# PROLONGATIONS AND CYCLIC VECTORS

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## 1. Introduction

In this paper we continue and refine a discussion begun in [19] concerning the continuation properties of the non-cyclic vectors for the backward shift operator

$$Bf = \frac{f - f(0)}{z}$$

on a general Banach space $X$ contained in $\mathrm{Hol}(\mathbb{D})$, the analytic functions on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here we say $f \in X$ is *cyclic* if

$$[f] := \bigvee\{B^n f : n = 0, 1, 2 \cdots\} = X$$

($\bigvee$ denotes the closed linear span). As observed in several papers including [1, 2, 8, 16, 18], non-cyclic vectors have 'continuations' of one sort or another to functions $\widetilde{f}$ which are meromorphic on the extended exterior disk $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$. Here $\mathbb{D}^- := \{|z| \leqslant 1\}$ is the closure of the disk and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Perhaps the first to observe this continuation phenomenon was Nyman [16, Ch. 4, p. 88 ff] who observed that if $f \in \ell^\infty_A$ (analytic functions on $\mathbb{D}$ with bounded Taylor coefficients, endowed with the weak-* topology of the set of two sided bounded sequences $\ell^\infty$) is non-cyclic and has an analytic continuation to a neighborhood of an arc $I$ of the unit circle $\mathbb{T} := \partial\mathbb{D}$, then $f$ has an analytic continuation across $I$ to a meromorphic function on $\mathbb{D}_e$.

When our Banach space $X$ is the Hardy space

$$H^2 := \Big\{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \Big\},$$

there is a result of [8] which says the following: To each non-cyclic vector $f \in H^2$, there is a unique meromorphic function $\widetilde{f}$ of bounded type (quotient of two bounded analytic functions) on $\mathbb{D}_e$ such that the non-tangential limits of $f$ (from $\mathbb{D}$) and $\widetilde{f}$ (from $\mathbb{D}_e$) agree almost everywhere with respect to Lebesgue measure on the unit circle $\mathbb{T} := \partial\mathbb{D}$. In language developed in [20], we say that $\widetilde{f}$ is a *pseudocontinuation* of $f$. In fact, the existence of such a pseudocontinuation $\widetilde{f}$ of bounded type completely characterizes the non-cyclic vectors of $H^2$. Moreover, $\widetilde{f} = \widetilde{f}_L$, where

$$\widetilde{f}_L(w) := L\left(\frac{f}{z-w}\right) \Big/ L\left(\frac{1}{z-w}\right), \tag{1.1}$$

and $L$ is any non-zero continuous linear functional on $H^2$ for which $L|[f] = 0$. That $\widetilde{f}_L$ is independent of $L$ follows from the Lusin-Privalov uniqueness theorem [12, p. 62]: If $g$ is meromorphic on $\mathbb{D}$ and has non-tangential limits equal to zero on a subset of $\mathbb{T}$ of positive measure, then $g \equiv 0$. The same pseudocontinuation results hold for the Hardy spaces $H^p$

$$H^p := \Big\{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty\Big\}, \ \ p > 1.$$

When $X = L^2_a$, the Bergman space

$$L^2_a := \Big\{f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} \frac{1}{n+1}|a_n|^2 = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\Big\},$$

where $dA$ is area measure, a similar result is true: Each non-cyclic $f \in L^2_a$ is of bounded type on $\mathbb{D}$ (Bergman space functions are, in general, not of bounded type) and moreover, there is a unique $\widetilde{f}$ of bounded type on $\mathbb{D}_e$ which is a pseudocontinuation of $f$ [18]. However, there are inner functions $\varphi$, which are cyclic for $B$ on $L^2_a$, and have pseudocontinuations given by

$$\widetilde{\varphi}(z) := \frac{1}{\overline{\varphi(1/\overline{z})}}$$

[5, p. 108]. Again, as is the case for the Hardy space, the pseudocontinuation $\widetilde{f}$ of $f$ is given by the formula in eq.(1.1) (and is independent of the annihilating $L$). The same result is true for the more general weighted Bergman spaces

$$L^p_a(w) = \Big\{f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p w(|z|) dA(z) < \infty\Big\}, \ \ p > 1,$$

for suitable weights $w$, for example $w(r) = (1-r)^s, s > -1$ [1, 2].

When $X = \mathcal{D}$, the classical Dirichlet space

$$\mathcal{D} := \Big\{f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty}(n+1)|a_n|^2 = \int_{\mathbb{D}} |(zf)'|^2 \frac{dA}{\pi} < \infty\Big\},$$

or more generally the spaces

$$\mathcal{D}_p := \Big\{f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |(zf)'(z)|^p dA < \infty\Big\}, \ \ p > 1,$$

$$D_\alpha := \Big\{f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty}(n+1)^\alpha |a_n|^2 < \infty\Big\}, \quad \alpha > 0,$$

the non-cyclic vectors $f$ need not have pseudocontinuations across any arc of $\mathbb{T}$ [1, Thm. 6.2]. Nevertheless, for non-cyclic $f$, the function $\widetilde{f}_L$ in eq.(1.1) can be thought of as a 'continuation' of $f$. For example, in $\mathcal{D}_p$,

$$L\big(\frac{1}{z-w}\big)\Big[\widetilde{f}_L(w) - f(e^{i\theta})\,\Big] \to 0$$

for almost every $e^{i\theta}$ as $w \to e^{i\theta}$ non-tangentially [1, Lemma 7.4]. So, for example, if $w \to L((z-w)^{-1})$ has non-tangential limits almost everywhere on $\mathbb{T}$, then $\widetilde{f}_L$ is a pseudocontinuation of $f$.

In this paper, we demonstrate both the ubiquity and the utility of the function $\widetilde{f}_L$, which we shall call the *$L$-prolongation of $f$*. In the above discussion, we have already seen the appearance of $L$-prolongations as 'continuations' of non-cyclic vectors. $L$-prolongations also appear in the analysis of $\sigma(B|\mathcal{M})$, the spectrum of $B$ restricted to one of its invariant subspaces $\mathcal{M}$. This has been observed in [1] and we review these results in § 2 of this paper. In § 3 we will show that for a wide class of Banach spaces $X$, that include $H^p$, $L^p_a(w)$, $D_\alpha$, and $\mathcal{D}_p$, the $L$-prolongation of $f$ is compatible with analytic continuation in the sense that if $f$ has an analytic continuation (also denoted by $f$) to an open neighborhood $U_\zeta$ of $\zeta \in \mathbb{T}$, then $f$ is equal to $\widetilde{f}_L$ on $U_\zeta \cap \mathbb{D}_e$. For these spaces $X$, the proof presented here will be simpler than the one presented in [19, p. 101]. From the other direction, we will examine the question: If $\widetilde{f}_L$ has an analytic continuation (also denoted by $\widetilde{f}_L$) to an open neighborhood $U_\zeta$ of $\zeta \in \mathbb{T}$, then is $\widetilde{f}_L$ equal to $f$ on $U_\zeta \cap \mathbb{D}$? These results yield interesting corollaries about the nature of non-cyclic vectors. For example, if $f \in X$ has an isolated winding point on the circle, then $f$ is cyclic. This is a well-known fact in the $H^2$ setting [8]. These results also make connections to certain convolution equations [4, 9] and to determining the orbit of an $f \in X$ under both the forward and backward shift operators, that is

$$\bigvee\{z^n f, B^n f : n = 0, 1, 2, \cdots\}.$$

Finally, in § 4 and § 5, we will show how $L$-prolongations relate to several questions, originally asked in [21], about overconvergence of rational functions. We will also show the relationship between $L$-prolongations and approximate spectral synthesis.

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## 2. Preliminaries

Let $X$ be a Banach space of analytic functions on $\mathbb{D}$ which satisfies the following properties:

$$X \hookrightarrow \mathrm{Hol}(\mathbb{D}) \tag{2.1}$$

$$M_z X \subset X, \quad M_z f := zf \tag{2.2}$$

$$1 \in X \tag{2.3}$$

$$\bigvee\{1, z, z^2, \cdots\} = X \tag{2.4}$$

$$\frac{f - f(\lambda)}{z - \lambda} \in X \text{ whenever } f \in X \text{ and } \lambda \in \mathbb{D} \tag{2.5}$$

$$\sigma(M_z) = \mathbb{D}^-. \tag{2.6}$$

$$\|M_{(z-\lambda)^{-1}}\| \to 0, \quad |\lambda| \to \infty. \tag{2.7}$$

*Remark* 2.8. (1) In eq.(2.1), the inclusion map from $X$ (with the norm topology) to $\mathrm{Hol}(\mathbb{D})$ (with the topology of uniform convergence on compact sets) is both injective and continuous. In particular, for each compact $K \subset \mathbb{D}$,

$$\sup\{|f(z)| : z \in K\} \leqslant C_K \|f\| \text{ for all } f \in X. \tag{2.9}$$

(2) Note that

$$\bigvee\{1, z, z^2, \cdots\} = \bigvee\Big\{\frac{1}{z - \lambda} : \lambda \in \mathbb{D}_e\Big\} = X$$

[1, Prop. 2.2].

(3) For an operator $T$ on a Banach space $V$, $\sigma(T)$, the spectrum of $T$, denotes the set of complex numbers $\lambda$ such that $(\lambda I - T)$ is not invertible.
(4) For $|\lambda| > 1$, the operator $M_{(z-\lambda)^{-1}} f := f/(z-\lambda)$ is well-defined by eq.(2.6).

Examples of such $X$ include the spaces $H^p$, $L^p_a(w)$ for suitable $w$, $\mathcal{D}_p$, $D_\alpha$ $(-\infty < \alpha < \infty)$ as well as spaces like

$$\ell^p_A(w) := \Big\{f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^p w_n < \infty\Big\}, \quad p \geqslant 1,$$

for suitable weight sequences $w = (w_n)_{n\geqslant 0}$ [22].

