

TOEPLITZ OPERATORS IN SEGAL–BARGMANN SPACES OF VECTOR-VALUED FUNCTIONS

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Abstract

We discuss new results concerning unbounded Toeplitz operators defined in Segal–Bargmann spaces of (vector-valued) functions, i.e. the space of all entire functions which are square summable with respect to the Gaussian measure in $\mathbb{C}^n$. The problem of finding adjoints of analytic Toeplitz operators is solved in some cases. Closedness of the range of analytic Toeplitz operators is studied. We indicate an example of an entire function inducing a Toeplitz operator, for which the space of polynomials is not a core though it is contained in its domain.

1. Introduction.

We begin by introducing basic definitions and notations. Let $\mathcal{H}$ be a complex separable Hilbert space. Consider $L^2(\mu) \otimes \mathcal{H}$, the Hilbert space of all complex Borel functions taking values in $\mathcal{H}$ which are square-integrable on $\mathbb{C}^n$ with respect to the measure μ given by $d\mu(z) = \pi^{-n} e^{-z \cdot \bar{z}} dV(z)$, where V is the Lebesgue measure in $\mathbb{C}^n$ and $z \cdot \bar{z} = |z_1|^2 + \dots + |z_n|^2$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. The inner product in $L^2(\mu) \otimes \mathcal{H}$ is given by

$$\langle f, g \rangle_{(\mathcal{H})} = \int_{\mathbb{C}^n} \langle f(\zeta), g(\zeta) \rangle_{\mathcal{H}} d\mu(\zeta), \quad f, g \in L^2(\mu) \otimes \mathcal{H},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ stands for the inner product in $\mathcal{H}$. The norm induced by the above defined inner product is denoted by $\| \cdot \|_{(\mathcal{H})}$ and in case $\mathcal{H} = \mathbb{C}^N$ by $\| \cdot \|_{(N)}$. The Segal–Bargmann space $\mathcal{B}_n \otimes \mathcal{H}$ (abbreviated $\mathcal{B} \otimes \mathcal{H}$) is a closed subspace of $L^2(\mu) \otimes \mathcal{H}$ composed of all entire functions belonging to $L^2(\mu) \otimes \mathcal{H}$. By $P \otimes I_{\mathcal{H}}$ we mean the orthogonal projection of $L^2(\mu) \otimes \mathcal{H}$ onto $\mathcal{B} \otimes \mathcal{H}$. We will use the following identifications: $L^2(\mu) = L^2(\mu) \otimes \mathbb{C}$, $\mathcal{B} = \mathcal{B} \otimes \mathbb{C}$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{(1)}$, $\| \cdot \| = \| \cdot \|_{(1)}$ and $P = P \otimes I_{\mathbb{C}}$. Observe that $\mathcal{B} \otimes \mathbb{C}^N$ can be identified with $\mathcal{B} \oplus \dots \oplus \mathcal{B}$ (N -times), and following this we have $\mathcal{B} \otimes \mathbb{C}^N = \{(f_1, \dots, f_N) : f_1, \dots, f_N \in \mathcal{B}\}$ and $\|(f_1, \dots, f_N)\|_{(N)}^2 = \int_{\mathbb{C}^n} (|f_1|^2 + \dots + |f_N|^2) d\mu$.

Given $f : \mathbb{C}^n \rightarrow \mathbb{C}$ and $h \in \mathcal{H}$ we define $(f \otimes h)(z) := f(z)h$, $z \in \mathbb{C}^n$. Put $e_a(z) := e^{z \cdot \bar{a}}$, where $z \cdot a := \sum_{k=1}^n z_k a_k$ and $\bar{a} := (\bar{a}_1, \dots, \bar{a}_n)$ for $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Denote by $\mathcal{E}$ ($= \mathcal{E}_n$) the linear span of the set of functions $\{e_a : a \in \mathbb{C}^n\}$. It can be checked that $\langle f(z), h \rangle_{\mathcal{H}} = \langle f, e_z \otimes h \rangle_{(\mathcal{H})}$ for $f \in \mathcal{B} \otimes \mathcal{H}$, $h \in \mathcal{H}$ and $z \in \mathbb{C}^n$ (cf. [4]), which is referred to as the reproducing property for $\mathcal{B} \otimes \mathcal{H}$. By $\mathcal{P}$ ($= \mathcal{P}_n$) we denote the space of all analytic polynomials in $\mathbb{C}^n$. Both $\mathcal{P}$ and $\mathcal{E}$ are dense subsets of $\mathcal{B}$. The sequence $f_k(z) := z^k / \sqrt{k!}$, $k \in \mathbb{N}^n$, $z \in \mathbb{C}^n$, forms an orthonormal basis for $\mathcal{B}$, where according to the standard multiindex notation $z^k := z_1^{k_1} \dots z_n^{k_n}$, $k! := k_1! \dots k_n!$ and $|k| := k_1 + \dots + k_n$ for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \dots\}$).

A is called a linear operator in a complex Hilbert space $\mathcal{H}$ if its domain $\mathcal{D}(A)$ is a linear subspace of $\mathcal{H}$ and $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ is a linear mapping. As usual, A^* , $\bar{A}$,

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$\mathcal{R}(A)$ stand for the adjoint (provided it exists), the closure of A and the range of A , respectively. We say that a linear subspace $\mathcal{D} \subseteq \mathcal{D}(A)$ is a core for a closable operator A if $(A|_{\mathcal{D}})^- = \overline{A}$. All these notions are also meaningful for linear operators acting between different Hilbert spaces. By $B(\mathcal{H}, \mathcal{K})$ ($\mathcal{H}, \mathcal{K}$ - Hilbert spaces) we denote the space of all bounded linear operators $A : \mathcal{H} \rightarrow \mathcal{K}$, and $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$. The graph norm $\|\cdot\|_A$ is defined by $\|f\|_A^2 = \|f\|_{\mathcal{H}}^2 + \|Af\|_{\mathcal{H}}^2$, $f \in \mathcal{D}(A)$.

Write $\mathcal{P} \otimes \mathcal{H}$ for the space of polynomials (in $\mathbb{C}^n$) taking values in $\mathcal{H}$, i.e. the totality of functions of type $\sum_{j=0}^K p_j \otimes h_j$, where $p_j \in \mathcal{P}_n$, $h_j \in \mathcal{H}$ and $K \in \mathbb{N}$ (in the case where $\mathcal{H} = \mathbb{C}^N$ the elements of this space are identified with N -tuples of polynomials). Spaces $\mathcal{E} \otimes \mathcal{H}$ and $\mathcal{P} \otimes B(\mathcal{H}, \mathcal{K})$ are defined analogously. Observe that $p \in \mathcal{P} \otimes B(\mathbb{C}^M, \mathbb{C}^N)$ can be written as a matrix $[p_{kl}]_{k=1, \dots, N}^{l=1, \dots, M}$ with some polynomials $p_{kl} \in \mathcal{P}$. Given $p \in \mathcal{P} \otimes B(\mathcal{H}, \mathcal{K})$, $p(z) = \sum_{|j| < K} A_j z^j$, $A_j \in B(\mathcal{H}, \mathcal{K})$, $z \in \mathbb{C}^n$, define $p^\# \in \mathcal{P} \otimes B(\mathcal{K}, \mathcal{H})$ via $p^\#(z) = p(\bar{z})^*$, $z \in \mathbb{C}^n$, and a differential operator $(p(D)F)(z) = \sum_{|j| < K} A_j D^j F(z)$, $z \in \mathbb{C}^n$, where $F : \mathbb{C}^n \rightarrow \mathcal{H}$ is an analytic function (here $D^j = \frac{\partial^{|j|}}{\partial z_1^{j_1} \dots \partial z_n^{j_n}}$ for $j = (j_1, \dots, j_n)$). We will regard $p(D)$ as an operator in $\mathcal{B} \otimes \mathcal{H}$ and its domain is defined as $\mathcal{D}(p(D)) = \{F \in \mathcal{B} \otimes \mathcal{H} : p(D)F \in \mathcal{B} \otimes \mathcal{K}\}$.

