[12]. We can express this in a more symmetric way by looking at the free *heat system* $\dot{u}+\frac{1}{2}u''=0$, $-\dot{v}+\frac{1}{2}v''=0$. (Here and below $\dot{u}=u_t$ and $u'=u_q$.) Then all the functionals $I_n := \frac{1}{2}(u^{(n)}v+(-1)^n uv^{(n)})$, for $n=0, 1, ...$ are *conservation laws* in involution w.r.t. a (well known) natural Lie bracket defined below. It is an elementary but deep fact that

$$v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u} = u^{(n)}v-(-1)^n uv^{(n)} = \frac{\partial}{\partial q}\left(u^{(n-1)}v - u^{(n-2)}v' + ...\right) \qquad (1)$$

and that the expression $u^{(n-1)}v - u^{(n-2)}v' + ...$ between the parentheses on the right is equivalent to $nI_{n-1} = \mathsf{D}I_n$, in the sense of conservation laws. The operator $\mathsf{D} = D^{-1}\left(v\frac{\delta}{\delta v} - u\frac{\delta}{\delta u}\right)$ can be extended to a derivation on the space of polynomials obtained from I_n, $n \geq 0$.

In the last part of the paper we compare this example with the classical case and a non-linear system of heat equations, viz the *Euclidean non-linear heat equation*

$$\dot{u}+\frac{1}{2}u''=u^2v, \qquad -\dot{v}+\frac{1}{2}v''=uv^2. \qquad (2)$$

We also study more generally the structure on the polynomials obtained from a derivation D. This is sketched below, details will appear elsewhere.

1. The heat Lie algebra in 1+1 dimensions. For $u=u(t,q)$, t, q real, we define the (backward) free *heat operator* by

$$Ku := \dot{u}+\frac{1}{2}u''. \qquad (3)$$

Consider all linear partial differential operators Λ of order at most one in (t,q):

$$\Lambda = T\frac{\partial}{\partial t} + Q\frac{\partial}{\partial q} + U, \qquad (4)$$

where T, Q and U are functions of (t,q), and where U acts as multiplication operator.

Definition: Λ belongs to the *heat Lie algebra* if, for some function $\Phi=\Phi_\Lambda(t,q)$ it holds that

$$[K,\Lambda] = K\Lambda - \Lambda K = \Phi\cdot K. \qquad (5)$$

Simple calculations lead to the following well-known facts: The heat Lie algebra consists of two parts, of which the first is generated by the operators

$$\Lambda_0 = 1 \quad \text{(the centre)}, \quad \Lambda_1 = \frac{\partial}{\partial q}, \quad \Lambda^* = \Lambda_1^* = t\frac{\partial}{\partial q} - q, \qquad (6)$$

forming the Heisenberg algebra (since $[\Lambda_1^*, \Lambda_1] = 1$), whereas the second, generated by

$$\Xi_1 = \Lambda_2 = \frac{\partial}{\partial t}, \quad \Xi_2 = t\frac{\partial}{\partial t} + \frac{1}{2}q\frac{\partial}{\partial q}(+\frac{1}{4}), \quad \Xi_3 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{t}{2}q\frac{\partial}{\partial q} - \frac{1}{4}(q^2-t) \quad (7)$$

form the Lie algebra sl_2. (We remark that Λ^* is not intended to suggest adjoint.)

2. The classical counterpart. Consider a two-dimensional phase space with coordinates p and q, and the usual Poisson bracket

$$\{\phi, \psi\} = \frac{\partial\phi}{\partial p}\frac{\partial\psi}{\partial q} - \frac{\partial\phi}{\partial q}\frac{\partial\psi}{\partial p}. \tag{8}$$

Regarding t as a parameter, the functions

$$1, \quad p \quad \text{and} \quad pt - q, \tag{9}$$

form the Heisenberg algebra, whereas

$$p^2, \quad p(pt-q) \quad \text{and} \quad (pt-q)^2 \tag{10}$$

form sl_2.