If $X'$ is the dual space of $X$ and $L \in X'$, the function

$$\lambda \to L\big(\frac{1}{z-\lambda}\big)$$

is analytic on $\mathbb{D}_e$ and vanishes at infinity. It follows from the Hahn-Banach theorem and basic complex function theory that

$$X = \bigvee \big\{\frac{1}{z-\lambda} : \lambda \in E\big\}, \tag{2.10}$$

whenever $E \subset \mathbb{D}_e$ has a cluster point in $\mathbb{D}_e$. Furthermore, for fixed $|\lambda| > 1$,

$$X = \bigvee \big\{\frac{1}{(z-\lambda)^n} : n = 1, 2, \cdots\big\}. \tag{2.11}$$

We will say the dual pair $(X, X')$ is an $\ell^2$ *dual pair* if $X'$ can be identified with a Banach space of analytic functions on $\mathbb{D}$ such that the dual pairing is given by

$$(f, g) := \sum_{n=0}^{\infty} a_n \overline{b_n}, \tag{2.12}$$

where the above series converges absolutely and $(a_n)_{n\geqslant 0}$ are the Taylor series coefficients of $f \in X$ and $(b_n)_{n\geqslant 0}$ are those of $g \in X'$. In this case, $X'$ also satisfies the conditions eq.(2.1) through eq.(2.7) [1, Prop. 5.2]. Spaces like $\ell^p_A(w)$, for $p > 1$ and $w = (w_n)_{n\geqslant 0}$, satisfy this condition since $(\ell^p_A)'$ can be identified with $\ell^q_A(w')$, where $q = p/(p-1)$ and $w' = (1/w_n)_{n\geqslant 0}$. For example, $(L^2_a, \mathcal{D})$ is an $\ell^2$ dual pair. Note that for $g \in X', \lambda \in \mathbb{D}$, and $n = 0, 1, 2, \cdots$,

$$\big(\frac{n! z^n}{(1-\overline{\lambda} z)^{n+1}}, g\big) = \overline{g^{(n)}(\lambda)}. \tag{2.13}$$

We will not require $(X, X')$ to always be an $\ell^2$ dual pair, but will impose it from time to time as needed.

From the hypothesis on $X$, it follows that the operators $M_z f = zf$ and

$$Bf = \frac{f - f(0)}{z}$$

are continuous on $X$. In fact, if $(X, X')$ is an $\ell^2$ dual pair, then $B$ on $X$ is the adjoint of $M_z$ on $X'$. We will denote the collection of (closed) $B$-invariant subspaces by $\mathrm{Lat}(B, X)$. In this general setting, one can prove (see [1, §2]) that

$$\sigma(B) = \mathbb{D}^-$$

and that for $w \in \mathbb{D}$,

$$(I - wB)^{-1} f = \frac{zf - wf(w)}{z - w} \tag{2.14}$$

$$(I - wB)^{-1}Bf = \frac{f - f(w)}{z - w}. \tag{2.15}$$

From the spectral radius formula, it follows that

$$\sigma(B|\mathcal{M}) \subset \mathbb{D}^- \text{ for all } \mathcal{M} \in \mathrm{Lat}(B, X). \tag{2.16}$$

Since, for $a \in \mathbb{D}$,

$$B\frac{1}{1 - az} = a\frac{1}{1 - az},$$

each $a \in \mathbb{D}$ is an eigenvalue of $B$ (with geometric multiplicity one) with corresponding eigenfunction $(1 - az)^{-1}$. Moreover,

$$\sigma_{ap}(B|\mathcal{M}) \cap \mathbb{D} = \sigma_p(B|\mathcal{M}) \cap \mathbb{D} = \Big\{a \in \mathbb{D} : \frac{1}{1 - az} \in \mathcal{M}\Big\}. \tag{2.17}$$

Here $\sigma_{ap}$ is the approximate point spectrum and $\sigma_p$ is the point spectrum (the set of eigenvalues). Furthermore, by eq.(2.10), the set in eq.(2.17) is either a countable subset of $\mathbb{D}$ with no cluster points in $\mathbb{D}$, or all of $\mathbb{D}$. Thus $\sigma_p(B|\mathcal{M}) \cap \mathbb{D} = \mathbb{D}$ if and only if $\mathcal{M} = X$. Under a mild regularity condition on $X$,

$$\sigma_{ap}(B|\mathcal{M}) \cap \mathbb{T} = \sigma(B|\mathcal{M}) \cap \mathbb{T} \tag{2.18}$$

and this set is the complement (in the unit circle) of the set of points $\zeta \in \mathbb{T}$ such that every $f \in \mathcal{M}$ extends to be analytic in a neighborhood of $1/\zeta$. Furthermore, since $\partial\sigma(B|\mathcal{M}) \subset \sigma_{ap}(B|\mathcal{M})$, one can prove the following dichotomy: either

$$\sigma(B|\mathcal{M}) \cap \mathbb{D} = \sigma_{ap}(B|\mathcal{M}) \cap \mathbb{D}$$

and is a countable subset of $\mathbb{D}$ with no cluster points in $\mathbb{D}$, or $\sigma(B|\mathcal{M}) = \mathbb{D}^-$.

For a set $A \subset X$, let $A^\perp$, the annihilator of $A$, denote the set

$$A^\perp := \{L \in X' : L|A = 0\}.$$

For a non-cyclic vector $f \in X$ and $L \in [f]^\perp \setminus \{0\}$, define the *$L$-prolongation* of $f$ to be the meromorphic function on $\mathbb{D}_e$ defined by

$$\widetilde{f}_L(w) := L\left(\frac{f}{z - w}\right) / L\left(\frac{1}{z - w}\right). \tag{2.19}$$

Observe from eq.(2.10) that the denominator of the above expression is not identically zero. One can show, when $T = B|[f]$ and $\sigma(T) \cap \mathbb{D}$ is discrete, that for each non-trivial annihilating $L$ and $|w| > 1$ with $1/w \notin \sigma(T)$,

$$(I - wT)^{-1}f = \frac{zf - \widetilde{f}_L(w)}{z - w}.$$

Compare this formula to the one in eq.(2.14). Furthermore, in this case, the meromorphic function $\widetilde{f}_L$ is independent of $L$. In fact, if $\mathcal{M}$ is any $B$-invariant subspace, $T := B|\mathcal{M}$, and $|w| > 1$ with $1/w \notin \sigma_{ap}(T)$, then $(I - wT)^{-1}$ exists if and only if for each $f \in \mathcal{M}$, $\widetilde{f}_L(w)$ is independent of the $L \in \mathcal{M}^\perp$ with $L((z - w)^{-1}) \neq 0$ [1, Prop. 2.6].

When $X$ is either $H^p$ or $L^p_a(w)$, $\widetilde{f}_L$ is a function of bounded type and is a pseudocontinuation of $f$. Thus, by the Lusin-Privalov uniqueness theorem, $\widetilde{f}_L$ is independent of $L \in [f]^\perp \setminus \{0\}$. For other spaces $X$ such as the Dirichlet-type spaces $\mathcal{D}_p$ or $D_\alpha$ ($\alpha > 0$), $\widetilde{f}_L$ need not be of bounded type nor be a pseudocontinuation of $f$. Indeed, it might depend on $L$ [19, p. 122]. For instance, there is a non-cyclic

$f \in \mathcal{D}$ and $L_1, L_2 \in [f]^{\perp} \setminus \{0\}$ such that $\widetilde{f}_{L_1}$ is a pseudocontinuation of $f$ while $\widetilde{f}_{L_2}$ is not.

The $L$-prolongation $\widetilde{f}_L$ can also be thought of as a 'continuation' via formal multiplication of series as in [19, Chap. 8]. In cases where $(X, X')$ is an $\ell^2$ dual pair, then

$$L\,\Big(\,\frac{1}{z-w}\,\Big)\,f(w) = L\,\Big(\,\frac{f}{z-w}\,\Big),$$

where we understand the above equation (which is not technically defined since $L((f/(z-w))$ and $L(1/(z-w))$ are defined on $\mathbb{D}_e$ while $f$ is defined on $\mathbb{D}$) as a formal multiplication of Laurent series. That is to say, if

$$f = \sum_{n=0}^{\infty} a_n w^n, \quad |w| < 1$$

$$L\,\Big(\,\frac{f}{z-w}\,\Big) = \sum_{n=0}^{\infty} \frac{A_n}{w^n}, \quad |w| > 1,$$

$$L\,\Big(\,\frac{1}{z-w}\,\Big) = \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \quad |w| > 1,$$

then a formal computation with Laurent series, along with the fact that

$$0 = L(B^N f) = \sum_{k=0}^{\infty} a_{N+k} B_k, \quad N = 0, 1, 2, \cdots,$$

yields

$$\big(B_0 + \frac{B_1}{w} + \cdots\big)\big(a_0 + a_1 w + \cdots\big) = A_0 + \frac{A_1}{w} + \cdots.$$

## 3. Compatibility with analytic continuation

In this section, we will prove two results about the compatibility of $L$-prolongations with ordinary analytic continuation. The first is the following.