Let $\varphi : \mathbb{C}^n \rightarrow B(\mathcal{H}, \mathcal{K})$ be an analytic function. Define $\varphi^\#(z) := \varphi(\bar{z})^*$, which amounts to taking adjoints of all coefficients in the series expansion of φ , and $\varphi^*(z) := \varphi(z)^*$ for $z \in \mathbb{C}^n$. A Toeplitz operator¹ with symbol φ is defined by $\mathcal{D}(T_\varphi) = \{f \in \mathcal{B} \otimes \mathcal{H} : \varphi f \in \mathcal{B} \otimes \mathcal{K}\}$ and $T_\varphi f = \varphi f$ for $f \in \mathcal{D}(T_\varphi)$, where $(\varphi f)(z) = \varphi(z)f(z)$, $z \in \mathbb{C}^n$. We need one more operator in $\mathcal{B} \otimes \mathcal{H}$ denoted by Π_{φ^*} , which is defined via

$$\Pi_{\varphi^*} f(z) = \int_{\mathbb{C}^n} \varphi^*(\zeta) f(\zeta) e^{z \cdot \bar{\zeta}} d\mu(\zeta), \quad z \in \mathbb{C}^n,$$

and its domain consists of all $f : \mathbb{C}^n \rightarrow \mathcal{K}$ such that the above integral exists² for all $z \in \mathbb{C}^n$ and the function in z defined by it lies in $\mathcal{B} \otimes \mathcal{H}$. Since $\Pi_{p^*} = p^\#(D)$ for $p \in \mathcal{P} \otimes B(\mathcal{H}, \mathcal{K})$ (cf. [4]), we may regard Π_{φ^*} as a generalized differential operator. It is known that $\Pi_{p^*} = (T_p|_{\mathcal{E} \otimes \mathcal{H}})^* = (T_p|_{\mathcal{P} \otimes \mathcal{H}})^*$ (cf. [4, Corollary 7.3]). The following abstract lemma reflects the connection between T_p and Π_{p^*} .

LEMMA 1.1. *Let A be a closed operator in $\mathcal{H}$ and $\mathcal{V} \subseteq \mathcal{D}(A)$ be a dense subspace of $\mathcal{H}$. Then for $B := (A|_{\mathcal{V}})^*$ the following conditions are equivalent:*

- (i) $A^* = B$;
- (ii) $\ker(I + AB) = \{0\}$;
- (iii) AB is symmetric;
- (iv) AB is non-negative;
- (v) AB is selfadjoint;
- (vi) $\mathcal{V}$ is a core for A .

Interchanging A and B in (i) – (v) gives another condition equivalent to (vi).

PROOF. Observe that $A^* \subseteq B$ and B is closed. Both these facts are in frequent use in the following proof. We begin with (i) $\Leftrightarrow$ (ii). Let $\Gamma(A)$ denote the graph of A , i.e. the set $\{(x, y) : x \in \mathcal{D}(A), Ax = y\}$. Then one can easily show that $\Gamma(B) \ominus \Gamma(A^*)$ consists of all (x, y) such that $x \in \mathcal{D}(B)$, $y = Ax \in \mathcal{D}(A)$ and $x + Ay = 0$, which immediately implies the desired equivalence. Equivalence (i) $\Leftrightarrow$ (vi) is implied by taking adjoints in (i). By standard operator theory (i) implies (iii), (iv) and (v). Since each of (iv) and (v) implies (iii) by definition, it suffices to prove implication (iii) $\Rightarrow$ (i). Observe that AA^* is a selfadjoint operator contained in AB . Hence by

¹What we have defined here is, in fact, an *analytic* Toeplitz operator, cf. [4] for the general case.

²i.e. it is weakly convergent, cf. [4]

(iii) $AA^* = AB$, which implies (ii) and, consequently, (i). So the equivalence of conditions (i)–(vi) is established.

Condition (i) may be rewritten as $A = B^*$. Thus the last statement of the assertion follows.

The *adjointness hypothesis* consists in equality $T_p^* = \Pi_{p^*}$ for $p \in \mathcal{P} \otimes B(\mathcal{H})$, which was proved in some cases, e.g. $p \in \mathcal{P}_n$, cf. [10], or $p \in \mathcal{P}_1 \otimes B(\mathcal{H})$ and its leading coefficient is surjective, cf. [4]. In the present paper we present new results concerning the hypothesis.

Toeplitz operators of type T_p as above, and their adjoints, play an important role in extending known results on partial differential operators and convolution operators in the space of entire functions to the context of Segal–Bargmann spaces. Suppose for simplicity we consider one-variable Segal–Bargmann space $\mathcal{B}$. In the classical theory of linear differential equations of infinite order, with constant coefficients (one variable), one starts with a “symbol function” φ , which is entire and of exponential type, and the thematic problem is to show that every entire function u which satisfies the equation $\varphi(D)u = 0$ is, in the sense of uniform approximation on compact sets, the limit of a sequence each term of which is a finite linear combination of “monomial exponential” solutions of this same equation, that is solutions of the form $m(z) = z^r \exp(wz)$, where $w \in \mathbb{C}$ and $r \in \mathbb{N}$. There are various ways, all equivalent, to define the operator $\varphi(D)$, for example as an infinite order differential operator, or as convolution with a certain compactly supported measure on $\mathbb{C}$, whose Fourier–Laplace transform is φ . This theory originated in the study of difference and differential-difference equations, and there is a solid treatment of this e.g. [6], for more modern versions see e.g. [5].

A more refined question (arising from Delsarte’s notion of mean periodic functions, originally in the context of functions of a real variable but extended to the holomorphic category by Laurent Schwartz and others) is this: Given u entire such that its translates (equivalently its derivatives) fail to span $\text{Hol}(\mathbb{C})$ (the entire functions), which is equivalent to the existence of some nontrivial φ of exponential type such that $\varphi(D)u = 0$, do those translates none the less span enough monomial exponentials to, in turn, span (“synthesize”) u ? This is the spectral synthesis question for $\text{Hol}(\mathbb{C})$. It is known that this question too has an affirmative answer; in higher dimensions however there are counterexamples to the spectral synthesis version, whereas the positive solution to the versions in the preceding paragraph holds in all dimensions.

Using this as a guide, one can formulate analogous questions in other topological vector spaces of entire functions. For example, in $\mathcal{B}$ let us ask whether the solutions to $\varphi^\#(D)u = 0$ are in the closure of the monomial exponential solutions. Here we may take as our definition of $\varphi^\#(D)$ the operator Π_{φ^*} . It is no longer necessary to suppose φ of exponential type, we get a sensible problem whenever the product of φ by each exponential is in $\mathcal{B}$; however, to fix ideas think first of the case where φ is of exponential type; even here the question to be posed is still unsolved. The monomial exponentials in question consist precisely of, for each zero w of φ , the functions $z^k \exp(\bar{w}z)$, $k = 0, \dots, r-1$ where r is the multiplicity of the zero.

The question whether these span all solutions to $\varphi^\#(D)u = 0$ in the *norm topology* of $\mathcal{B}$, then translates, by duality, to this: suppose h in $\mathcal{B}$ is orthogonal to the monomial exponentials belonging to $\ker \varphi^\#(D)$; this is equivalent to it vanishing on all zeros of φ , with the appropriate multiplicities, i.e. to the assertion: h/φ is an entire function. We ask if h must then be orthogonal to each u that is, in turn, orthogonal to all $\varphi(z) \exp(tz)$ with $t \in \mathbb{C}$. Since $h = \varphi F$ for some entire F , the question is whether φF is approximable in $\mathcal{B}$ -norm by exponential multiples of φ .

And this is just the question: do the finite linear combinations of exponentials span every entire function, in the metric of $L^2(|\varphi|^2 d\mu)$?

In [10] an affirmative solution was given for φ which are exponential polynomials, and indeed in any number of variables. But, for other classes of entire functions φ this is unsolved.

The counterexample of Borichev [1] (and our version of it below, in Section 4) are a borderline case: they can be said to disprove spectral synthesis for some “symbol function” φ of very large growth — so large, indeed, that the equation $\varphi^\#(D)u = 0$ has to be interpreted in a generalized sense, namely u (in $\mathcal{B}$) is orthogonal to all the *polynomial* multiples of φ (this is all we have to work with, since the *exponential* multiples, needed for the natural definition, do not lie in $\mathcal{B}$). So, it is of interest to know whether a corresponding counterexample exists where φ is of smaller growth, at least multiplying exponentials into $\mathcal{B}$.