We now turn to the *free Newton equation*

$$\ddot{q} = 0, \tag{11}$$

in which case $p := \dot{q}$ and $pt - q$ are obvious constants of motion (CMs). The functions $1, ..., (pt-q)^2$ is a basis for the CMs which are of order at most two in the variable p. We call it the *classical algebra.*

The first five functions correspond to the same operators as in the heat case. The sixth function corresponds to the operator

$$\frac{t^2}{2}\frac{\partial}{\partial t} + \frac{t}{2}q\frac{\partial}{\partial q} - \frac{q^2}{4} \quad \text{(no } t\text{-term)} \tag{12}$$

The zero-order term $q^2 - t$ in the heat case is the “quantum version”,
$q^2 - t =: q^2 :$ in physicist notation. Readers familiar with stochastic analysis will no doubt recognise the extra term, e.g. from Ito’s formula.

The heat Lie algebra and the classical algebra satisfy the same commutator relations. The Heisenberg algebra is an ideal.

Clearly

$$I_0 = 1, \quad I_1 = p, \quad I_2 = p^2, \ldots \tag{13}$$

all commute. With

$$I^* = I_1^* = pt - q, \tag{14}$$

we get

$$\{I^*, I_n\} = nI_{n-1} = \frac{d}{dp}I_n. \tag{15}$$

We see that $I_n \to I_{n+1}$ is the “creation operator” multiplication with p, whereas $I_n \to I_{n-1}$ is the “annihilation operator” d/dp.

3. The free heat system. We consider the system of two equations

$$\dot{u} + \frac{1}{2}u'' = 0 \qquad \text{and} \qquad -\dot{v} + \frac{1}{2}v'' = 0, \tag{16}$$

obtained from the Lagrangian

$$L = \frac{1}{2}(u\dot{v} - \dot{u}v) + \frac{1}{2}u'v'. \tag{17}$$

The symmetry Lie algebra contains the vector fields

$$\Lambda_0 = v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u},\ \Lambda_1 = \frac{\partial}{\partial q},\ \Lambda_1^* = t\frac{\partial}{\partial q} - q\left(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\right),\ \Lambda_2 = \frac{\partial}{\partial t}. \tag{18}$$

By Noether's theorem we get the conservation laws

$$I_0 = uv, \quad I_1 = \tfrac{1}{2}(u'v - uv'), \quad I^* = tI_1 - qI_0, \quad I_2 = \tfrac{1}{2}(u''v + uv''), \tag{19}$$

where I_0, I_1 and $I^* = I_1^*$ form a Heisenberg algebra with respect to the (field theory) bracket

$$\{F, G\} := \int \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u}\right) dq, \tag{20}$$

t being looked upon as a parameter. Here, the variational derivative refers to the space variable only:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dq}\frac{\partial F}{\partial u'} + \frac{d^2}{dq^2}\frac{\partial F}{\partial u''} - \ldots\ldots \tag{21}$$

with a corresponding expression for $\delta F/\delta v$.

There are two more vector fields associated with the remaining elements of sl_2, viz. $\Xi_2 = t\frac{\partial}{\partial t} + \frac{q}{2}\frac{\partial}{\partial q}$, and $\Xi_3 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{tq}{2}\frac{\partial}{\partial q} - \frac{q^2}{4}\Lambda_0 - \frac{t}{4}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)$ but our main interest is with I_0, I_1, I^* and I_2. We remark however, that in this more symmetric setting the quantum correction disappears: the conservation law corresponding to Ξ_3 is 1/2 times $t^2 I_2 + tqI_1 - \frac{1}{2}q^2 I_0$.

Now, defining

$$I_n := \tfrac{1}{2}\left(u^{(n)}v + (-1)^n uv^{(n)}\right), \qquad n \geq 0, \tag{22}$$

i.e.

$$I_{n+1} = CI_n = \frac{1}{2}D \cdot I_n \qquad \text{(Hirota bilinear derivative)} \tag{23}$$

one finds

$$\{I_m, I_n\} = 0 \quad \text{and} \quad \{I^*, I_n\} = nI_{n-1}, \qquad m,\, n \geq 0 \tag{24}$$

(with $I_{-1} = 0$).