**Theorem 3.1.** *Let $X$ satisfy eq.(2.1) through eq.(2.7) along with the following additional condition: If $f \in X$ extends analytically to an open neighborhood of a point $\zeta \in \mathbb{T}$, then*

$$\frac{f - f(w)}{z - w} \to \frac{f - f(\zeta)}{z - \zeta} \quad \text{in the norm of } X \text{ as } w \to \zeta. \tag{3.2}$$

*Suppose that $f \in X$ extends analytically, to a function also denoted by $f$, to an open neighborhood $U_\zeta$ of a boundary point $\zeta \in \mathbb{T}$. Then any $L$-prolongation $\widetilde{f}_L$ agrees with $f$ on $U_\zeta \cap \mathbb{D}_e$.*

*Proof.* Let $T = B|[f]$. Since, by eq.(2.16), $\sigma(T) \subset \mathbb{D}^-$, then $(I - wT)^{-1}$ exists for all $w \in \mathbb{D}$. Moreover, by eq.(2.15),

$$(I - wT)^{-1} T f = \frac{f - f(w)}{z - w}.$$

So for any $L \in [f]^{\perp} \setminus \{0\}$,

$$L\big(\frac{f - f(w)}{z - w}\big) = 0 \text{ for all } w \in \mathbb{D}.$$

By eq.(2.6),

$$\frac{f}{z-w} \text{ and } \frac{1}{z-w} \in X \text{ for all } w \in U_\zeta \cap \mathbb{D}_e$$

and moreover, since $f$ is analytic on $U_\zeta$, we can use our hypothesis in eq.(3.2) to observe that

$$w \to \frac{f-f(w)}{z-w}$$

is an analytic $X$-valued function on $U_\zeta$. Thus,

$$H(w) := L\Big(\frac{f-f(w)}{z-w}\Big)$$

is analytic on $U_\zeta$. But since $H$ is zero on $U_\zeta \cap \mathbb{D}$, it is zero on all of $U_\zeta$. It follows now that

$$L\,\Big(\frac{f}{z-w}\Big) = f(w) L\,\Big(\frac{1}{z-w}\Big) \quad \text{for all } w \in U_\zeta \cap \mathbb{D}_e$$

and so $\widetilde{f}_L = f$ on $U_\zeta \cap \mathbb{D}_e$. □

The hypothesis in eq.(3.2) seems a bit mysterious. Nevertheless, spaces $X$ with a norm satisfying

$$\|f\|^p \asymp \sum_{k=0}^{K} \int_{\mathbb{D}} |f^{(k)}(z)|^p w(|z|) dA(z) \tag{3.3}$$

satisfy this hypothesis. Indeed, if $f \in X$ continues analytically to $B(\zeta, 2r) := \{w : |w-\zeta| < 2r\}$ and $C = \{w : |w-\zeta| = \frac{3}{2}r\}$, then for $a, b, z \in B(\zeta, r)$,

$$f(z;a) := \frac{f(z)-f(a)}{z-a} = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z)(t-a)} dt.$$

For each $k = 0, 1, \cdots, K$,

$$\frac{d^k}{dz^k}\big[f(z;a) - f(z;b)\big] = \frac{k!}{2\pi i}(a-b)\int_C \frac{f(t)}{(t-z)^{k+1}(t-a)(t-b)} dt.$$

Thus

$$\sup_{z \in B(\zeta,r)} \Big|\frac{d^k}{dz^k}\big[f(z;a) - f(z;b)\big]\Big| \leqslant C(k,r)$$

and is independent of the points $a$ and $b$. Now estimate $\|f(z;a) - f(z;b)\|$ by computing the integrals in eq.(3.3) separately over $\mathbb{D} \cap B(\zeta, r)$ and $\mathbb{D} \setminus B(\zeta, r)$ and use the dominated convergence theorem.

Certainly all the Bergman-type spaces $L^p_a(w)$ or even $\mathcal{D}_p$ satisfy the extra condition of eq.(3.2). The Dirichlet-type spaces $D_\alpha$ $(-\infty < \alpha < \infty)$ also satisfy the extra hypothesis of Theorem 3.1 since

$$\sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 \asymp \int_{\mathbb{D}} |f|^2 (1-|z|^2)^{-1-\alpha} dA, \quad \alpha < 0, \tag{3.4}$$

[24, Lemma 2] and for any $\alpha \in \mathbb{R}$, $f \in \mathcal{D}_\alpha$ if and only if $f' \in \mathcal{D}_{\alpha-2}$. For other spaces such as $\ell^p_A(w)$, the condition in eq.(3.2) seems difficult to verify, though, for a wide class of weight sequences $w = (w_n)_{n \geqslant 0}$, there is an alternate proof of Theorem 3.1 which is more involved [19, p. 101].

Since $L$-prolongations must be single-valued, the following useful corollary follows.

**Corollary 3.5.** *Any $f \in X$ which has an isolated winding point on the circle, must be cyclic.*

For example, the function $\log(1-z)$, whenever this belongs to $X$, is cyclic. Other functions such as

$$\exp\big(\frac{1}{z-2}\big) \text{ and } e^z$$

are also cyclic since they have analytic continuations across $\mathbb{T}$ but are not meromorphic on $\mathbb{D}_e$. Note that $\infty$ lies in the interior of $\mathbb{D}_e$ (the *extended* exterior disk) and so $e^z$ is not meromorphic on $\mathbb{D}_e$.

Now we come from the other direction and start with the $L$-prolongation $\widetilde{f}_L$. If $\widetilde{f}_L$ has an analytic continuation across a boundary point $\zeta \in \mathbb{T}$, must this continuation be equal to $f$? For a general Banach space $X$, this question seems difficult so let us first focus on the important special case when $\widetilde{f}_L$ is identically zero. We restrict ourselves to spaces $X$ which satisfy our usual properties eq.(2.1) through eq.(2.7) along with the following two additional conditions: First we assume that $(X, X')$ is an $\ell^2$ dual pair. Second, we assume that if $f = \sum_n a_n z^n \in X \setminus \{0\}$ and $g = \sum_n b_n z^n \in X'$, then

$$\sum_{l=0}^{\infty} a_{l-k}\overline{b_l} = 0 \text{ for all } k \in \mathbb{Z} \ \ \Rightarrow (b_n)_{n \geqslant 0} \equiv (0), \tag{3.6}$$

(where $a_s := 0$ when $s < 0$). The sequence defined by the left-hand side of the above is the *convolution* $(a_{-n}) * (\overline{b_n})$ of the sequences $(a_{-n})_{n\in\mathbb{Z}}$ and $(\overline{b_n})_{n\in\mathbb{Z}}$ (where $b_n := 0$ for $n < 0$). In a moment, we will discuss examples of these spaces.

**Theorem 3.7.** *Let $X$ satisfy the conditions eq.(2.1) through eq.(2.7) along with the additional assumptions that $(X, X')$ is an $\ell^2$ dual pair and the condition in eq.(3.6) holds. If $f \in X$ is a non-cyclic vector and $\widetilde{f}_L \equiv 0$ for some $L \in [f]^{\perp} \setminus \{0\}$, then $f \equiv 0$.*

Assuming the condition in eq.(3.6) holds, here is the proof of the theorem: If $\widetilde{f}_L$ is the zero function, that is to say, in our $\ell^2$ pairing notation (equating $L \in X'$ with $g = \sum_n b_n z^n$),

$$\big(\frac{f}{z-\lambda}, g\,\big) \,/\, \big(\,\frac{1}{z-\lambda}, g\,\big) = 0, \quad |\lambda| > 1,$$

then

$$\big(\frac{f}{z-\lambda}, g\,\big) = 0, \quad |\lambda| > 1$$

and a computation with power series shows that for all $|\lambda| > 1$,

$$0 = \big(\frac{f}{z-\lambda}, g\,\big) = -\sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} \sum_{l=k}^{\infty} a_{l-k}\overline{b_l}$$

and so

$$(M_z^k f, g) = \sum_{l=k}^{\infty} a_{l-k}\overline{b_l} = 0, \quad k = 0, 1, 2, \cdots. \tag{3.8}$$

But $g$ annihilates the $B$-invariant subspace generated by $f$ and so

$$0 = (B^k f, g) = \sum_{l=0}^{\infty} a_{l+k}\overline{b_l}, \quad k = 0, 1, 2, \cdots . \tag{3.9}$$

Combining eq.(3.8) and eq.(3.9) we see that the convolution $(a_{-n}) * (\overline{b_n})$ is the zero sequence (where we understand that $a_n, b_n = 0$ when $n < 0$). But if $(a_n)_{n\geqslant 0}$ is not the zero sequence, then, by our assumption on the space $X$, $(b_n)_{n\geqslant 0}$ must be the zero sequence, which we are assuming is not the case. Thus $f \equiv 0$.

*Remark* 3.10. Theorem 3.7 was stated in a slightly different way in [19, p. 134] for the spaces $D_\alpha$, $\alpha \in \mathbb{R}$, but the proof was deficient. This proof corrects this.

As it turns out, the spaces $D_\alpha$ with $\alpha > 0$, and many others as well, satisfy eq.(3.6) (see remark below). For the Hardy and weighted Bergman spaces, a stronger result than Theorem 3.7 is true (see Proposition 3.13 below) and so we focus our efforts here on the Dirichlet-type spaces, or more generally, spaces of analytic functions on $\mathbb{D}$ which are 'smooth up to the boundary'. For the sake of completeness, and to give the reader a flavor of what goes on here, we will show that spaces like $\ell^1_A(w)$ with, for example, $w_n = (1 + n)^s$, $s > 0$, satisfy eq.(3.6). The following argument was kindly communicated to us by Yngve Domar.

To do this, let $w = (w_n)_{n\in\mathbb{Z}}$ be a (two-sided) weight sequence such that

$$w_n = w_{-n}$$

$$w_0 = 1, \quad w_n \geqslant 1$$

$$w_{n+m} \leqslant w_m w_n$$

$$\sum_{n\in\mathbb{Z}} \frac{\log w_n}{1+n^2} < \infty. \tag{3.11}$$

Weights like $w_n = (1 + |n|)^s$, with $s > 0$, satisfy these conditions. Let $\ell^1(w)$ be the space of sequences $x = (x(n))_{n\in\mathbb{Z}}$ with

$$\sum_{n\in\mathbb{Z}} |x(n)| w_n < \infty.$$

The dual space of $\ell^1(w)$ can be identified with $\ell^\infty(w)$, the space of sequences $y = (y(n))_{n\in\mathbb{Z}}$ with

$$\sup_{n\in\mathbb{Z}} \frac{|y(n)|}{w_n} < \infty$$

via the pairing

$$\langle x, y\rangle := \sum_{n\in\mathbb{Z}} x(n) y(-n).$$

One can check that $\ell^1(w)$ is a Banach algebra with respect to the operations of pointwise addition and convolution: For $x_1, x_2 \in \ell^1(w)$,

$$(x_1 + x_2)(n) := x_1(n) + x_2(n)$$

$$(x_1 * x_2)(n) := \sum_{m\in\mathbb{Z}} x_1(n - m) x_2(m).$$

The Banach algebra $\ell^1(w)$ is also *regular*: Given any open arc $I \subset \mathbb{T}$, there is an $x \in \ell^1(w)$ whose Gelfand transform

$$\widehat{x}(\zeta) := \sum_{n \in \mathbb{Z}} x(n)\zeta^n, \quad \zeta \in \mathbb{T}$$

(which is continuous on $\mathbb{T}$ since its coefficients are absolutely summable) has support in $I$. In fact, given $N \in \mathbb{Z}$, we can even choose $x$ to satisfy the extra condition that $x(N) \neq 0$. For weight sequences like $w_n = (1 + |n|)^s$ with $s > 0$, sequences like $(x(n))_{n \in \mathbb{Z}}$, where $x(n)$ is the $n$-th Fourier coefficient of a $C^\infty$ function whose support is contained in $I$, are contained in $\ell^1(w)$. For general weights $w_n$ satisfying the 'non-quasianalyticity condition' in eq.(3.11), the problem is a bit more delicate [7].

