ON ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS GENERATED BY VECTOR FIELDS WHICH DEGENERATE AT THE BOUNDARY OF THE DOMAIN

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Abstract: In this paper we study some elliptic systems of partial differential equations in a bounded domain generated by a vector field which degenerates on the boundary of the domain.

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1. Elliptic systems in a plane domain generated by a given vector field, which degenerates at the boundary

1.1 Let $\big(a(x,y), b(x,y)\big)$ be a real vector field of class C^1 in a bounded domain G of the real (x,y)-plane $\mathbf{R}^2$. We shall consider the following three first order systems in G which are all generated by this vector field:

$$\begin{aligned} &\frac{\partial u}{\partial x} - a(x,y)\,Pu - \frac{\partial v}{\partial y} + b(x,y)\,Pv = f_1(x,y)\,, \\ &\frac{\partial u}{\partial y} - b(x,y)\,Pu - \frac{\partial v}{\partial x} - a(x,y)\,Pv = g_1(x,y)\,, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\frac{\partial u}{\partial x} - a(x,y)\,Pu - \frac{\partial v}{\partial y} = f_2(x,y)\,, \\ &\frac{\partial u}{\partial y} - b(x,y)\,Pu + \frac{\partial v}{\partial x} = g_2(x,y)\,, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + b(x,y)\,Pv = f_3(x,y)\,, \\ &\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - a(x,y)\,Pv = g_3(x,y)\,, \end{aligned} \tag{1.3}$$

where we have used the notation $P :\equiv a(x,y)\,\partial/\partial x + b(x,y)\,\partial/\partial y$. The characteristic form of both system (1.2) and (1.3) is equal to

$$(1-a^2)\,\xi^2 - 2ab\,\xi\eta + (1-b^2)\,\eta^2$$

and the characteristic form of (1.1) is given by

$$\big((1-a^2)\,\xi - ab\,\eta\big)^2 + \big(-ab\,\xi + (1-b^2)\,\eta\big)^2.$$

The systems (1.1)-(1.3) are thus evidently elliptic inside the domain G and degenerate on its boundary $\Gamma = \partial\Omega$ if we assume the vector field to be such that

$$a^2(x,y) + b^2(x,y) < 1\,, \quad (x,y) \in G\,, \tag{1.4}$$

and

$$a^2(x,y) + b^2(x,y) \equiv 1\,, \quad (x,y) \in \Gamma\,. \tag{1.5}$$

Multiplying the second equation by i and adding it to the first, we may replace the system (1.1) by the single equation

$$\frac{\partial w}{\partial \bar{z}} - q(z)\,\frac{\partial w}{\partial z} = f(z)\,, \tag{1.6}$$

for the complex valued function $w(z) = u(x,y) + i\,v(x,y)$, where

$$q(z) = \frac{(a+ib)^2}{2 - a^2 - b^2}\,. \tag{1.7}$$

In the same way both systems (1.2) and (1.3) may be put into the following single equation

$$\frac{\partial w}{\partial \bar{z}} - q_1(z)\frac{\partial w}{\partial z} - q_2(z)\overline{\frac{\partial w}{\partial z}} = f(z) \tag{1.8}$$

where we have the coefficient

$$q_1(z) = \frac{(a+ib)^2}{2(2 - a^2 - b^2)} \tag{1.9}$$

in both systems (1.2) and (1.3), whereas the other coefficient is

$$q_2(z) = \frac{a^2 + b^2}{2(2 - a^2 - b^2)} \tag{1.10}$$

for the system (1.2) and

$$q_2(z) = -\frac{a^2 + b^2}{2(2 - a^2 - b^2)} \tag{1.11}$$

for the system (1.3). From (1.4) and (1.5) it is seen that the Beltrami equation (1.6) as well as equation (1.8) are elliptic inside G and degenerate on the boundary Γ.

Here we point out the following few examples of equation (1.6) in the unit disc $|z| < 1$ obtained from (1.1) for various explicit choices of vector fields. With $a(x,y) \equiv x$ and $b(x,y) \equiv y$ one obtains

$$\frac{\partial w}{\partial \bar{z}} - \frac{z^2}{2 - |z|^2}\,\frac{\partial w}{\partial z} = f(z)\,, \tag{1.12}$$

with

$$a(x,y) \equiv \frac{x}{(1 + \sqrt{1 - x^2 - y^2})^{1/2}}\,, b(x,y) \equiv \frac{y}{(1 + \sqrt{1 - x^2 - y^2})^{1/2}}$$

one gets the equation

$$\frac{\partial w}{\partial \bar{z}} - \frac{z^2}{(1+\sqrt{1-x^2-y^2})^{1/2}}\,\frac{\partial w}{\partial z} = f(z)\,, \tag{1.13}$$