Exactly as in the classical case, C is the creation and $\{I^*, \cdot\}$ the annihilation operator.

4. Constants of motion for the free heat system. Assume that u and v satisfy (16) (we could add interaction terms here). Then

$$\frac{d}{dt}\int uv\,dq = 0, \tag{25}$$

assuming, as always, sufficently rapid decrease at infinity, or the boundary, of u and v. Let $f = f(t, q)$. It is easy to show that

$$\frac{d}{dt}\int f\,uv\,dq = \int Df\,uv\,dq = \int D^* f\,uv\,dq, \tag{26}$$

where

$$Df := \frac{1}{u}K(fu) = \dot{f} + \tfrac{1}{2}f'' + \frac{u'}{u}f, \tag{27}$$

$$D^* f := -\frac{1}{v}K^\dagger(fv) = \dot{f} - \tfrac{1}{2}f'' - \frac{v'}{v}f. \tag{28}$$

Then, if $\Lambda = T\partial_t + Q\partial_q + U$ belongs to the heat Lie algebra, we have

$$uD(\Lambda u/u) = K\Lambda u = ([K, \Lambda] + \Lambda K)u = (\Phi_\Lambda + \Lambda)Ku = 0. \tag{29}$$

This is an alternative way to express that $\Lambda u \cdot v$ is the density of a conservation law. In more detail, the preceding equation may be written

$$D(\Lambda u/u) = T\frac{\dot{u}}{u} + Q\frac{u'}{u} + U = 0, \tag{30}$$

very much as in the classical case. The coefficients $\frac{\dot{u}}{u}$ and $\frac{u'}{u}$ are, respectively, the *energy density* and the *momentum density* in a form that emphasises the backward motion. The density is $I_0 = uv$, and e.g. $u'v = \frac{u'}{u} \cdot I_0$ is an equivalent form for I_1.

5. Euclidean non-linear Schrödinger system. ENS may be looked upon as an extension of the free heat system with a 'potential' V that depends on u and v. Let us start somewhat more generally with

$$\dot{u} + \frac{1}{2}u'' = Vu \qquad \text{and} \qquad -\dot{v} + \frac{1}{2}v'' = Vv, \tag{31}$$

obtained from the Lagrangian

$$L = \frac{1}{2}(u\dot{v} - \dot{u}v) + \frac{1}{2}u'v' + \Phi(uv), \tag{32}$$

provided $V = \phi(uv)$, with $\phi = \Phi'$.

The following are always conservation laws:

$$I_0 = uv, \quad I_1 = \tfrac{1}{2}(u'v - uv'), \quad I^* = tI_1 - qI_0, \tag{33}$$

$$I_2 = \tfrac{1}{2}(u''v + uv'') - 2\Phi(uv) \tag{34}$$

for the same reasons as in the free heat case. They also commute.

One can prove that there is a third order conservation law,

$$\frac{1}{2}(u'''v - uv''') + \text{ terms of lower order,} \tag{35}$$

only when $\Phi''' = 0$. Leaving the linear case ($\Phi'' = 0$) aside, we choose $\Phi(s) = \frac{1}{2}s^2$ so that our heat equations become

$$\dot{u} + \frac{1}{2}u'' = u^2 v \qquad \text{and} \qquad -\dot{v} + \frac{1}{2}v'' = uv^2, \tag{36}$$

corresponding to $V = uv$. This can be seen as a two-dimensional field theory with quartic interaction.