The result here is the following: Suppose $x \in \ell^1(w) \setminus \{0\}$, $y \in \ell^\infty(w)$ with $y(n) = 0$ for all $n > 0$, and $\langle x(\cdot - m), y\rangle = 0$, that is,

$$\sum_{n \in \mathbb{Z}} x(n - m)y(-n) = 0$$

for all $m \in \mathbb{Z}$. Then $y = 0$. In other words, $\ell^1_A(w)$ satisfies the condition in eq.(3.6).

To see this, note that since $x$ is non-trivial and $\widehat{x}$ is continuous on $\mathbb{T}$, there is an arc $I \subset \mathbb{T}$ so that $\widehat{x}(\zeta) \neq 0$ for all $\zeta \in I$. Fix $N \geqslant 0$ and choose $v \in \ell^1(w)$ with $\widehat{v}$ supported in $I$ so that $v(N) \neq 0$ (regularity of $\ell^1(w)$). For real $t$, define $u_t \in \ell^1(w)$ by

$$u_t(n) := v(n)e^{int}$$

and observe that for small positive $c$, $\widehat{u_t}$ is again supported in $I$ provided $|t| < c$. The quotient $\widehat{u_t}/\widehat{x}$ is the Gelfand transform of an $r_t \in \ell^1(w)$ with $u_t = r_t * x$. Thus since $\langle x(\cdot - m), y\rangle = 0$ for all $m \in \mathbb{Z}$, then $\langle u_t(\cdot - m), y\rangle = 0$ for all $m \in \mathbb{Z}$, $|t| < c$, or equivalently,

$$\sum_{n \in \mathbb{Z}} y(-n)v(n - m)e^{int}e^{-imt} = 0 \quad m \in \mathbb{Z}, |t| < c.$$

Letting $m = 0$ and noting that $y(-n) = 0$ for all $n < 0$, we conclude

$$\sum_{n=0}^{\infty} y(-n)v(n)e^{int} = 0, \quad |t| < c.$$

But since the sequence $(y(-n)v(n))_{n \geqslant 0}$ is absolutely summable,

$$h(z) := \sum_{n=0}^{\infty} y(-n)v(n)z^n$$

is analytic on $\mathbb{D}$, continuous on $\mathbb{D}^-$, and is zero on an arc of $\mathbb{T}$. Thus $h \equiv 0$ and so $y(-n)v(n) = 0$ for all $n \geqslant 0$. But we are assuming $v(N) \neq 0$ and so $y(-N) = 0$. Since $N$ was arbitrary, $y(n) = 0$ for all $n \leqslant 0$. But we are already assuming $y(n) = 0$ for all $n > 0$ and so $y = 0$.

*Remark* 3.12.
(1) There are some spaces which are not of the form $\ell^1_A(w)$ which satisfy the condition in eq.(3.6), for example, $D_\alpha$ $(\alpha > 0)$ [4].
(2) Recently, Jean Esterle [9, Thm. 4.10] showed there are $\ell^2_A(w)$ spaces with weights $w_n$ increasing to infinity as slowly as desired, containing nontrivial functions whose orbit under all backward and forward shifts fails to span $\ell^2_A(w)$, or in our terminology, using eq.(3.8) and eq.(3.9): A noncyclic vector

> $f$ for the backward shift for which some $L$-prolongation is identically zero (both $f$ and $L$ of course being nontrivial). The crucial feature of this example is that the weights, while monotone, lack finer regularity properties like convexity which are needed in the theorems of [4].

For our three spaces $H^p$, $L^p_a(w)$ with a suitable weight, and $\mathcal{D}_p$, the following stronger compatibility result is true. Note that the Dirichlet-type spaces $D_\alpha$ ($\alpha < 0$) can be included in this list since $L^2_a((1-|z|)^{-1-\alpha}) \cong D_\alpha$ for $\alpha < 0$ (recall eq.(3.4)).

**Proposition 3.13.** *Suppose that $X$ is one of the three spaces mentioned above. If $f \in X$ is a non-cyclic vector and $\widetilde{f}_L$ has an analytic continuation, also denoted by $\widetilde{f}_L$, to an open neighborhood $U_\zeta$ of $\zeta \in \mathbb{T}$, then $\widetilde{f}_L$ agrees with $f$ on $U_\zeta \cap \mathbb{D}$.*

*Proof.* For the Hardy and Bergman spaces, $\widetilde{f}_L$ is of bounded type on $\mathbb{D}_e$ and is a pseudocontinuation of $f$. Thus if $\widetilde{f}_L$ has an analytic continuation $F$ across an arc $I \subset \mathbb{T}$, then $\widetilde{f}_L$ will have two pseudocontinuations across $I$, $f$ and $F$. By the Lusin-Privalov uniqueness theorem $f$ and $F$ must be the same.

For the Dirichlet spaces $\mathcal{D}_p$, the proof is somewhat similar. In [1, Lemma 7.4], the authors show, using very special calculations with the local Dirichlet integral, that

$$L\Big(\frac{1}{z-w}\Big)\Big[\widetilde{f}_L(w) - f(e^{i\theta})\Big] \to 0$$

for almost every $e^{i\theta}$ as $w \to e^{i\theta}$ non-tangentially. If $\widetilde{f}_L$ has finite non-tangential limits on an arc $I \subset \mathbb{T}$, then since

$$L\Big(\frac{1}{z-w}\Big)$$

cannot go to zero as $w \to e^{i\theta}$ (non-tangentially) on a subset of $I$ of positive measure (Lusin-Privalov uniqueness theorem), it must be the case that

$$\widetilde{f}_L(w) \to f(e^{i\theta})$$

as $w \to e^{i\theta}$ non-tangentially for almost every $e^{i\theta} \in I$. Thus $\widetilde{f}_L$ must be a pseudocontinuation of $f$ across $I$. So if $\widetilde{f}_L$ has an analytic continuation across an arc $I \subset \mathbb{T}$, then, by combining the above along with the Lusin-Privalov uniqueness theorem, $\widetilde{f}_L$ must be an analytic continuation of $f$ across $I$. $\square$

To prove the analog of Proposition 3.13 for a wide class of Banach spaces $X$, say even something like $\ell^p_A(w)$, remains an open problem. As pointed out in Remark 3.12 (Esterle's example), Proposition 3.13 does not hold for certain pathological $\ell^2_A(w)$ spaces since there are non-cyclic $f \in \ell^2_A(w) \setminus \{0\}$ and $L \in [f]^\perp \setminus \{0\}$ for which $\widetilde{f}_L \equiv 0$.

## 4. Overconvergence and spectral synthesis

Let $T$ be a bounded linear operator on a Banach space $V$. Given $t \in \mathbb{C}$ and $r \in \mathbb{N}$, we say that $t$ is an eigenvalue of algebraic multiplicity $r$, and the non-zero vector $v \in V$ is a *root vector* of order $r$ corresponding to $t$ if

$$(T - tI)^r v = 0 \ \text{ but } \ (T-tI)^{r-1} v \neq 0.$$

The subspace

$$\bigvee\{(T-tI)^j v : j = 0, 1, \cdots, r-1\}$$

is called a root space of $V$ and one can show it has dimension $r$. An invariant subspace $\mathcal{M}$ of $T$ has the *spectral synthesis property* if $\mathcal{M}$ is the closed linear span of the root spaces it contains, where the root spaces are of the type $\ker(T - tI)^r$ as above. From elementary linear algebra, every finite dimensional $\mathcal{M}$ has the spectral synthesis property. When $\mathcal{M}$ is infinite dimensional, this is no longer the case, though there is a reasonable substitute to spectral synthesis (see below).

For example, if $T$ is the backward shift operator $B$ on our Banach space $X$ of analytic functions on $\mathbb{D}$, observe that for $a \in \mathbb{D}_e$ and $r(a) \in \mathbb{N}$,

$$\ker(aI - B)^{r(a)} = \bigvee \Big\{ \frac{1}{(z-a)^s} : s = 1, \cdots, r(a) \Big\}. \tag{4.1}$$

Note, from eq.(2.10) and eq.(2.11), that if $\mathcal{M} \neq X$ has the spectral synthesis property, there is a sequence $(a_n)_{n \geqslant 1} \subset \mathbb{D}_e$ with no cluster points in $\mathbb{D}_e$ and $r(a_n) \in \mathbb{N}$ so that

$$\mathcal{M} = \bigvee \Big\{ \frac{1}{(z-a_n)^s} : s = 1, \cdots, r(a_n) \Big\}. \tag{4.2}$$

Moreover, if $\lambda \notin (a_n)_{n \geqslant 1}$, then $(z - \lambda)^{-1} \notin \mathcal{M}$.

*Remark* 4.3. To avoid needless technicalities, we will assume the eigenvalues of $B$ do not lie on the unit circle. If they do, what was stated above, and what follows below can be suitably altered.

Since $(H^2, H^2)$ is an $\ell^2$ dual pair, we can use eq.(2.13) to see that $\mathcal{M} = (\varphi H^2)^{\perp}$ does not have the spectral synthesis property whenever $\varphi$ is an inner function with a non-constant singular inner factor. A thorough discussion of the spectral synthesis property can be found in [13]. An interesting fact worth pointing here is the following.