with

$$a(x,y) \equiv \frac{\sqrt{2}x}{\sqrt{1+x^2+y^2}}\,, b(x,y) \equiv \frac{\sqrt{2}y}{\sqrt{1+x^2+y^2}}$$

the equation becomes

$$\frac{\partial w}{\partial \bar{z}} - z^2\frac{\partial w}{\partial z} = f(z), \tag{1.14}$$

and, finally, with

$$a(x,y) \equiv \frac{-\sqrt{2}y}{\sqrt{1+x^2+y^2}}\,, b(x,y) \equiv \frac{\sqrt{2}x}{\sqrt{1+x^2+y^2}}$$

equation (1.6) takes the form

$$\frac{\partial w}{\partial \bar{z}} + z^2\frac{\partial w}{\partial z} = f(z)\,. \tag{1.15}$$

The homogeneous equation corresponding to (1.15) posseses the solution

$$\zeta_1 = \frac{2z}{1+|z|^2}\,, \tag{1.15'}$$

which maps the unit disc $|z| < 1$ onto the unit disc $|\zeta| < 1$ homeomorphically, and in the same way the homogeneous equation corresponding to (1.13) admits the solution

$$\zeta_2 = \frac{z}{1+\sqrt{1-|z|^2}}\,,$$

which again maps the unit disc $|z| < 1$ homoemorphically onto the unit disc $|\zeta| < 1$. As a consequence of this, the general solutions, continuous in the closed disc $|z| \leq 1$, of these two homogeneous equations, are superpositions of an arbitrary holomorphic function $\phi(\zeta)$ in the unit disc $|\zeta| < 1$ and these two respective homeomorphisms: $\phi \circ \zeta_1$ and $\phi \circ \zeta_2$. In contrast to (1.13) and (1.15), the homogeneous equations corresponding to (1.14) and (1.12) do not posses any solutions that map the unit disc onto the unit disc. Instead, they have the respective solutions

$$\zeta = \frac{z}{1-|z|^2}$$

and

$$\zeta = \frac{z}{1+\sqrt{1-|z|^2}}\,,$$

which map the unit disc $|\zeta| < 1$ onto the whole complex ζ-plane $\mathbf{C}$. From this it follows that the equations (1.14) and (1.12) are solvable without any boundary conditions and that the corresponding homogeneous equations have no other solutions, continuous in the closed disc $|z| \leq 1$, except the constant ones. What is the cause for such a difference? At first sight all these equations have the same property: they are elliptic inside the disc $|\zeta| < 1$ and degenerate on the circle $|\zeta| = 1$. One of the reasons for the differences between the equations (1.12)-(1.14) and equation (1.15) might be the circle being the characteristic set of degeneracy for (1.12)-(1.14), while it is not a characteristic set for (1.15). But then - what is the cause for the differences between (1.13) and the two equations (1.12) and (1.14), these three equations all having this circle as their characeristic set? Even in this case there is a cause: the degenerataion for (1.12) and (1.14) is of order one, while for (1.13) it is of order one half, that is, less than one.

1.2 Let us now consider the equations (1.14) and (1.12) in $|\zeta| < 1$ perturbed by lower terms:

$$\frac{\partial w}{\partial \bar{z}} - z^2 \frac{\partial w}{\partial z} + \lambda\, zw = f(z)\,, \tag{1.14$'$}$$

and

$$\frac{\partial w}{\partial \bar{z}} - \frac{z^2}{2-|z|^2}\frac{\partial w}{\partial z} + \lambda \frac{\partial}{\partial z}(\frac{z^2}{2-|z|^2})w = f(z)\,. \tag{1.12$'$}$$

As we have already seen, the corresponding homogeneous equations have no other solutions bounded in $|z| \leq 1$ except the constant ones, and the inhomogeneous equations are unconditionally solvable if $\lambda = 0$. What happens if $\lambda = \alpha + i\beta \neq 0$? Making the change of variables

$$\zeta = \frac{z}{1-|z|^2}\,,$$

we get instead of (1.14$'$) the new equation

$$\frac{\partial w}{\partial \bar{\zeta}} + \frac{2\lambda\,\zeta}{\sqrt{1+4|\zeta|^2}(1+\sqrt{1+4|\zeta|^2})}w = \frac{f(\zeta)}{\sqrt{1+4|\zeta|^2}}$$

in the whole complex ζ-plane $\mathbf{C}$, and this may also be written as the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial w}{\partial \bar{\zeta}} = \frac{f(\zeta)(1+\sqrt{1+4|\zeta|^2})^{\lambda}}{\sqrt{1+4|\zeta|^2}} \tag{1.16}$$