2. Adjointness.

In what follows $\Delta(w, R) \subseteq \mathbb{C}^n$ stands for the open polydisk with radius R centered at w . Let $P_k : \mathbb{C}^n \rightarrow \mathbb{C}$, $k = 1, \dots, n$, denote projection onto the k -axis, i.e. $P_k(z) := z_k$ for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

LEMMA 2.1. *Let X be a Borel measurable set such that $P_k(X)$ is bounded for some $k \in \{1, \dots, n\}$ and $R := \inf\{r > 0 : P_k(X) \text{ is contained in } \Delta(w, r) \text{ with some } w\}$. Then there exists a constant C depending only on R such that*

$$(2.1) \quad \int_{\mathbb{C}^n} |f|^2 d\mu \leq C \int_{\mathbb{C}^n \setminus X} |f|^2 d\mu$$

for all entire functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$.

PROOF. First consider the one-dimensional case. Fix $R > 0$, $w \in \mathbb{C}$ and an entire function f defined on $\mathbb{C}$. Put $I(r) := \int_0^{2\pi} |f(re^{it} + w)|^2 dt$, $r \geq 0$. Since $I(r)$ is increasing with respect to r we have

$$re^{-r^2} I(r) \leq (R+r)e^{3R^2} e^{-(R+r)^2} I(R+r), \quad r \in [0, R].$$

Both sides of this inequality are functions in r , so integrating over the interval $[0, R]$ (with respect to the Lebesgue measure) yields

$$(2.2) \quad \int_{\Delta(w, R)} |f|^2 d\mu \leq e^{3R^2} \int_{\Delta(w, 2R) \setminus \Delta(w, R)} |f|^2 d\mu.$$

Adding $\int_{\mathbb{C} \setminus \Delta(w, R)} |f|^2 d\mu$ to both sides of (2.2) implies the assertion with $X = \Delta(w, R)$ and $C := e^{3R^2} + 1$.

We now proceed with the multi-dimensional case. Assume that X satisfies the assumptions of the lemma with some k . Let R be the constant defined in the statement of Lemma, then $P_k(X) \subseteq \Delta(w, R+1)$ with some $w \in \mathbb{C}$. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be an entire function. Set $Y_k := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_k - w| > R+1\}$. An easy application of the one-dimensional case leads to

$$\int_{\mathbb{C}^n} |f|^2 d\mu \leq C \int_{Y_k} |f|^2 d\mu,$$

with $C := e^{3(R+1)^2} + 1$. Since $Y_k \subseteq \mathbb{C}^n \setminus X$ the result follows.

LEMMA 2.2. *Given $f \in \mathcal{B}_1$, $p \in \mathcal{P}_1 \setminus \{0\}$ and $\epsilon > 0$ there exists a polynomial $q \in \mathcal{P}_1$ such that $\|f - q\| \leq \epsilon$ and $f = q \pmod{p}$, i.e. there exists $h \in \text{Hol}(\mathbb{C})$ satisfying $f - q = ph$.*

PROOF. We first state the following auxiliary fact:

LEMMA 2.3. *Let $p \in \mathcal{P}_1 \setminus \{0\}$ and let $\rho > 0$ be such that $\Delta(0, \rho)$ contains all zeroes of p . Then there exists a constant $C > 0$ depending only on p and ρ such that for every $F \in \text{Hol}(\Delta(0, \rho))$ one can find a unique $Q \in \mathcal{P}_1$ such that*

- (i) $\deg Q \leq \deg p - 1$,
- (ii) $\frac{F-Q}{p} \in \text{Hol}(\Delta(0, \rho))$,
- (iii) $\sup_{z \in \Delta(0, \rho)} |Q(z)| \leq C \sup_{z \in \Delta(0, \rho)} |F(z)|$.

To prove this suppose that $z_1, \dots, z_r$ are distinct zeroes of p with multiplicities $m_1, \dots, m_r$ (resp.). Observe that $m := m_1 + \dots + m_r = \deg p$. By the Cauchy inequality there exists $C_1 > 0$ (depending on ρ) such that

$$|g^{(l)}(z_j)| \leq C_1 \sup_{z \in \Delta(0, \rho)} |g(z)|, \quad l = 1, \dots, m_j,$$

for every $j \in \{1, \dots, r\}$ and $g \in \text{Hol}(\Delta(0, \rho))$. We can find another constant $C_2 > 0$, which depends only on p and ρ , such that

$$\sup_{z \in \Delta(0, \rho)} |h(z)| \leq C_2 \max\{|h^{(l_j)}(z_j)| : l_j = 1, \dots, m_j \text{ and } j = 1, \dots, r\},$$

for every $h \in \mathcal{P}_1$ with $\deg h \leq m - 1$, because by the Lagrange–Hermite interpolation the right-hand side of the above inequality defines a norm in the space of polynomials of degree at most $m - 1$.

Choose arbitrary $F \in \text{Hol}(\Delta(0, \rho))$. Applying the Lagrange–Hermite interpolation we infer that there exists a unique $Q \in \mathcal{P}_1$ of degree at most $m - 1$ such that

$$Q^{(l)}(z_j) = F^{(l)}(z_j), \quad l = 1, \dots, m_j.$$

for every $j = 1, \dots, r$. Hence Q satisfies (i) and (ii). By the choice of C_1 and C_2 we see that (iii) holds with $C := C_1 C_2$. This completes the proof of Lemma 2.3.

We now turn to the proof of Lemma 2.2. Fix $f \in \mathcal{B}_1$, $p \in \mathcal{P}_1$ and $\delta > 0$. Choose a polynomial g such that $\|f - g\| \leq \delta$ and $\rho > 0$ such that $\Delta(0, \rho)$ contains all zeroes of p . Set $F := f - g$. Then, by the reproducing property for $\mathcal{B}$, $|F(z)| \leq \delta e^{|z|^2/2}$, $z \in \mathbb{C}$, and, consequently,

$$|F(z)| \leq \delta e^{\rho^2/2}, \quad z \in \Delta(0, \rho).$$

We can now apply Lemma 2.3 to obtain the polynomial Q , $\deg Q \leq m - 1$ (here $m := \deg p$), such that $\frac{F-Q}{p}$ is analytic in $\Delta(0, \rho)$ (hence entire) and

$$|Q(z)| \leq C \delta e^{\rho^2/2}, \quad z \in \Delta(0, \rho),$$

with $C > 0$ depending only on p and ρ . This implies that

$$|Q(z)| \leq C \delta e^{\rho^2/2} \max \left\{ \left(\frac{|z|}{\rho} \right)^{m-1}, 1 \right\}, \quad z \in \mathbb{C}.$$

This inequality leads to $\|Q\| \leq C' \delta$ with a new constant $C' > 0$ depending only on p and ρ . Setting $q := g + Q$ we obtain a polynomial, for which $f - q$ is divisible by p . Moreover,

$$\|f - q\| = \|f - g - Q\| \leq \|f - g\| + \|Q\| \leq \delta(1 + C').$$

Since δ can be made arbitrarily small, the result follows.

THEOREM 2.4. *Let $p \in \mathcal{P}_1 \otimes B(\mathbb{C}^M, \mathbb{C}^N)$ with $M > N \geq 1$ and $f \in \mathcal{D}(T_p)$. Assume that $p(z)$ is of maximal rank for at least one point $z \in \mathbb{C}$. Suppose that $T_p f = 0$. Then for any $\epsilon > 0$ there exists $q \in \mathcal{P}_1 \otimes \mathbb{C}^M$ such that $\|f - q\|_{(M)} \leq \epsilon$ and $T_p q = 0$.*

PROOF. We can represent p by means of the matrix $[p_{kl}]_{k=1,\dots,N}^{l=1,\dots,M}$, where $p_{kl} \in \mathcal{P}_1$. Put $\tilde{p} := [p_{kl}]_{k=1,\dots,N}^{l=1,\dots,N}$. By assumption on the rank of p , rearranging columns if necessary, we can assume that $d(z) := \det \tilde{p}(z)$ is not identically zero; moreover, it can be done so that $\deg d$ is maximal among degrees of all possible minors of dimension N , which arise from the matrix representing p .