**Proposition 4.4.** *Let $A = (a_n)_{n \geqslant 1}$ be a sequence of distinct points of $\mathbb{D}$ and let $(c_n)_{n \geqslant 1}$ be a sequence of non-zero complex numbers for which*

$$\varlimsup_{n \to \infty} |d_n|^{1/n} < 1,$$

*where $d_n = c_n \|(1 - a_n z)^{-1}\|_X$. If*

$$f := \sum_{n=1}^{\infty} \frac{c_n}{1 - a_n z},$$

*then $f \in X$ and*

$$[f] = \mathcal{M}(A) := \bigvee \big\{ \frac{1}{1 - a_n z} : n = 1, 2, \cdots \big\}. \tag{4.5}$$

*That is to say, the spectral synthesis invariant subspace $\mathcal{M}(A)$ is singly generated.*

To prove this, we begin with a theorem of Beurling [3].

**Lemma 4.6.** *Suppose $(z_j)_{j \geqslant 1}$ is a sequence of distinct points of $\mathbb{D}$ and $V$ is a matrix of Vandermonde type, that is to say, the $j$-th column of $V$ is $1, z_j, z_j^2, z_j^3, \cdots$. If $w = (w_n)_{n \geqslant 1}$ is a column vector of complex numbers such that*

$$\varlimsup_{n \to \infty} |w_n|^{1/n} < 1$$

*and $Vw$ is the zero vector, then $w$ is the zero vector.*

*Proof of Proposition 4.4.* Suppose that $L \in [f]^{\perp}$, that is to say $L(B^N f) = 0$ for all $N = 0, 1, 2, \cdots$. Then, using the identity

$$B^N \frac{1}{1 - a_n z} = a_n^N \frac{1}{1 - a_n z},$$

wee see that

$$0 = L(B^N f) = \sum_{n=1}^{\infty} c_n a_n^N L\big(\frac{1}{1 - a_n z}\big) \text{ for all } N = 0, 1, 2, \cdots.$$

By our hypothesis and Lemma 4.6,

$$L\big(\frac{1}{1 - a_n z}\big) = 0 \text{ for all } n.$$

But this means that $L \in \mathcal{M}(A)^{\perp}$ and so $[f] \supset \mathcal{M}(A)$. The other inclusion is obvious. $\square$

*Remark* 4.7. For a general bounded operator $T$ on a Banach space $V$ for which the eigenvectors (corresponding to distinct eigenvalues) of $T$ span $V$, Nordgren and Rosenthal [15], in an unpublished paper brought to our attention by M. Putinar, prove the existence of a vector $v \in V$ such that

$$V = \bigvee \{T^n v : n = 0, 1, \cdots\} = V.$$

In other words, $T$ has a cyclic vector. We suspect that the same is true when $V$ is spanned by root vectors (corresponding to distinct eigenvectors, which we assume all have one dimensional eigenspaces) and, under certain technical conditions, this is indeed the case.

Our next observation here concerns the backward shift $B$. If $(f_n)_{n \geqslant 1} \subset \mathcal{M}$ is a sequence of finite linear combinations of root vectors of $X$ (which in this case are rational functions whose poles lie in $\mathbb{D}_e$) which converge in norm to $f$, then certainly $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$ (see eq.(2.9)). This next result says that this sequence of rational functions 'overconverges'. This result is found in [21] but we include a proof anyway, both for completeness and since the version stated here is slightly more general. We will also be using some of the techniques of the proof in the next section.

**Theorem 4.8.** *Let $\mathcal{M} \in Lat(B, X)$, $\mathcal{M} \neq X$ have the spectral synthesis property and $(f_n)_{n \geqslant 1}$ be a sequence of finite linear combinations of root vectors in $\mathcal{M}$ with $f_n \to f$ in norm. Then there is a meromorphic function $S_f$ on $\mathbb{D}_e$ such that $f_n \to S_f$ uniformly on compact subsets of $\mathbb{D}_e \setminus (a_n)_{n \geqslant 1}$.*

*Proof.* Let $\lambda \in \mathbb{D}_e \setminus (a_n)_{n \geqslant 1}$ and $L \in \mathcal{M}^{\perp} \setminus \{0\}$ with

$$L\big(\frac{1}{z - \lambda}\big) \neq 0.$$

This is possible since $(z - \lambda)^{-1} \notin \mathcal{M}$. Since $L((z - \lambda)^{-1})$ is analytic, there is an open neighborhood $U_\lambda$ of $\lambda$ so that

$$L\big(\frac{1}{z - w}\big) \neq 0 \text{ for all } w \in U_\lambda.$$

Fix $w \in U_\lambda$, $a = a_n$, and $k \leqslant r(a)$ (see eq.(4.2)). Select constants $c_j = c_j(a, w)$ (independent of $z$) so that

$$\frac{1}{z-w} + \frac{c_1}{z-a} + \cdots + \frac{c_k}{(z-a)^k} = \frac{(w-a)^k}{(z-w)(z-a)^k}.$$

Thus, since

$$L(\frac{1}{(z-a)^j}) = 0, \ \ j = 1, \cdots, k,$$

then

$$L(\frac{1}{z-w}) = (w-a)^k L(\frac{1}{(z-w)(z-a)^k})$$

and so

$$L(\frac{1}{(z-w)(z-a)^k}) = \frac{1}{(w-a)^k} L(\frac{1}{z-w}).$$

It follows now, if $f$ is a finite linear combination of root vectors in $\mathcal{M}$, that for $w \in U_\lambda$,

$$L(\frac{f}{z-w}) = f(w) L(\frac{1}{z-w}). \tag{4.9}$$

Thus for any $w \in U_\lambda$,

$$|f(w)| \leqslant \frac{\|L\| \|M_{1/(z-w)}\| \|f\|}{|L((z-w)^{-1})|} \leqslant C_\lambda \|f\|.$$

The result now follows. $\square$

*Remark* 4.10. (1) We actually get a bit more here. Indeed if $f_n \to f$ in norm as above, then for any $L \in \mathcal{M}^\perp \setminus \{0\}$ (note that $L \in [f_n]^\perp \setminus \{0\}$) eq.(4.9) shows that

$$f_n = \widetilde{f_n}_L$$

on $\mathbb{D}_e$. Using the norm convergence of $f_n \to f$ and the definitions of $\widetilde{f_n}_L$ and $\widetilde{f}_L$ (from eq.(2.19)), one can show that

$$\widetilde{f}_L = \lim_{n\to\infty} \widetilde{f_n}_L$$

pointwise on $\mathbb{D}_e$ (minus appropriate poles). Thus the limit function $S_f$ in Theorem 4.8 is $\tilde{f}_L$, the $L$-prolongation of $f$.

(2) Since $S_f = \widetilde{f}_L$ for any $L \in \mathcal{M}^\perp \setminus \{0\}$, then $\widetilde{f}_L$ is independent of the $L \in \mathcal{M}^\perp \setminus \{0\}$ and so, using the results mentioned in § 2 (see also [1, Prop. 2.6]),

$$\sigma(B|\mathcal{M}) \cap \mathbb{D} = (1/a_n)_{n\geqslant 1}.$$

(3) As a small technical matter, notice that $f_n = \widetilde{f_n}_L$ for all $L \in \mathcal{M}^\perp \setminus \{0\}$. It is true that $f_n = \widetilde{f_n}_L$ for all $L \in [f_n]^\perp \setminus \{0\}$. To see this, one can use a variant of the proof of Proposition 4.4 to show that if $\{b_1, \cdots b_k\} \subset (a_n)_{n\geqslant 1}$, and

$$h = \sum_{j=1}^{k} \frac{c_j}{(z-b_j)^{s(j)}},$$

then

$$[h] = \bigvee \{\frac{1}{(z-b_j)^s} : 1 \leqslant j \leqslant k, 1 \leqslant s \leqslant s(j)\}.$$

Thus, by replacing $\mathcal{M}$ with $[f_n]$ in eq.(4.9), we see that $f_n = \widetilde{f_n}_L$ for all $L \in [f_n]^{\perp} \setminus \{0\}$. When $X$ satisfies the condition in eq.(3.2), this follows directly from Theorem 3.1.

(4) We know, for $f \in \mathcal{M}$, that $\widetilde{f}_L$ is independent of $L \in \mathcal{M}^{\perp} \setminus \{0\}$ and that $\widetilde{f_n}_L$ is independent of $L \in [f_n]^{\perp} \setminus \{0\}$. One might be tempted to say that $\widetilde{f}_L$ is independent of $L \in [f]^{\perp} \setminus \{0\}$. However, this is not the case. By a construction in [11], there are two sequences $A_1, A_2$ contained in $\mathbb{D}$ so that

$$\mathcal{I}(A_j) := \{f \in L^2_a : f|A_j = 0\} \neq \{0\}$$

and the $M_z$-invariant subspace $\mathcal{I} := \mathcal{I}(A_1) \bigvee \mathcal{I}(A_2)$ has index equal to two, that is to say $\dim(\mathcal{I}/M_z\mathcal{I}) = 2$. It is well-known [17, Thm. 4.5] that the $B$-invariant subspace (of the Dirichlet space $\mathcal{D}$) $\mathcal{I}^{\perp}$ satisfies $\sigma(B|\mathcal{I}^{\perp}) = \mathbb{D}^{-}$. Using the $\ell^2$ dual pairing between $L^2_a$ and $\mathcal{D}$, one can show, using eq.(2.13), that

$$\mathcal{I}(A_1)^{\perp} = \bigvee \Big\{ \frac{1}{1 - \overline{a} z} : a \in A_1 \Big\}$$

and thus has the spectral synthesis property. By using [1, Prop. 2.6], there is an $f \in \mathcal{I}^{\perp}$ and annihilating $L_1, L_2$ of $\mathcal{I}^{\perp}$ (and hence $L_1, L_2 \in [f]^{\perp} \setminus \{0\}$) so that $\widetilde{f}_{L_1} \neq \widetilde{f}_{L_2}$. Notice that $f \in \mathcal{I}^{\perp} \subset \mathcal{I}(A_1)^{\perp}$.