One can prove [8] , [16], that there is an operator C such that

$$I_n := C^n I_0, \qquad n \geq 0, \tag{37}$$

satisfy the same relations as in the free case:

$$\{I_m, I_n\} = 0 \quad \text{and} \quad \{I^*, I_n\} = nI_{n-1}, \qquad m, n \geq 0. \tag{38}$$

After I_2, the next two are

$$I_3 = \tfrac{1}{2}(u'''v - uv''') - \tfrac{3}{2}uv(u'v - uv'), \tag{39}$$

$$I_4 = \tfrac{1}{2}(u^{\text{iv}}v + uv^{\text{iv}}) + u'^2v^2 + u^2v'^2 + 6uu'vv' + 2u^3v^3. \tag{40}$$

Since all I_n commute with I_0, we have

$$v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u} = \frac{d}{dq}a_n \tag{41}$$

for some functional a_n. More generally, one sees that for each I that commutes with I_0, the operator

$$\mathsf{D}I := \left(\frac{d}{dq}\right)^{-1}\left(v\frac{\delta I}{\delta v} - u\frac{\delta I_n}{\delta u}\right) \tag{42}$$

is well defined. One can show that for functions of I_0, I_1, I_2, ..., D is a *derivation* in that

$$\mathsf{D}\{f(I_0, I_1, ..., I_n)\} = \sum_{\mu=0}^{n} \partial_\mu f(I_0, I_1, ..., I_n)\mathsf{D}I_\mu, \tag{43}$$

for any C^1 function f. This holds also in the free heat case.

6. Invariant polynomials. We assume, without reference to the particular cases considered above, that we are given variables I_0, I_1, $I_2, ...$, and an operator D such that

$$\mathsf{D}I_\mu = \mu I_{\mu-1} \quad \text{for all} \quad \mu = 0,\, 1,\, ... \tag{44}$$

We also assume that

$$\mathsf{D}\{f(I_0, I_1, ..., I_n)\} = \sum_{\mu=0}^{n} \partial_\mu f(I_0, I_1, ..., I_n)\mu I_{\mu-1} \tag{45}$$

for any $n \in \mathbb{N}$ and for any function $f \in C^1(\mathbb{R}^{n+1})$.

Definition: A function $M = M_\alpha$ of the form

$$M = I_0^{\alpha_0} I_1^{\alpha_1} \cdots I_n^{\alpha_n}, \quad \alpha_0,\, \alpha_1, ..., \alpha_n \in \mathbb{N}, \tag{46}$$

is a *monomial of order*

$$N = \sum \mu\alpha_\mu = ||\alpha||. \tag{47}$$

Definition: A function P of the form

$$P = \sum_{||\alpha||=N} c_\alpha M_\alpha, \tag{48}$$

where c_α are constants, is a *polynomial of order N*.

Definition: A polynomial P is *invariant* if

$$\mathsf{D}P = 0. \tag{49}$$

For $N = 0$ every polynomial, in fact, every differentiable function, of I_0 is invariant. These functions should be looked upon as scalars.

For $N = 1$ there are no invariant polynomials.

For $N = 2$,

$$K_2 = I_1^2 - I_0 I_2 \tag{50}$$

is invariant, and for $N = 3$,

$$K_3 = 2I_1^3 - 3I_0 I_1 I_2 + I_0^2 I_3 \tag{51}$$

is invariant. Up to multiplication with functions of I_0, K_2 and K_3 are unique.

For $N = 4$, of course K_2^2 is invariant. There is another one, unique up to multiplication with functions of I_0, viz.

$$K_4 = 4I_1 I_3 - 3I_2^2 - I_0 I_4. \tag{52}$$

K_4 is *irreducible.*

For $N = 5$ we get the obviously invariant polynomial $K_2 K_3$ and a new, irreducible, invariant polynomial, K_5. For $N = 6$ we get K_2^3, K_3^2 and $K_2 K_4$ in addition to the new, irreducible, invariant polynomial K_6.