Given $\epsilon_0 > 0$ by Lemma 2.2 we can find polynomials $q_{N+1}, \dots, q_M \in \mathcal{P}_1$ such that $f_j = q_j \pmod{d}$ and $\|f_j - q_j\| < \epsilon_0$, $j = N+1, \dots, M$. By the Cramer formulas there exist rational functions $q_1, \dots, q_N$ such that $T_p(q_1, \dots, q_M) = 0$. Thus every q_j , $j = 1, \dots, N$, can be expressed by $q_{N+1}, \dots, q_M$ via

$$(2.3) \quad q_j = d^{-1} \det \begin{bmatrix} p_{11} & \dots & p_{1j-1} & -\sum_{l=N+1}^M p_{1l} q_l & p_{1j+1} & \dots & p_{1N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{N1} & \dots & p_{Nj-1} & -\sum_{l=N+1}^M p_{Nl} q_l & p_{Nj+1} & \dots & p_{NN} \end{bmatrix}$$

for $j = 1, \dots, N$.

Observe that all q_j 's are polynomials! Indeed, one can verify that the difference $f_j - q_j$ is an entire function. To see this it suffices to check that the formula (2.3) holds if q_j is replaced by f_j , $j = 1, \dots, M$, in both members of (2.3). Then it turns out that for each $j \in \{1, \dots, N\}$ function $f_j - q_j$ is a linear combination with polynomial coefficients of $f_l - q_l$, $l = N+1, \dots, M$, divided by d . By the choice of q_j the difference $f_l - q_l$ is divisible by d , $l = N+1, \dots, M$. Thus $f_j - q_j$ is entire and, consequently, q_j is entire. Since every entire rational function has to be a polynomial, we deduce that every q_j is a polynomial.

It remains to estimate the norm of $f_j - q_j$ for $j = 1, \dots, N$. Note that by (2.3) for fixed $j \in \{1, \dots, N\}$ we get

$$f_j - q_j = \sum_{l=N+1}^M \alpha_l \frac{M_l}{d} (f_l - q_l),$$

where M_l is a properly chosen minor of dimension N in p (obviously, M_l depends on j , which has been omitted in the notation) and α_l is equal to 1 or -1 . Choose $R > 0$ such that all zeros of d lie in Δ_R . By the assumption on $\deg d$ we infer that every quotient $\frac{M_l}{d}$ is bounded outside Δ_R . Set $\tilde{c}_l = \sup\{|\frac{M_l(z)}{d(z)}| : z \in \mathbb{C} \setminus \Delta_R\}$. Then

$$|f_j(z) - q_j(z)| \leq \sum_{l=N+1}^M \tilde{c}_l |f_l(z) - q_l(z)|, \quad z \in \mathbb{C} \setminus \Delta_R.$$

Taking squares, integrating over $\mathbb{C} \setminus \Delta_R$ and applying Lemma 2.1 yield

$$\|f_j - q_j\|^2 \leq C_R \epsilon_0^2 (M - N) \sum_{l=N+1}^M \tilde{c}_l^2,$$

where C_R is some positive constant. If ϵ_0 is small enough, then polynomial $q := (q_1, \dots, q_M)$ satisfies all the required conditions.

NOTE: The authors thank John McCarthy for a valuable suggestion in connection with the proof of Theorem 2.4.

THEOREM 2.5. *Let $p \in \mathcal{P}_1 \otimes B(\mathbb{C}^M, \mathbb{C}^N)$. Then $T_p^* = \Pi_{p^*}$.*

PROOF. It suffices to check the density of $\mathcal{P}_1 \otimes \mathbb{C}^N$ in $\mathcal{D}(T_p)$ with respect to the graph norm $\|\cdot\|_{T_p}$. Choose $f \in \mathcal{D}(T_p)$. We are going to show that there exists $q \in \mathcal{P}_1 \otimes \mathbb{C}^N$ such that $\|f - q\|_{T_p}$ is arbitrarily small. Define $\tilde{p} \in \mathcal{P}_1 \otimes B(\mathbb{C}^{M+N}, \mathbb{C}^N)$ by $\tilde{p}(z) = [p(z), -I_N]$, where I_N stands for the identity matrix of dimension N . It is easily seen that $\tilde{p}$ satisfies the assumptions of Theorem 2.4.

Since $T_{\bar{p}}(f, T_p f) = 0$, given $\epsilon > 0$ we can find $\tilde{q} = (q_1, \dots, q_{M+N}) \in \mathcal{P}_1 \otimes \mathbb{C}^{M+N}$ such that $T_{\bar{p}}\tilde{q} = 0$ and $\|(f, T_p f) - \tilde{q}\|_{(M+N)} \leq \epsilon$. Setting $q := (q_1, \dots, q_N)$ we see that $T_p q = (q_{N+1}, \dots, q_{N+M})$, as a consequence of the equality $T_{\bar{p}}\tilde{q} = 0$, and $\|f - q\|_{T_p}^2 = \|(f, T_p f) - \tilde{q}\|_{(M+N)}^2 \leq \epsilon^2$.

It seems that the case of homogeneous polynomials is much easier to deal with, even in the case of several complex variables and values being operators on (possibly) infinite-dimensional Hilbert space $\mathcal{H}$.

PROPOSITION 2.6. *If $p \in \mathcal{P} \otimes B(\mathcal{H})$ is a homogeneous polynomial then $T_p^* = \Pi_{p^*}$.*

PROOF. The main idea of the proof is to write $\mathcal{B} \otimes \mathcal{H}$ as an orthogonal sum of subspaces reducing $p^\#(D)T_p$ to a non-negative operator. Let $\mathcal{F}_k \subseteq \mathcal{P} \otimes \mathcal{H}$ denote the space of homogeneous polynomials of degree k (with the zero polynomial included). Obviously, $\mathcal{B} \otimes \mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{F}_k$. It is easily seen that $p^\#(D)T_p(\mathcal{F}_k) \subseteq \mathcal{F}_k$ for all $k \geq 0$. Hence, each $\mathcal{F}_k$ reduces $p^\#(D)T_p$. Pick arbitrary $f \in \mathcal{F}_k$ and compute

$$\begin{aligned} \langle p^\#(D)T_p f, f \rangle_{(\mathcal{H})} &= \int_{\mathbb{C}^n} \langle p^\#(D)(pf)(z), f(z) \rangle_{\mathcal{H}} d\mu(z) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle p(\zeta)^* p(\zeta) f(\zeta), f(z) \rangle_{\mathcal{H}} e^{z \cdot \bar{\zeta}} d\mu(\zeta) d\mu(z). \end{aligned}$$

The last equality follows from [4, Lemma 2.3], but the reader may obtain it applying the theorem on differentiating under the integral sign. We may now change the order of integration, which is allowed because the function under the integral sign is summable with respect to $\mu \otimes \mu$. Then it suffices to apply the reproducing property for $\mathcal{B} \otimes \mathcal{H}$ to see that

$$\langle p^\#(D)T_p f, f \rangle_{(\mathcal{H})} = \langle p^* p f, f \rangle_{(\mathcal{H})} \geq 0.$$

Since³ $p^\#(D)T_p = \bigoplus_{k=0}^{\infty} p^\#(D)T_p|_{\mathcal{F}_k}$ we infer that $p^\#(D)T_p$ is non-negative, so by Lemma 1.1 the assertion follows.

REMARK 2.7. The above proof works also for the wider class of *t-homogeneous polynomials* (cf. [9]).

PROPOSITION 2.8. *Assume that $p = [p_{kl}]_{k,l=1,\dots,N} \in \mathcal{P} \otimes B(\mathbb{C}^N)$ is such that all p_{kl} are homogeneous. If $\deg p_{kl}$ depends only on l , then $T_p^* = \Pi_{p^*}$.*

PROOF. (Based on an idea of J. Janas.) We are going to prove that $T_p p^\#(D)$ is non-negative. Let $\mathcal{F}_{k,j} \subseteq \mathcal{P} \otimes \mathbb{C}^j$ denote the space of homogeneous polynomials of degree k for $k \geq 0$. Observe that $\mathcal{B} \otimes \mathbb{C}^N = \bigoplus_{k=0}^{\infty} \mathcal{F}_{k,N}$. We claim that each $\mathcal{F}_{k,N}$ is invariant under $T_p p^\#(D)$ (hence reducing for this operator). Operator $T_p p^\#(D)|_{\mathcal{F}_{k,N}} : \mathcal{F}_{k,N} \rightarrow \mathcal{B} \otimes \mathbb{C}^N$ may be written as an operator matrix $[A_{kl}]_{k,l=1,\dots,N}$, where

$$A_{kl} = \sum_{j=0}^N T_{p_{kj}} p_{lj}^\#(D), \quad \mathcal{D}(A_{kl}) = \mathcal{F}_{k,1}.$$

By assumption on p we see that if f is homogeneous of degree k then so is $A_{kl}f$.