(5) If $X'$ has the property that every (non-zero) $M_z$-invariant subspace has index equal to one (for example $H^2$, $\mathcal{D}$, and many others [1, Cor. 5.10]), then $\sigma(B|[f]) \cap \mathbb{D}$, where $f$ is any non-cyclic vector, is a countable subset of $\mathbb{D}$ with no accumulation points in $\mathbb{D}$ and so [1, Prop. 2.6] $\widetilde{f}_L$ is independent of $L \in [f]^{\perp} \setminus \{0\}$.

(6) For the above Borel series $f$, in eq.(4.5), that generates $\mathcal{M}(A)$ (which we shall assume is not all of $X$), there are two functions defined on $\mathbb{D}_e$ that, in some sense, can be associated with $f$. There is the function

$$\sum_{n=0}^{\infty} \frac{c_n}{1 - a_n w}$$

which is just the Borel series defined on $\mathbb{D}_e \setminus (1/a_n)_{n \geqslant 0}$ and then there is an $L$-prolongation of $f$ for any $L \in [f]^{\perp} \setminus \{0\}$. By Theorem 4.8, these are the same.

(7) In a way, the function $S_f$ can be thought of as a 'continuation' of $f$ across $\mathbb{T}$, though not necessarily an analytic continuation. At the end of the paper [21], the author posed the following compatibility questions about this continuation $S_f$: (i) If $S_f \equiv 0$, must $f \equiv 0$? (ii) If $f$ has an analytic continuation across $e^{i\theta}$, must this analytic continuation be equal to $S_f$ near $e^{i\theta}$? (iii) If $S_f$ has an analytic continuation across $e^{i\theta}$, must $S_f$ equal $f$ near $e^{i\theta}$? The identity $\widetilde{f}_L = S_f$, along with Theorem 3.1, Theorem 3.7, and Proposition 3.13, give answers to these questions.

(8) When the space $X$ satisfies certain mild technical conditions, which spaces like $H^2, L^p_a(w), D_\alpha, \mathcal{D}_p$ do, then the sequence $(f_n)_{n \geqslant 1}$ converges uniformly on compact subsets of $\widehat{\mathbb{C}} \setminus (a_n)^{-}_{n \geqslant 1}$ (see [21] or Remark 5.11 below).

## 5. Overconvergence and approximate spectral synthesis

As we have seen earlier, there are invariant subspaces which do not have the spectral synthesis property. The following reasonable substitution was proposed by

Nikolskii [14] (see also [23] for a nice exposition): If $T$ is a bounded operator on a Banach space $V$ and $(\mathcal{M}_j)_{j\geqslant 1}$ is a sequence of $T$-invariant subspaces of $V$, define

$$\mathcal{M} = \varliminf_{j\to\infty} \mathcal{M}_j := \{v \in V : \lim_{j\to\infty} \operatorname{dist}(v, \mathcal{M}_j) = 0\}. \tag{5.1}$$

The above 'liminf' space is closed and $T$-invariant. It is important to note here that $\mathcal{M}_j$ need not lie in $\mathcal{M}$. We say a $T$-invariant subspace $\mathcal{M}$ has the *approximate spectral synthesis* property if $\mathcal{M}$ can be written as in eq.(5.1) for a sequence $(\mathcal{M}_j)_{j\geqslant 1}$ of $T$-invariant subspaces with $\dim(\mathcal{M}_j) < \infty$. In this case, $\mathcal{M}_j$ is a linear span of root spaces (and thus satisfies the approximate spectral synthesis property), and hence the name 'approximate spectral synthesis'. Note that if $\mathcal{M}_j \subset \mathcal{M}_{j+1}$ for all $j \geqslant 1$, then $\mathcal{M}$ has the spectral synthesis property.

When $T$ is the backward shift operator $B$ on $X$, two important questions are: (i) When is this 'liminf space' all of $X$?; (ii) Can every $B$-invariant subspace of $X$ be written as a 'liminf space'? For the first question, there are theorems of [10, 25, 26] and others that give specific conditions for this to happen. For the second question, the answer is yes when $X = H^2$ (the Hardy space) [25] and when $X = \mathcal{D}$ (the Dirichlet space) [23]. For other spaces, question (ii) remains unanswered.

In the previous section, we showed that if $(f_n)_{n\geqslant 1}$ is a norm-convergent sequence of root vectors in some non-trivial $B$-invariant subspace satisfying the spectral synthesis property, then this sequence overconverges on $\mathbb{D}_e$ (minus the appropriate poles). In a moment, we will show an analogous result for $B$-invariant subspaces satisfying the approximate spectral synthesis property.

Our set up is as follows: For each $n = 1, 2, \cdots$, choose a sequence

$$E_n := \{z_{n,1}, \cdots, z_{n,N(n)}\}$$

of points of $\mathbb{D}$ (multiplicities are allowed) to create the tableau $\mathcal{S}$

$$\begin{gathered} z_{1,1}, z_{1,2}, \cdots, z_{1,N(1)} \\ z_{2,1}, z_{2,2}, \cdots, z_{2,N(2)} \\ \vdots \end{gathered}$$

For each $n$, create the finite dimensional subspace $B$-invariant subspace (see eq.(4.1))

$$R_n := \bigvee \Big\{ \frac{1}{(1-\overline{z_{n,j}}z)^s} : 1 \leqslant j \leqslant N(n), 1 \leqslant s \leqslant \operatorname{mult}(z_{n,j}) \Big\},$$

where $\operatorname{mult}(z_{n,j})$ is the number of times $z_{n,j}$ appears in $E_n$, the $n$-th row of the tableau. One can now form the 'liminf space'

$$R(\mathcal{S}) := \varliminf R_n$$

associated with this tableau $\mathcal{S}$. When $X = H^2$, there is a condition that determines when $R(\mathcal{S}) \neq H^2$ [25, 26]: If

$$\gamma(E_n) := \sum_{j=1}^{N(n)} (1 - |z_{n,j}|),$$

then

$$R(\mathcal{S}) \neq H^2 \quad \Leftrightarrow \quad \varliminf_{n\to\infty} \gamma(E_n) < \infty. \tag{5.2}$$

There are analogous results for other spaces [13] where the quantity $\gamma(E_n)$ is replaced by another 'capacity' suitable for the particular Banach space $X$. Recall also the results of [23, 25] that say, when our Banach space is either $H^2$ or the Dirichlet space $\mathcal{D}$, that every $B$-invariant subspace can be written as an $R(\mathcal{S})$ for some tableau $\mathcal{S}$.

We say a tableau $\mathcal{S}$ is *ample* if $R(\mathcal{S}) = X$, *deficient* if $R(\mathcal{S}) \neq X$, and *uniformly deficient* if

$$\varliminf_{n\to\infty} \operatorname{dist}\big(\frac{1}{z-\lambda}, R_n\big) > 0$$

for some $\lambda \in \mathbb{D}_e$. These uniformly deficient tableaux will be the focus of our attention. Before stating our overconvergence theorem, we make a few remarks.

*Remark* 5.3. (1) Assuming $(X, X')$ is an $\ell^2$ dual pair, recall from eq.(2.13) that for $g \in X', \lambda \in \mathbb{D}$, and $n = 0, 1, 2, \cdots$,

$$\big(\frac{n! z^n}{(1-\overline{\lambda} z)^{n+1}}, g\big) = \overline{g^{(n)}(\lambda)}.$$

Thus $g$ belongs to $R_n^{\perp}$ if and only if $g$ vanishes on $E_n$ (up to appropriate orders).

(2) With our $\ell^2$ dual pairing, we equate a functional $L$ with an analytic function $g$. With this set up, the $L$-prolongation $\widetilde{f}_L$ of a non-cyclic vector $f$ looks like

$$\widetilde{f}_L(w) = \big(\frac{f}{z-w}, g\ \big) \Big/ -\frac{1}{w}\overline{g(\frac{1}{\overline{w}})}. \tag{5.4}$$

(3) If

$$d_n(\lambda) := \operatorname{dist}\big(\frac{1}{1-\overline{\lambda} z}, R_n\big)$$

(note from eq.(2.13) that $d_n(\lambda) = 0$ if and only if $\lambda \in E_n$) and

$$d(\lambda) := \varliminf_{n\to\infty} d_n(\lambda),$$

then $\mathcal{S}$ is uniformly deficient if and only if $d(\lambda) > 0$ for some $\lambda \in \mathbb{D}$. It is well-known, and easily shown, that the distance $d(x, A)$ from a point $x$ in a metric space to some fixed set $A$ satisfies $|d(x, A) - d(y, A)| \leqslant d(x, y)$. So, for $z, w \in \mathbb{D}$,

$$|d_n(w) - d_n(z)| \leqslant |w - z|$$

and so

$$d(\lambda) > 0 \ \Rightarrow\ d(w) > 0 \ \text{ for all } w \in U_\lambda, \tag{5.5}$$

where $U_\lambda$ is some open neighborhood containing $\lambda$. Therefore, the set

$$Q := \{\lambda \in \mathbb{D} : d(\lambda) > 0\}$$

is an open subset of $\mathbb{D}$ which is non-empty whenever the tableau $\mathcal{S}$ is uniformly deficient.

(4) The liminf subspaces arising from uniformly deficient tableaux need not satisfy the spectral synthesis property. For instance, if $\phi$ is a singular inner function, then the $B$-invariant subspace $(\phi H^2)^{\perp}$ does not satisfy the spectral synthesis property. Indeed $\phi(\lambda) \neq 0$ for any $\lambda \in \mathbb{D}$ and so by eq.(2.13), $(\phi H^2)^{\perp}$ contains no eigenvectors of $B$. However, it arises as a liminf space

from a uniformly deficient tableau. One can see by using Frostman's theorem [12, p. 85] to obtain a sequence $(B_n)_{n\geqslant 1}$ of finite Blaschke products with $B_n \to \phi$ weakly in $H^2$ (equivalently $B_n \to \phi$ pointwise in $\mathbb{D}$). From here, it is routine to prove that

$$\varliminf_{k\to\infty} (B_{n_k} H^2)^{\perp} = (\phi H^2)^{\perp}$$

for any subsequence $(B_{n_k})_{k\geqslant 1}$. Since, for fixed $\lambda \in \mathbb{D}$

$$\frac{1}{1-\overline{\lambda}z} \notin (\phi H^2)^{\perp},$$

then

$$\operatorname{dist}\Big(\frac{1}{1-\overline{\lambda}z}, (B_{n_k} H^2)^{\perp}\Big) \geqslant t > 0$$

for some subsequence. Thus the tableau

$$E_k = B_{n_k}^{-1}(\{0\}), \quad k = 1, 2, \cdots$$

is uniformly deficient.