Theorem: *For each $N \geq 2$ there is an irreducible invariant polynomial K_N, unique up to multiplication with functions of I_0.*

Denote by $\mathcal{P}$ all polynomials, and by $\mathcal{M}$ the quotient space

$$\mathcal{M} = \mathcal{P}/(K_2, K_3, ...) \tag{53}$$

That $K_2 \equiv 0$ means that $I_0 I_2 \equiv I_1^2$ or

$$\frac{I_2}{I_0} \equiv \left(\frac{I_1}{I_0}\right)^2 \tag{54}$$

Using also $K_3 \equiv 0$ we find $I_3/I_0 \equiv (I_1/I_0)^3$, and so on:

$$\frac{I_n}{I_0} \equiv \left(\frac{I_1}{I_0}\right)^n, \quad n = 2, 3, ... \tag{55}$$

Note that

$$\mathsf{D}\frac{I_1}{I_0} = 1, \quad \text{and} \quad \mathsf{D}\frac{I_n}{I_0} \equiv \mathsf{D}\left(\frac{I_1}{I_0}\right)^n = n\left(\frac{I_1}{I_0}\right)^{n-1}, \quad n = 0, 1, ... \tag{56}$$

Hence I_1/I_0 correspond to p in our first example, the free equation $\ddot{q} = 0$. We remark that I_1/I_0 is the momentum density mentioned at the end of §4 above.

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References

[1] Brandão, A, Symplectic structure for Gaussian diffusions. *J. Math. Phys.* **39** (1998), no. 9, 4257–4283.

[2] Brandão, A. and Kolsrud T., Phase space transformations of Gaussian diffusions. *Potential Anal.* **10** (1999), no. 2, 119–132.

[3] Brandão, A. and Kolsrud T., Time-dependent conservation laws and symmetries for classical mechanics and heat equations. Harmonic morphisms, harmonic maps, and related topics (Brest, 1997), 113–125, Chapman & Hall/CRC Res. Notes Math., 413, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[4] Djehiche, B. and Kolsrud T., Canonical transformations for diffusions. *C. R. Acad. Sci. Paris* **321, I** (1995), 339–44.

[5] Goldstein, H., *Classical mechanics*, 2nd ed, Addison-Wesley, New York, 1980.

[6] Ibragimov, N.H., *Transformation Groups Applied to Mathematical Physics,* Nauka, Moscow, 1983 (English translation by D. Reidel, Dordrecht, 1985).

[7] Ibragimov, N.H. and Kolsrud T., Lagrangian approach to evolution equations: symmetries and conservation laws. *Nonlinear Dynam.* **36**, 2004, no 1, 29-40.

[8] Kolsrud, T., Quantum constants of motion and the heat Lie algebra in a Riemannian manifold. Preprint TRITA-MAT 1996 (Stockholm)

[9] Kolsrud, T., The Hierarchy of the Euclidean Non-linear Schrödinger Equation is a Harmonic Oscillator Containing KdV. Preprint TRITA-MAT 2004 (Stockholm)

[10] Kolsrud, T. and Zambrini, J. C., The general mathematical framework of Euclidean quantum mechanics. Stochastic analysis and applications (Lisbon 1989), 123-43, Birkhäuser 1991.

[11] Lie, S. Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen, *Arch. Math.* **6** (1881), 328-68.

[12] Morse, P. M. and Feshbach, H., *Methods of theoretical physics, vol I-II* McGraw-Hill New York 1953.

[13] Olver, P. J., *Applications of Lie groups to differential equations, Second edition,* Springer, Berlin, Heidelberg, New York 1993.

[14] Thieullen, M. and Zambrini, J.C., Probability and quantum symmetries I. The theorem of Noether in Schrödinger's euclidean quantum mechanics. *Ann. Inst. Henri Poincaré, Phys. Théorique* **67:3** (1997), 297–338.

[15] Thieullen, M. and Zambrini, J.C., Symmetries in the stochastic calculus of variations. *Probab. Theory Relat. Fields* **107** (1997), 401–27.

[16] Faddeev, L.D. and Takhtajan, L.A., *Hamiltonian Methods in the Theory of Solitons,* Nauka, Moscow, 1986 (English translation by Springer-Verlag, Berlin Heidelberg, 1987).