It now remains to show that $T_p p^\#(D)|_{\mathcal{F}_{k,N}}$ is non-negative. Note first that

$$P \otimes I_{\mathbb{C}^N} (p^* f)(z) = \int_{\mathbb{C}^n} p^*(\zeta) f(\zeta) e^{z \cdot \bar{\zeta}} d\mu(\zeta) = p^\#(D) f(z), \quad z \in \mathbb{C}^n, f \in \mathcal{F}_{k,N},$$

for all $k \geq 0$ (cf. [4, Lemma 2.3]). Thus

$$\langle T_p p^\#(D) f, f \rangle_{(\mathcal{H})} = \langle p p^\#(D) f, f \rangle_{(\mathcal{H})} = \langle p^\#(D) f, P \otimes I_{\mathbb{C}^N} (p^* f) \rangle_{(\mathcal{H})} \geq 0$$

³Note that $p^\#(D)T_p$ is closed.

for all $f \in \mathcal{F}_{k,N}$ and $k \geq 0$. Applying Lemma 1.1 completes the proof.

One of the natural questions which arise when studying the adjointness hypothesis is whether it is possible to find the solution to this problem only by means of coefficients of p . This idea is presented in the following theorem. The set of operators $\mathcal{A} \subseteq B(\mathcal{H})$ is called *jointly subnormal* if there exists a set $\mathcal{A}' \subseteq B(\mathcal{K})$ of commuting normal operators defined on a larger Hilbert space containing $\mathcal{H}$ as a closed subspace such that for every $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ satisfying $A = A'|_{\mathcal{H}}$.

THEOREM 2.9. *Suppose that $p = \sum_{k=0}^N p_k \otimes A_k \in \mathcal{P} \otimes B(\mathcal{H})$ with $p_k \in \mathcal{P}$ and $A_k \in B(\mathcal{H})$. If the sequence $\{A_k\}_{k=0}^N$ is jointly subnormal, then $T_p^* = \Pi_{p^*}$.*

The proof of this theorem requires the following lemma

LEMMA 2.10. *Suppose that $q \in \mathcal{P} \otimes B(\mathcal{H})$ takes values only among normal operators, $f \in \mathcal{D}(T_q)$ and $g \in \mathcal{B} \otimes \mathcal{H}$. Then the function*

$$\mathbb{C}^n \times \mathbb{C}^n \ni (z, w) \rightarrow \langle g(w), q(w)f(z) \rangle e^{z \cdot \bar{w}} \in \mathbb{C}$$

is summable with respect to the measure $\mu \otimes \mu$ in $\mathbb{C}^n \times \mathbb{C}^n$.

PROOF. We begin with a change of variables $w = z + \zeta$, which yields

$$\int_{\mathbb{C}^n} |\langle g(w), q(w)f(z) \rangle e^{z \cdot \bar{w}}| d\mu(w) = \int_{\mathbb{C}^n} |\langle g(z + \zeta), q(z + \zeta)f(z) \rangle e^{-z \cdot \bar{\zeta}}| d\mu(\zeta)$$

Thus

$$\begin{aligned} J &:= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle g(w), q(w)f(z) \rangle e^{z \cdot \bar{w}}| d\mu(w) d\mu(z) \\ &\leq \int_{\mathbb{C}^n} \|g(\cdot + \zeta)e_{-\zeta}\|_{(\mathcal{H})} \|q(\cdot + \zeta)f\|_{(\mathcal{H})} d\mu(\zeta). \end{aligned}$$

But $\|g(\cdot + \zeta)e_{-\zeta}\|_{(\mathcal{H})} = e^{\frac{1}{2}\|\zeta\|^2} \|g\|_{(\mathcal{H})}$, which together with the normality of $q(z)$ for all $z \in \mathbb{C}^n$ implies that

$$J \leq \|g\|_{(\mathcal{H})} \int_{\mathbb{C}^n} \|q(\cdot + \zeta)^* f\|_{(\mathcal{H})} e^{\frac{1}{2}\|\zeta\|^2} d\mu(\zeta).$$

Applying the isometry theorem (cf. [10], [4]), which states that⁴

$$\|p^* h\|_{(\mathcal{H})}^2 = \sum_{j \geq 0} \frac{1}{j!} \|(D^j p^\#)(D)h\|_{(\mathcal{H})}^2, \quad p \in \mathcal{P} \otimes B(\mathcal{H}), h \in \mathcal{B} \otimes \mathcal{H},$$

we infer that $\|q(\cdot + \zeta)^* f\|_{(\mathcal{H})}$ is of polynomial growth, so $J < \infty$.

PROOF OF THEOREM 2.9. Let $q(z) := \sum_{j=0}^N p_j \otimes B_j$, where $\{B_k\}_{k=0}^N$ is the sequence of commuting normal operators defined on a larger Hilbert space $\mathcal{K}$ such that $A_k = B_k|_{\mathcal{H}}$, $k = 0, \dots, N$. Thus q takes its values only among normal operators and, consequently, $T_q^* = q^\#(D)$, according to [4, Theorem 7.8]. We will show that $\langle p^\#(D)T_p f, f \rangle_{(\mathcal{H})} \geq 0$ for every $f \in \mathcal{D}(p^\#(D)T_p)$. Observe that

$$\begin{aligned} \langle p^\#(D)T_p f, f \rangle_{(\mathcal{H})} &= \int_{\mathbb{C}^n} \langle p^\#(D)(pf)(z), f(z) \rangle d\mu(z) \\ &= \int_{\mathbb{C}^n} \langle q^\#(D)(pf)(z), f(z) \rangle d\mu(z) \\ (2.4) \quad &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle q(w)^* p(w)f(w), f(z) \rangle e^{z \cdot \bar{w}} d\mu(w) d\mu(z). \end{aligned}$$

⁴The symbol “ $\sum_{j \geq 0}$ ” should be read “sum over all multiindices j ”. In the following equality we put $\|F\|_{(\mathcal{H})} = \infty$ whenever $F \notin L^2(\mu) \otimes \mathcal{H}$

We know that $pf \in B \otimes \mathcal{H}$, which implies that $qf \in \mathcal{B} \otimes \mathcal{K}$, because $p(z) = q(z)|_{\mathcal{H}}$ for all $z \in \mathbb{C}^n$. By Lemma 2.10 we are allowed to change the order of integration in (2.4). So we arrive at

$$\begin{aligned} \langle p^\#(D)T_p f, f \rangle_{(\mathcal{H})} &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \langle q(w)^* p(w) f(w), f(z) \rangle e^{z \cdot \bar{w}} d\mu(z) d\mu(w) \\ &= \int_{\mathbb{C}^n} \langle q(w)^* p(w) f(w), f(w) \rangle d\mu(w) \\ &= \int_{\mathbb{C}^n} \|p(w) f(w)\|^2 d\mu(w) \geq 0. \end{aligned}$$

Thus $\ker(I_{\mathcal{H}} + p^\#(D)T_p) = \{0\}$, which implies that $T_p^* = p^\#(D)$.

REMARK 2.11. If the property $T_p^* = \Pi_{p^*}$ is proved for all polynomials, whose coefficients commute, then it can easily be shown for arbitrary polynomial p . To see this pick arbitrary $p \in \mathcal{P} \otimes B(\mathcal{H})$ and define polynomial $q \in \mathcal{P} \otimes B(\mathcal{H} \oplus \mathcal{H})$ via

$$q(z) = \begin{bmatrix} 0 & 0 \\ p(z) & 0 \end{bmatrix}, \quad z \in \mathbb{C}^n.$$

Note that $q(z)$ can be written as $\sum_{|j| \leq N} B_j z^j$, $z \in \mathbb{C}^n$, where $B_j \in B(\mathcal{H} \oplus \mathcal{H})$ and $B_j B_k = 0$ for all admissible j and k . Suppose that $T_q = q^\#(D)$. Then for any $g \in \mathcal{D}(T_q)$ one can find a sequence $\{h_j\}_{j=0}^\infty$ in $\mathcal{P} \otimes (\mathcal{H} \oplus \mathcal{H})$ such that $h_j \rightarrow g$ and $qh_j \rightarrow qg$. One can readily check that $\|q(z)(f \oplus g)\| = \|p(z)f\|$ for all $z \in \mathbb{C}^n$ and $f, g \in \mathcal{H}$.