Our overconvergence will take place on the set $W^{-1} = \{1/w : w \in W\}$ (see Theorem 5.9), where

$$W = \mathbb{D} \setminus \bigcap_{n=1}^{\infty} S_n,$$

$$S_n := \Big(\bigcup_{k=n}^{\infty} E_k\Big)^{-}.$$

However, unlike Theorem 4.8, where the overconvergence took place on the set $\mathbb{D}_e \setminus (a_n)_{n\geqslant 1}$ and the removed sequence does not have any cluster points in $\mathbb{D}_e$, the overconvergence will take place on a much smaller set. In fact, without the hypothesis of uniformly deficient, the overconvergence need not take place at all. For example, let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ be a Blaschke sequence, that is

$$\sum_{n=1}^{\infty} (1-|\lambda_n|) < \infty,$$

and $(a_n)_{n\geqslant 1}$ be a countable dense subset of $\mathbb{D}$. Letting

$$E_{2n+1} := \{\lambda_1, \cdots, \lambda_n\}, \quad E_{2n} := \{a_1, \cdots, a_n\}, \tag{5.6}$$

we see that

$$\gamma(E_{2n+1}) = \sum_{j=1}^{n} (1-|\lambda_j|)$$

is uniformly bounded and so by eq.(5.2), $R(\mathcal{S}) \neq H^2$. Note that $R(\mathcal{S}) \neq \{0\}$ since $(1-\overline{\lambda_j}z)^{-1} \in R(\mathcal{S})$ for all $j$. Indeed choose a subsequence of $(a_n)_{n\geqslant 1}$ converging to $\lambda_j$ and use that fact that the sets $E_{2n+1}$ and $E_{2n}$ are increasing.

Before proceeding, we also mention that overconvergence in the liminf setting will not come as smoothly as it did in the spectral synthesis case (Theorem 4.8). In that case, $f$ was the norm limit of the rational functions $f_n$ and for any chosen $L \in \mathcal{M}^{\perp} \setminus \{0\}$, one saw that $\widetilde{f}_L$ was the pointwise limit of the functions $\widetilde{f_n}_L$. What made that work was the fact that $f_n \in \mathcal{M}$ and so whenever $L \in \mathcal{M}^{\perp} \setminus \{0\}$, then $L \in [f_n]^{\perp}$, making $\widetilde{f_n}_L$ a bone fide $L$-prolongation of $f_n$. When $R(\mathcal{S}) = \varliminf R_n$

and $f$ is the norm limit of $f_n \in R_n$, then $f_n$ may not belong to $R(\mathcal{S})$ and so $L \in R(\mathcal{S})^{\perp} \setminus \{0\}$ may not annihilate $[f_n]$, making the expression

$$L(\frac{f_n}{z-\lambda}) / L(\frac{1}{z-\lambda})$$

a meromorphic function on $\mathbb{D}_e$ but not an $L$-prolongation of $f_n$. The rub here is to judiciously choose $L_n \in [f_n]^{\perp} \setminus \{0\}$ converging weak-* to $L \in R^{\perp} \setminus \{0\}$.

Using the definitions of the sets $Q$, $W$, and the function $d(\lambda)$, one can show that

$$Q \subset W.$$

Indeed, if $d(\lambda) > 0$, then $|d_k(\lambda)| \geqslant t > 0$ for all $k \geqslant n$. This certainly means that

$$\lambda \notin \bigcup_{k \geqslant n} E_k.$$

If $\lambda_{k_j} \in E_{k_j}$ converges to $\lambda$, then

$$\frac{1}{1-\overline{\lambda_{k_j}} z} \to \frac{1}{1-\overline{\lambda} z}$$

in $X$ norm (since by eq.(2.6), $M_{(1-az)^{-1}} = (I - aM_z)^{-1}$ is an operator-valued analytic function on $\mathbb{D}$). But this says that $d(\lambda) = \lim_{j\to\infty} d_{k_j}(\lambda) = 0$ which is a contradiction. Thus $\lambda \notin S_n$ for some $n$ and so $\lambda \in W$.

**Lemma 5.7.** *If $Q$ is non-empty, i.e., $\mathcal{S}$ is uniformly deficient, and $W$ is connected, then $Q = W$.*

*Proof.* Suppose $b \in Q$, that is $d(b) > 0$. We need to show that

$$d(c) = \varliminf_{n\to\infty} d_n(c) > 0$$

for each $c \in W$. By the Hahn-Banach theorem, there is, for each (large) $n$, some $g_n \in R_n^{\perp}$ (equivalently, by eq.(2.13), $g_n$ vanishing on $E_n$ up to appropriate orders), $\|g_n\| = 1$, and $g_n(b) = d_n(b)$. The hypothesis of the theorem say that for large enough $n$, $|g_n(b)| \geqslant t' > 0$.

Assume the conclusion of the theorem is false, in that there is a subsequence $n_1, n_2, n_3, \cdots$ so that

$$d_{n_k}(c) \to 0.$$

Join the points $b$ and $c$ with a smooth arc in $W$ (note that $W$ is connected) and let $V$ be an open neighborhood containing that arc and with $V^- \subset W$. Let $V'$ be an open set with $V^- \subset V' \subset V'^- \subset W$. Since $\|g_{n_k}\| = 1$, $g_{n_k}$ is uniformly bounded on compact subsets of $\mathbb{D}$ (Since $(X, X')$ is an $\ell^2$ dual pair, then $X'$ also satisfies eq.(2.1) through eq.(2.7) [1, Prop. 5.2]) and so there is a subsequence (still denoted by $g_{n_k}$) converging uniformly on compact subsets of $V'$ to an analytic function $g$ on $V$. Since $|g_{n_k}(b)| \geqslant t' > 0$, $g$ is not identically zero on $V'$.

Observe that the number of zeros of $g_{n_k}$ in $V$ is uniformly bounded for all $k$. Indeed, since $g$ is not identically zero, there is a Jordan curve $C \subset V'$ surrounding $V$ on which $g$ is non-vanishing. So, for large $k$, $g_{n_k}$ is also non-vanishing on $C$ and its change in argument as one goes around $C$ is equal to that of $g$. Thus for all large $k$, $g_{n_k}$ has the same number of zeros, say $r$, in $V$.

Let $B_{n_k}$ denote the Blaschke product formed from these $r$ zeros of $g_{n_k}$ in $V$ and let

$$G_{n_k} = \frac{g_{n_k}}{B_{n_k}}.$$

One can check that $G_{n_k} \in X'$ and is uniformly bounded in norm. To see this, note again that $X'$ satisfies eq.(2.1) through eq.(2.7) and so, in particular, the spectrum of $B$ on $X'$ is $\mathbb{D}^-$. Thus from eq.(2.15),

$$\frac{f}{z-\lambda} = (I - \lambda B)^{-1} B f$$

whenever $f \in X'$ with $f(\lambda) = 0$. Since $\|(I-\lambda B)^{-1}B\|$ is continuous on $\mathbb{D}$, one can argue, using the fact that for each $k$, $B_{n_k}$ has only $r$ zeros on $V$, that $g_{n_k}/B_{n_k}$ is uniformly bounded in norm.

So, by eq.(2.9), $G_{n_k}$ is uniformly bounded in $V'$ and thus has a subsequence (also denoted by $G_{n_k}$) converging uniformly on $V$ to $G$ which, since $|G_{n_k}(b)|$ is bounded away from zero, is not identically zero. By Hurwitz's theorem, $G$ vanishes nowhere in $V$ since all the $G_{n_k}$ are zero free. Hence $G(c) \neq 0$.

Since $G_{n_k}$ vanishes on $E_{n_k}$ (up to appropriate orders), $G_{n_k}$ belongs to $R_{n_k}^{\perp}$. For any $f \in R_{n_k}$,

$$|G_{n_k}(c)| = \big|\big(\frac{1}{1-\bar{c}z} + f, G_{n_k}\big)\big| \leqslant \big\|\frac{1}{1-\bar{c}z} + f\big\|_X \|G_{n_k}\|_{X'}.$$

Now use that fact that $G_{n_k}$ is norm-bounded and the definition of $d_{n_k}(c)$ to conclude

$$|G_{n_k}(c)| \leqslant C d_{n_k}(c) \to 0 \;\text{ as }\; n_k \to \infty.$$

Thus $G(c) = 0$ which is a contradiction. □

*Remark* 5.8. If $W$ is disconnected, the above argument can be adjusted to show that if $Q$ meets any component of $W$, it contains that whole component.

Our overconvergence theorem is the following.

**Theorem 5.9.** *Let $X$ be a Banach space satisfying the conditions eq.(2.1) through eq.(2.7) and such that $(X, X')$ is an $\ell^2$ dual pair. For a uniformly deficient tableau $\mathcal{S}$ such that $W$ is connected, and $f \in R(\mathcal{S})$ with*

$$f = \lim_{n\to\infty} f_n, \quad f_n \in R_n,$$

*there is an analytic function $S_f$ on $W^{-1} := \{1/w : w \in W\}$ such that $f_n \to S_f$ uniformly on compact subsets of $W^{-1}$.*

*Proof.* Let $b \in W$ (which equals the set $Q$) and note that $d(b) > 0$. By eq.(5.5), $d_n(\lambda) \geqslant t' > 0$ on an open neighborhood $U_b$ of $b$ with $U_b \subset W$ for all large enough $n$.

By the Hahn-Banach theorem, there is, for each (large) $n$, some $g_n \in R_n^{\perp}$ (equivalently $g_n$ vanishing on $E_n$ up to appropriate orders), $\|g_n\| = 1$, and $g_n(b) = d_n(b)$. In fact,

$$\varliminf_{n\to\infty} |g_n(\lambda)| \geqslant t'$$

for all $\lambda \in U_b$. Proceed as in the proof of Theorem 4.8 (using the $\ell^2$ dual pairing and equating $L$ with $g$, see eq.(5.4)) to see that for all $\lambda \in U_b$,

$$|f_n(\frac{1}{\lambda})| \leqslant \frac{\|g_n\| \|M_{(1-\bar{\lambda}z)^{-1}}\| \|f_n\|}{|g_n(\lambda)|} \leqslant C_b \|f_n\|.$$

The result now follows. □

If

$$f = \lim_{n\to\infty} f_n, \quad f_n \in R_n,$$

then $f_n \to S_f$ uniformly on compact subsets of $W^{-1}$ when $W^{-1}$ is a connected subset of $\mathbb{D}_e$.

**Corollary 5.10.** *Let $X$ satisfy the conditions of Theorem 5.9. For $f \in R(\mathcal{S})$, $S_f = \widetilde{f}_L$ for some $L \in R(\mathcal{S})^\perp \setminus \{0\}$.*

*Proof.* Since $W^{-1}$ is connected, it suffices to show that $S_f = \widetilde{f}_L$ for some $L \in R(\mathcal{S})^\perp \setminus \{0\}$ on some open set $U \subset W^{-1}$. For $b \in W$, let $(g_n)_{n\geqslant 1} \subset X'$ be as in the proof of Lemma 5.7. By Remark 4.10,

$$f_n = \widetilde{(f_n)}_{L_n}.$$

Here we equate $g_n$ with $L_n$. Since $g_n$ is norm-bounded, $g_{n_k} \to g$ weak-$^*$ for some subsequence and in fact, $g \in R(\mathcal{S})^\perp$. Indeed for any

$$h = \lim_{n\to\infty} h_n, \quad h_n \in R_n,$$

$$|(h, g)| = \lim_{n_k\to\infty} |(h - h_{n_k}, g_{n_k})| \leqslant \overline{\lim_{n_k\to\infty}} \|h - h_{n_k}\|_X \|g_{n_k}\|_{X'} = 0.$$

Furthermore, also by weak-$^*$ convergence, $g_{n_k} \to g$ pointwise on $\mathbb{D}$ (see eq.(2.13)) and since $|g_{n_k}(b)|$ is bounded away from zero, $g \not\equiv 0$. Using eq.(5.4), it is easy to show that

$$\widetilde{(f_{n_k})}_{L_{n_k}} \to \widetilde{f}_L$$

pointwise on $U_b^{-1}$. Again, we are equating $L$ with $g$. By Theorem 5.9,

$$\widetilde{(f_{n_k})}_{L_{n_k}} = f_{n_k} \to S_f$$

pointwise on $U_b^{-1}$ and so $\widetilde{f}_L = S_f$ on $U_b^{-1}$. □

*Remark* 5.11. (1) When $W$ is disconnected, one can make the obvious changes to the above results to show that given a connected component $W'$ of $W$, there is an $L \in R(\mathcal{S})^\perp \setminus \{0\}$ so that $f_n \to \widetilde{f}_L$ uniformly on compact subsets of $W'^{-1}$.

(2) For general deficient (but not necessarily uniformly deficient) tableaux, there is a weaker result [19, Thm. 8.7.6]: If $f = \lim_n f_n, f_n \in R_n$, and $w \in W$ (note that $W$ might be the empty set), then there is an open neighborhood $U_w$ of $w$, an $L \in R(\mathcal{S})^\perp \setminus \{0\}$, and a subsequence $(f_{n_k})_{k\geqslant 1}$ converging uniformly compact subsets of $U_w^{-1}$ to $\widetilde{f}_L$.

(3) A theorem of Beurling (see [6]) says that if $\mathcal{F}$ is a family of analytic functions on a domain $H$ and there exists a function $\rho(z)$ on $H$ such that $|f(z)| \leqslant \rho(z)$ for all $f \in \mathcal{F}$, where for some $r > 1$,

$$\int_H \left[\log^+ \log^+ \rho(z)\right]^r dA(z) < \infty,$$

then $\mathcal{F}$ is a normal family on $H$. One can show that (under certain technical conditions) if

$$U = \widehat{\mathbb{C}} \setminus \bigcap_{n=1}^{\infty} S_n,$$

and $f_n \to f$ in norm (as in either Theorem 4.8 or Theorem 5.9), then $f_n$ converges uniformly on compact subsets of $U$. This means that if $\bigcap_n S_n$ does not meet $\mathbb{T}$, then functions from $\mathcal{M}$ (in Theorem 4.8) or $R$ (in Theorem 5.9) have analytic continuations across parts of the circle. What are these 'certain technical conditions'? We just need to check, for all rational $f \in \mathcal{M}$ (or $R$), that

$$|f(z)| \leqslant \rho(z), \quad |z| < 1$$

$$|\widetilde{f}_L(z)| \leqslant \rho(z), \quad |z| > 1$$

for a $\rho$ defined as above on some open neighborhood of a point of the circle. For the standard spaces, $L^p_a(w)$ (with reasonable weights), $\mathcal{D}_p$, $D_\alpha$, $\ell^p_A(w)$ (with reasonable weights), one can often take $\rho$ to be something like

$$\rho(z) = \frac{1}{|1 - |z||^s}.$$

Such $\rho$ easily satisfy the hypothesis of Beurling's theorem. See [21] where this was done for $\mathcal{M}$ as in Theorem 4.8.

## 6. A final Comment

The example of Esterle in Remark 3.12 of an $f \in \ell^2_A(w)\setminus\{0\}$ with $\widetilde{f}_L \equiv 0$ for some $L \in [f]^\perp \setminus \{0\}$ inspires several intriguing questions. Assuming that $(X, X')$ is an $\ell^2$ dual pair, we know from eq.(3.8) and eq.(3.9) that $\widetilde{f}_L \equiv 0$ for some $L \in [f]^\perp \setminus \{0\}$ if and only if $O_f$, the orbit of $f$, satisfies

$$O_f := \bigvee\{S^n f, B^m f : m, n \geqslant 0\} \neq X,$$

where $S = M_z$. For a set $E \subset X$, let

$$[E]_B := \bigvee\{B^n h : n \geqslant 0, h \in E\}$$

$$[E]_S := \bigvee\{S^n h : n \geqslant 0, h \in E\}.$$

It is easy to show that for any non-trivial $f \in X$,

$$[[f]_B]_S = X.$$

Indeed, $[f]_B$ contains a $g$ with $g(0) \neq 0$ (look at $B^n f$ for a suitable choice of $n$). Then $SBg = g - g(0) \in [[f]_B]_S$ and so $1 \in [[f]_B]_S$. Thus $S^n 1 = z^n \in [[f]_B]_S$ for all $n$. What is more interesting is that

$$[[f]_S]_B = O_f.$$

To see this, first notice that the containment $\supset$ is obvious (since clearly $B^n f$ and $S^n f$ belong to $[[f]_S]_B$). For the other direction, let $g \in [[f]_S]_B$ and $\varepsilon > 0$ be given. Then there is an $h \in [f]_S$ and a polynomial $p$ such that $\|p(B)h - g\| < \varepsilon$. Moreover, there is another polynomial $q$ such that $h = q(S)f + k$, where $\|p(B)k\| < \varepsilon$. Hence

$$\|p(B)(q(S)f + k) - g\| < \varepsilon,$$

which implies that

$$\|p(B)q(S)f - g\| < \varepsilon.$$

But $p(B)q(S)f \in O_f$, since every monomial $B^m S^n$ reduces to either $B^{m-n}$ or to $S^{n-m}$ (because $BS = I$).

So, $\widetilde{f}_L \equiv 0$ for some $L \in [f]_B^\perp$ if and only if

$$[f]_B \subset [[f]_S]_B \neq X.$$

Of course, for many spaces this never happens. But when it does, the question is the following: If $\widetilde{f}_L \equiv 0$ for some $L \in [f]_B^\perp \setminus \{0\}$, does $\widetilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$?

If the answer to the above question is yes, then $[f]_B^\perp \subset O_f^\perp$ and so

$$[f]_B = [[f]_S]_B.$$

Conversely, if $[f]_B = [[f]_S]_B$, then $[f]_S \subset [f]_B$ and so

$$\frac{f}{z-\lambda} \in [f]_B, \quad \text{for all } |\lambda| > 1.$$

Hence $\widetilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$.

It seems rather odd at this point that either a 'yes' or 'no' answer to our question gives us interesting information. If, in fact, there is an $f \in X$ with $\widetilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$, then this $f$ has the very peculiar property (which certainly does not hold with our 'usual' Hardy, Bergman, Dirichlet spaces) that $O_f = [f]_B$, or in other words, the vectors $S^n f$, $n \geqslant 0$, do not add anything new to the closed linear span of the vectors $B^n f$, $n \geqslant 0$. Furthermore, by [1, Prop. 2.6], $\sigma(B|[f]) \subset \mathbb{T}$. In fact, using eq.(2.18) and Theorem 3.1, $\sigma(B|[f]) = \mathbb{T}$. If, on the other hand, $\widetilde{f}_{L_1} \equiv 0$ but $\widetilde{f}_{L_2} \not\equiv 0$, for different $L_1, L_2$, then, again by [1, Prop. 2.6], $\sigma(B|[f]) = \mathbb{D}^-$. But from here, it follows [17, Thm. 4.5], that the $S$-invariant subspace $[f]_B^\perp$ has index greater than one. This would be an example of a Hilbert space of analytic functions which is only 'slightly' bigger than $H^2$ (remember that one can choose the weights in Esterle's example to increase to infinity as slowly as desired) for which there is an $S$-invariant subspace with index two. For spaces which are 'much bigger' than $H^2$, for example the Bergman space, this is fact is well known.

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