Pick arbitrary $f \in \mathcal{D}(T_p)$. Then $f \oplus 0$ belongs to the domain of T_q . Let $\{h_j\}_{j=0}^\infty \subseteq \mathcal{P} \otimes (\mathcal{H} \oplus \mathcal{H})$ be the sequence chosen so that $h_j \rightarrow f \oplus 0$ and $qh_j \rightarrow q(f \oplus 0)$. Let $P_{\mathcal{H} \oplus 0}$ be the orthogonal projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H} \oplus 0$. By the choice of q it is apparent that the sequence $\{P_{\mathcal{H} \oplus 0} h_j\}_{j=0}^\infty \subseteq \mathcal{P} \otimes \mathcal{H}$ tends to f in the graph norm of T_p . Thus $\mathcal{P} \otimes \mathcal{H}$ is a core for T_p and, consequently, $T_p^* = p^\#(D)$.

3. Closedness of range.

Let $p \in \mathcal{P} \otimes B(\mathbb{C}^N)$. In what follows, $\det p$ stands for the polynomial defined in the natural way as $(\det p)(z) = \det(p(z))$, and $p^{kl} \in B(\mathbb{C}^{N-1})$ originates from p by removing the k -th row and the l -th column.

THEOREM 3.1. *Assume that $p \in \mathcal{P}_1 \otimes B(\mathbb{C}^N)$, $p \neq 0$. Then T_p is bounded below if and only if $\deg \det p \geq \max_{k,l=1,\dots,N} \deg \det p^{kl} \geq 0$.*

PROOF. The “only if” part of assertion was proved in [4, Proposition 8.6]. It remains to consider the “if” part. Suppose that $p = [p_{kl}]_{k,l=1,\dots,N}$ with $p_{kl} \in \mathcal{P}_1$ and $T_p(f_1, \dots, f_N) = (g_1, \dots, g_N)$ for $(f_1, \dots, f_N) \in \mathcal{D}(T_p)$, which means that

$$\begin{bmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & & \vdots \\ p_{N1} & \cdots & p_{NN} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}.$$

Note that by assumption $\det p$ is not identically 0, so the above equation may be solved for $f_1, \dots, f_N$. Choose $R > 0$ such that all zeroes of $\det p$ lie in $\Delta(0, R)$. Hence, by the Cramer formulas, we infer that

$$|f_j(z)| \leq \sup \left\{ \left| \frac{\det p^{kl}(\zeta)}{\det p(\zeta)} \right| : k, l = 1, \dots, N, |\zeta| \geq R \right\} (|g_1(z)| + \dots + |g_N(z)|)$$

for all $j = 1, \dots, N$ and $z \in \mathbb{C} \setminus \Delta(0, R)$. Let M denote the supremum in the above inequality. Then it follows that

$$\int_{\mathbb{C} \setminus \Delta(0, R)} \sum_{j=1}^N |f_j|^2 d\mu \leq MN \sum_{j=1}^N \|g_j\|^2.$$

Applying Lemma 2.1 yields

$$\|T_p(f_1, \dots, f_N)\|_{(N)} = \|(g_1, \dots, g_N)\|_{(N)} \geq C \|(f_1, \dots, f_N)\|_{(N)}$$

with an appropriate constant $C > 0$ depending only on p and R . This means that T_p is bounded below.

Theorem 3.1 is the one variable refinement of [4, Proposition 8.6], which gives the necessary condition for boundedness below of T_p in the multivariable case. Although it does not seem likely one can obtain the multivariable version of the above theorem we have not been able to give any counterexample.

The following proposition deals with non-injective operators with closed range.

PROPOSITION 3.2. *Let $p \in \mathcal{P}_1 \otimes B(\mathbb{C}^N, \mathbb{C})$. Then $\mathcal{R}(T_p)$ is closed.*

PROOF. Assume that $p = [p_1, \dots, p_N]$ and q is the greatest common divisor of all p_j 's, $j = 1, \dots, N$. We may choose p such that $p_N \neq 0$, which involves no loss of generality. We are going to establish the following description of the range:

$$\mathcal{R}(T_p) = \left\{ g \in \mathcal{B}_1 : \frac{g}{q} \text{ extends to an entire function} \right\}.$$

Since $T_p(f_1, \dots, f_N) = \sum_{j=1}^N p_j f_j$ the inclusion " $\subseteq$ " is obvious. To prove the reverse inclusion pick $g \in \mathcal{B}_1$ such that $\frac{g}{q}$ is entire. By easy algebra there exist polynomials $q_1, \dots, q_j$ such that $\sum_{j=1}^N q_j p_j = qr$, where r is a polynomial chosen so that $\frac{g-qr}{p_N}$ is entire (use the interpolation property). Put

$$f_j := q_j \text{ for } j = 1, \dots, N-1, \text{ and } f_N := \frac{g-qr}{p_N} + q_N.$$

It is easily seen that $f_N \in \mathcal{B}_1$ and $T_p(f_1, \dots, f_N) = g$, thus $g \in \mathcal{R}(T_p)$.

Observe that $\mathcal{R}(T_p) = \mathcal{R}(T_q)$, where T_q is Toeplitz operator defined in $\mathcal{B}_1$. By the Newman–Shapiro Isometry Theorem [10, 4] operator T_q is bounded below, hence $\mathcal{R}(T_q)$ is closed.

We now turn to an example showing that the above proposition is not true in the multivariable case.

EXAMPLE 3.3. $\mathcal{R}(T_p)$ need not be closed even in case of homogeneous polynomials. Define $p = [p_1 \ p_2]$, where $p_1(z, w, \zeta) = z^3 w^3 - \zeta^6$ and $p_2(z, w, \zeta) = w^6$, $(z, w, \zeta) \in \mathbb{C}^3$. Observe that $\mathcal{R}(T_p)$ contains all functions of type $w^3 \zeta^6 q$, where q is an arbitrary polynomial depending only on z . This follows from $T_p(-w^3 q, z^3 q) = w^3 \zeta^6 q$. Suppose $\mathcal{R}(T_p)$ is closed. Then it follows that the closure of $\{w^3 \zeta^6 q : q \text{ is a polynomial depending only on } z\}$ is contained in $\mathcal{R}(T_p)$. But this closure is equal to $\{w^3 \zeta^6 f : f \in \mathcal{B}_3, f \text{ depends only on } z\}$, which is an immediate consequence of $\|w^3 \zeta^6 q\|^2 = 3!6! \|q\|^2$, with q depending only on z . Put $f_0(z) = (e^{z^2/2} - 1)(z - z_1)^{-1}(z - z_2)^{-1}$, where z_1, z_2 are two different zeroes of function $e^{z^2/2} - 1$. Observe that $f_0 \in \mathcal{B}_1$. Hence, there are $f, g \in \mathcal{B}_3$ such that $T_p(f, g)(z, w, \zeta) = w^3 \zeta^6 f_0(z)$, or more explicitly

$$(z^3 w^3 - \zeta^6) f(z, w, \zeta) + w^6 g(z, w, \zeta) = w^3 \zeta^6 \frac{e^{z^2/2} - 1}{(z - z_1)(z - z_2)}, \quad z, w, \zeta \in \mathbb{C}.$$

Fix arbitrary $x > 0$. Substituting $z := x$, $w := \frac{1}{x}$ and $\zeta := 1$ we get

$$g(x, \frac{1}{x}, 1) = x^3 \frac{e^{x^2/2} - 1}{(x - z_1)(x - z_2)}, \quad x > 0.$$

This leads to a contradiction, because the left hand side of the above equality can be estimated from above by $\|g\| \exp(\frac{x^2}{2} + \frac{1}{2x^2} + \frac{1}{2})$, $x > 0$, whereas this is impossible for the right hand side. So we have proved that the range of T_p is not closed.

By a similar argument one can show that the range of operator T_q with $q(z, w, \zeta) = [q_1 \quad q_2]$, $q_1(z, w, \zeta) = z^3 w^3 - \zeta^6$, $q_2(z, w, \zeta) = w^4$, $(z, w, \zeta) \in \mathbb{C}^3$, is not closed. Observe that $\deg q_1 \neq \deg q_2$ in opposition to the previous example. Hence, if we drop the assumption on equality of degrees of coordinate polynomials, then it does not follow that the range of T_p is closed in case of several complex variables.

4. The non-density example.

Below we show an example of an analytic function φ for which polynomials *do not* form a core for the operator T_φ though they are contained in its domain. In other words: we will indicate an analytic φ for which the space of all entire functions square-integrable with respect to the measure $(1 + |\varphi|^2) d\mu$ contains polynomials as a non-dense subset. Before we go into details we need the following theorem, which is of independent interest.

THEOREM 4.1. *Let $\{z_k\}_{k=1}^\infty \subseteq \mathbb{C}$ be a sequence of non-zero numbers satisfying $|\frac{z_{k+1}}{z_k}| \geq \lambda$ for all $k \in \mathbb{N}$ with some $\lambda > 1$. Then the formula*

$$\psi(z) := \prod_{k=1}^\infty \left(1 - \frac{z}{z_k}\right), \quad z \in \mathbb{C},$$

defines an entire function for which there exists $c > 0$ such that $\|T_\psi f\| \geq c\|f\|$ for all $f \in \mathcal{D}(T_\psi)$.

PROOF. Since we want to impose some additional conditions on the given sequence $\{z_k\}_{k=1}^\infty \subseteq \mathbb{C}$, we will show that it suffices to deal only with a modification of this sequence obtained by removing a few initial terms. Indeed, if the theorem is proved for $\psi_1(z) := \prod_{k=1}^\infty (1 - \frac{z}{z_{k+j}})$ with some integer $j \geq 0$, then applying the boundedness below of T_p with the polynomial $p := \frac{\psi}{\psi_1}$ we get

$$\|T_\psi f\| = \|T_p \psi_1 f\| \geq c_1 \|\psi_1 f\|, \quad f \in \mathcal{D}(T_\psi),$$

where c_1 is a positive constant. Thus if T_{ψ_1} is bounded below, then so is T_ψ .

From now on we are going to assume that $|z_1| \geq 1$,

$$(4.5) \quad \lambda \geq \frac{3}{|z_k|} + 1, \quad k = 1, 2, \dots,$$

and

$$(4.6) \quad |z_k - z_j| > 4, \quad k \neq j,$$

which can be easily obtained by omitting a suitable number of initial terms in the sequence $\{z_k\}_{k=1}^\infty$.

Denote by $\Delta(w, r)$ the open disk with radius r centered at $w \in \mathbb{C}$. We claim that there exists a positive constant a such that

$$(4.7) \quad |\psi(z)| \geq a \text{ for } z \in \mathbb{C} \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1).$$

To prove this we first show that there exists $a > 0$ such that

$$(4.8) \quad |\psi(z_k + \zeta)| \geq a \text{ whenever } |\zeta| = 1 \text{ and } k = 1, 2, \dots$$

By assumption (4.6) the unit disks centered at the z_k 's are disjoint so

$$\min\{|\psi(z_k + \zeta)| : k = 1, 2, |\zeta| = 1\} > 0.$$

Fix $k \geq 3$ and observe that

$$\psi(z_k + \zeta) = \left(1 - \frac{z_k + \zeta}{z_1}\right) \prod_{j=2}^{k-1} \left(1 - \frac{z_k + \zeta}{z_j}\right) \left(-\frac{\zeta}{z_k}\right) \prod_{j=k+1}^{\infty} \left(1 - \frac{z_k + \zeta}{z_j}\right)$$

We are going to find uniform (in k) estimates for terms appearing in the above equality. We have

$$\begin{aligned} \left| \left(1 - \frac{z_k + \zeta}{z_1}\right) \left(-\frac{\zeta}{z_k}\right) \right| &\geq \frac{|z_k| - |z_1| - 1}{|z_1 z_k|} \\ &\geq \frac{1}{|z_1|} - \frac{1}{|z_2|} - \frac{1}{|z_1 z_2|} \\ &> 0. \end{aligned}$$

The next term is estimated as follows

$$\begin{aligned} \left| \prod_{j=2}^{k-1} \left(1 - \frac{z_k + \zeta}{z_j}\right) \right| &= \left| \prod_{j=1}^{k-2} \left(1 - \frac{z_k}{z_{k-j}} - \frac{\zeta}{z_{k-j}}\right) \right| \\ &\geq \prod_{j=1}^{k-2} \left(\left| \frac{z_k}{z_{k-j}} \right| - \frac{1}{|z_{k-j}|} - 1 \right) \\ &\geq \prod_{j=1}^{k-2} \left(\lambda^j - \frac{1}{|z_1|} - 1 \right). \end{aligned}$$

The last term tends to infinity as $k \rightarrow \infty$. By (4.5) we see that it is always positive, so there exists a uniform positive lower bound for all k 's. We now turn to the remaining part:

$$\begin{aligned} \left| \prod_{j=k+1}^{\infty} \left(1 - \frac{z_k + \zeta}{z_j}\right) \right| &= \prod_{j=1}^{\infty} \left| 1 - \frac{z_k + \zeta}{z_{k+j}} \right| \\ &\geq \prod_{j=1}^{\infty} \left(1 - \frac{|z_k| + 1}{\lambda^j |z_k|} \right) \\ &= \prod_{j=1}^{j_0} \left(1 - \frac{|z_k| + 1}{\lambda^j |z_k|} \right) \prod_{j=j_0+1}^{\infty} \left(1 - \frac{|z_k| + 1}{\lambda^j |z_k|} \right), \end{aligned}$$

where j_0 is the minimal integer for which $1 - \frac{2}{\lambda^j} > 0$ for $j \geq j_0$. Hence

$$\prod_{j=j_0+1}^{\infty} \left(1 - \frac{|z_k| + 1}{\lambda^j |z_k|} \right) \geq \prod_{j=j_0+1}^{\infty} \left(1 - \frac{2}{\lambda^j} \right) > 0.$$

To complete the proof of (4.8) we proceed to show that

$$(4.9) \quad 1 - \frac{|z_k| + 1}{\lambda^j |z_k|} > \tilde{c}, \quad j = 1, \dots, j_0,$$

with some $\tilde{c} > 0$. Indeed,

$$\begin{aligned} 1 - \frac{|z_k| + 1}{\lambda^j |z_k|} &\geq 1 - \frac{|z_k| + 1}{\lambda |z_k|} \\ &= 1 - \frac{1}{\lambda} - \frac{1}{\lambda |z_k|} \\ &\geq 1 - \frac{1}{\lambda} - \frac{1}{\lambda |z_2|}, \end{aligned}$$

which implies (4.9). So we have proved (4.8).

To deduce (4.7) from (4.8) we need to apply the Wiman theorem stating that for an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ of sufficiently small growth (i.e. dominated by $e^{\alpha|z|^\beta}$ with some $\alpha > 0$ and $\beta \in (0, \frac{1}{2})$) there exists a strictly increasing sequence of radii $\{R_j\}_{j=1}^\infty$ such that $R_j \rightarrow \infty$ and $\min\{|f(z)| : |z| = R_j\} \rightarrow \infty$ when $j \rightarrow \infty$ (cf. [2, Thm 3.1.1]). In particular, we are able to find a sequence $\{R_j\}_{j=1}^\infty$ satisfying $\min\{|\psi(z)| : |z| = R_j, j \geq 1\} \geq a$, where a is a constant appearing in (4.8). In order to prove (4.7) it suffices to notice that

$$|\psi(z)| \geq a, \quad z \in \Delta(0, R_j) \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)$$

for each $j \geq 1$. This in turn follows from the minimum principle applied to ψ restricted to the interior of $\Delta(0, R_j) \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)$. So the property (4.7) is proved.

Having established (4.7) we are in a position to finish the proof. Suppose that $F : \mathbb{C} \rightarrow \mathbb{C}$ is entire and chosen so that $\frac{F}{\psi}$ is again entire. Applying (2.2) and (4.7) we obtain

$$\begin{aligned} \int_{\mathbb{C}} \left| \frac{F}{\psi} \right|^2 d\mu &= \int_{\mathbb{C} \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)} \left| \frac{F}{\psi} \right|^2 d\mu + \sum_{k=1}^\infty \int_{\Delta(z_k, 1)} \left| \frac{F}{\psi} \right|^2 d\mu \\ &\leq \int_{\mathbb{C} \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)} \left| \frac{F}{\psi} \right|^2 d\mu + e^3 \sum_{k=1}^\infty \int_{\Delta(z_k, 2) \setminus \Delta(z_k, 1)} \left| \frac{F}{\psi} \right|^2 d\mu \\ &\leq \frac{1}{a^2} \int_{\mathbb{C} \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)} |F|^2 d\mu + \frac{e^3}{a^2} \sum_{k=1}^\infty \int_{\Delta(z_k, 2) \setminus \Delta(z_k, 1)} |F|^2 d\mu \\ &= \frac{1}{a^2} \int_{\mathbb{C} \setminus \bigcup_{k=1}^\infty \Delta(z_k, 1)} |F|^2 d\mu + \frac{e^3}{a^2} \int_{\bigcup_{k=1}^\infty \Delta(z_k, 2) \setminus \Delta(z_k, 1)} |F|^2 d\mu \\ &\leq \frac{e^3 + 1}{a^2} \int_{\mathbb{C}} |F|^2 d\mu. \end{aligned}$$

This implies that $\|T_\psi f\|^2 \geq a^2(e^3 + 1)^{-1} \|f\|^2$ for $f \in \mathcal{D}(T_\psi)$. The proof is complete.

We now proceed with the example described in the beginning of this section. Let $\{z_k\}_{k=1}^\infty$ be any sequence satisfying assumptions of Theorem 4.1 and such that $e^{z_k^2/2} = 1$, $k \geq 1$. Define the entire function φ by

$$\varphi(z) := \frac{e^{z^2/2} - 1}{\prod_{k=1}^\infty (1 - \frac{z}{z_k})}.$$

In the next proposition we list some properties of the operator T_φ .

- PROPOSITION 4.2. (i) $\mathcal{P} \subseteq \mathcal{D}(T_\varphi)$.
(ii) $\mathcal{P}$ is not dense in $\mathcal{D}(T_\varphi)$ with respect to the graph norm of T_φ .
(iii) $\mathcal{E} \not\subseteq \mathcal{D}(T_\varphi)$.

PROOF. Throughout the proof we will use the notation: $r(z) := z$ and $\psi_j(z) := \prod_{k=j}^{\infty} (1 - \frac{z}{z_k})$, $j \geq 1$, $z \in \mathbb{C}$.

(i) Given arbitrary $j \geq 1$ we show that $r^j \varphi \in \mathcal{B}$. First observe that

$$\left| \frac{z^j (e^{z^2/2} - 1)}{\prod_{k=1}^{j+2} (1 - \frac{z}{z_k})} \right| \leq C \frac{e^{|z|^2/2}}{(|z|+1)^2}, \quad z \in \mathbb{C},$$

with a properly chosen $C > 0$. Since the right hand side of this inequality is square summable with respect to μ we infer that the function $r^j \varphi \psi_{j+3}$ belongs to $\mathcal{B}$. Applying Theorem 4.1 to ψ_{j+3} yields

$$\|r^j \varphi \psi_{j+3}\| \geq c \|r^j \varphi\|$$

with some $c > 0$. So $r^j \in \mathcal{D}(T_\varphi)$.

(ii) We claim that the function $g(z) := e^{-z^2/4}$ belongs to the domain of T_φ but it cannot be approximated by polynomials with respect to the induced graph norm. Indeed, $g \in \mathcal{D}(T_\varphi)$, because $(g\varphi\psi_1)(z) = e^{z^2/4} - e^{-z^2/4}$ belongs to $\mathcal{B}$ and

$$\|g\varphi\psi_1\| \geq c \|g\varphi\|$$

with some $c > 0$ obtained in virtue of Theorem 4.1.

We are now going to disprove the possibility of approximating g by polynomials in the graph norm of T_φ . Suppose, contrary to our claim, that one can find a sequence $\{p_k\}_{k=1}^{\infty}$ such that $p_k \rightarrow g$ and $p_k \varphi \rightarrow g\varphi$, $k \rightarrow \infty$. We have

$$|p_k(z)\varphi(z)| = |\langle p_k \varphi, e_z \rangle| \leq \|p_k \varphi\| \|e_z\|, \quad z \in \mathbb{C}, k \geq 1.$$

Since the sequence $\{p_k \varphi\}_{k=1}^{\infty}$ is bounded in $\mathcal{B}$, we deduce that

$$|p_k(z)\varphi(z)| \leq D e^{|z|^2/2}, \quad z \in \mathbb{C}, k \geq 1,$$

with some constant $D > 0$. Taking $z = x \in \mathbb{R}$ we get

$$|p_k(x)\varphi(x)| \leq D e^{x^2/2}, \quad x \in \mathbb{R}, k \geq 1,$$

which together with uniform boundedness of the sequence $\{p_k\}_{k=1}^{\infty}$ near the origin (this follows from convergence in $\mathcal{B}$ of the sequence) implies that

$$|p_k(x)| \leq D_1 |\psi_1(x)|, \quad x \in \mathbb{R}, k \geq 1.$$

with a new constant $D_1 > 0$. Consider the function $h_k(z) := p_k(z) e^{-\sqrt[4]{z}}$ defined in $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, where the branch is chosen so that $\sqrt[4]{z} \in [0, \infty)$ for $z \in [0, \infty)$ and $\sqrt[4]{z} \in \{a(1+i) : a \in [0, \infty)\}$ for $z \in (-\infty, 0]$. Observe that every h_k is bounded on the real line. Indeed, this is a consequence of the following inequality

$$|h_k(x)| \leq D_1 e^{-\sqrt[4]{|x|}/2} |\psi_1(x)|, \quad x \in \mathbb{R}, k \geq 1,$$

and the fact that $\psi_1(z)$ is dominated by every function of the form $e^{A|z|^\alpha}$ with $A > 0$ and $\alpha \in (0, 1)$. Since h_k is continuous in $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$, analytic in the interior of this set and bounded by $e^{A\sqrt{|z|}}$ with some $A > 0$, we can apply the Phragmen-Lindelöf principle (cf. [2, Thm. 6.2.4]) to deduce that

$$|h_k(z)| \leq D_2, \quad \operatorname{Im} z > 0, k \geq 1,$$

where $D_2 := \sup\{D_1 e^{-\sqrt[4]{|x|}/2} |\psi_1(x)| : x \in \mathbb{R}\}$. This means that

$$|p_k(z)| \leq D_2 e^{\sqrt[4]{z}}, \quad \operatorname{Im} z > 0, k \geq 1,$$

which is in contradiction with the convergence $p_k(z) \rightarrow e^{-z^2/4}$, $k \rightarrow \infty$.

(iii) We will check that $e_a \notin \mathcal{D}(T_\varphi)$ when $\operatorname{Re} a > 0$. Suppose that $e_a \in \mathcal{D}(T_\varphi)$, then

$$|e_a(z)\varphi(z)| = |\langle e_a \varphi, e_z \rangle| \leq \|e_a \varphi\| e^{|z|^2/2}, \quad z \in \mathbb{C}.$$

It follows that

$$|e^{x\bar{a}}(e^{x^2/2} - 1)| \leq \|e_a\varphi\|e^{x^2/2}|\psi_1(x)|, \quad x \in \mathbb{R},$$

which is a contradiction, since $|\psi_1(x)|$ is dominated by $e\sqrt{|x|}$ on the real axis. The proof is complete.

REMARK 4.3. It is known that if P is any polynomial satisfying

$$|P(x)| \leq M(x), \quad x \in \mathbb{R},$$

where

$$(4.10) \quad \int_{-\infty}^{+\infty} \frac{\log^+ M(x)}{1+x^2} dx < +\infty,$$

then a majorant of exponential growth in $|z|$ can be given for $|P(z)|$ in the whole complex plane (the condition (4.10) is sharp). However, instead of just quoting this theorem, we preferred to give an ad hoc proof for (ii), for the reader's convenience.

The above example was found by the second named author as an attempt to find a “constructive” approach to a result published in [3] (Theorem 2.2), proof of which required existence of an entire function φ such that it belongs to $\mathcal{B}$ with all polynomial multiples, yet some exponential multiple does not belong to $\mathcal{B}$. Later on, it occurred that this function is also a good example for disproving the polynomial approximation property as it is stated in Proposition 4.2(ii). This in turn may be regarded as a contribution to the topic of paper [1], which deals with polynomial approximation in the Segal–Bargmann type spaces. However, it would be far more significant to establish whether there exists an entire function φ satisfying conditions (i) and (ii) of Proposition 4.2 together with $\mathcal{E} \subseteq \mathcal{D}(T_\varphi)$. In this way we would get the answer to the following open problem: is Π_φ always adjoint to T_φ with $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ entire and such that $\varphi e_z \in \mathcal{B}$ for all $z \in \mathbb{C}$?

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