for the function $v(\zeta) = (1+\sqrt{1+4|\zeta|^2})^{\lambda} w$, which grows at infinty $|\zeta| \to \infty$ for $w(z)$ bounded in $|z| \leq 1$ if $\alpha > 0$ and vanishes at infinity if $\alpha < 0$. Hence the inhomogeneous equation (1.16) is solvable and the general solution of the corresponding homogeneous equation is a polynomial of order $m = [\alpha]$:

$$(1+\sqrt{1+4|\zeta|^2})^{\lambda} w(\zeta) = p_m(\zeta) = a_0 + a_1\zeta + \ldots + a_m\zeta^m\,,$$

or in the original variable:

$$w(z) = b_0(1-|z|^2)^{\lambda} + b_1 z(1-|z|^2)^{\lambda-1} + \ldots + b_m z^m (1-|z|^2)^{\lambda-m}\,.$$

From this we conclude the following: if $\mathrm{Re}\,\lambda > 0$, then equation (1.14') is solvable, within the class of functions bounded in $|z| \leq 1$, for any right-hand side, and the corresponding homogeneous equation has exactly $m+1$ linearly independent solutions: $w_k(z) = z^k(1-|z|^2)^{\lambda-k}$, $k = 0, 1, \ldots, m$; if on the other hand $Re\lambda < -1$, then from (1.16) it follows that corresponding homogeneous equation has no non-zero solutions, and that the inhomogeneous equation is solvable if and only if

$$\int_{\mathbf{C}} \frac{f(\zeta)P_{m-2}(\zeta)\, d\mathbf{C}\zeta}{\sqrt{1+4|\zeta|^2}(1+\sqrt{1+4|\zeta|^2})^{-\lambda}} = 0\,. \tag{1.17}$$

Indeed, according to (1.16) the integral on the left-hand side of (1.17) is equal to

$$\int_{\mathbf{C}} \frac{\partial}{\partial \bar{\zeta}} \Big[w(\zeta) \frac{P_{m-2}(\zeta)}{(1+\sqrt{1+4|\zeta|^2})^{-\lambda}} \Big] d\mathbf{C}_\zeta = \lim_{R\to\infty} \int_{|\zeta|<R} \frac{\partial}{\partial \bar{\zeta}} \Big[\frac{w(\zeta)P_{m-2}(\zeta)}{(1+\sqrt{1+4|\zeta|^2})^{-\lambda}} \Big] d\mathbf{C}_\zeta$$

$$= \lim_{R\to\infty} \frac{1}{2i} \int_{|\zeta|=R} \frac{w(\zeta)P_{m-2}(\zeta)\, d\zeta}{(1+\sqrt{1+4|\zeta|^2})^{-\lambda}} = 0\,,$$

because

$$\left| \frac{1}{2i} \int_{|\zeta|=R} \frac{w(\zeta)P_{m-2}(\zeta)\, d\zeta}{(1+\sqrt{1+4|\zeta|^2})^{-\lambda}} \right| \leq \frac{M}{R}\,,$$

where M is a bound for w. But the left-hand side of (1.17) is

$$\int_{|z|<1} \frac{f(z)(1-|z|^2)^{1-\lambda}}{2^{-\lambda}(1+|z|^2)} P_{m-2}(z)\big(|\zeta_z|^2 - |\zeta_{\bar{z}}|^2\big)\, dxdy$$

$$= \int_{|z|<1} f(z)(1-|z|^2)^{-\lambda-2} \Big(b_0 + b_1 \frac{z}{1-|z|^2} + \ldots + b_{m-2} \frac{z^{m-2}}{(1-|z|^2)^{m-2}} \Big)\, dxdy\,,$$

so (1.17) means that the right-hand side $f(z)$ of $(1.14')$ satisfies the equalities

$$\int_{|z|<1} f(z)\, z^k\, (1-|z|^2)^{-\lambda-2-k} dxdy = 0\,, \qquad k = 0, 1, \ldots, m-2\,, \tag{1.18}$$

that is, $f(z)$ is orthogonal to the functions

$$\Psi_k(z) = \bar{z}^k\, (1-|z|^2)^{-\bar{\lambda}-2-k}\,, \qquad k = 0, 1, \ldots, m-2\,,$$

which are solutions to the homogeneous adjoint equation

$$\Psi_z - (\bar{z}\Psi)_{\bar{z}} - \bar{\lambda}\bar{z}\Psi = 0\,.$$

We now turn our attention to equation $(1.12')$, and we begin by making the change of variables

$$\zeta = \frac{z}{\sqrt{1-|z|^2}}$$

which transforms (1.12′) to the equation

$$\frac{\partial w}{\partial \bar\zeta} + \frac{\lambda\zeta(4+3|\zeta|^2)}{2(1+|\zeta|^2)(2+|\zeta|^2)} w = \frac{f(\zeta)(2+|\zeta|^2)}{2(1+|\zeta|^2)^{3/2}}$$

in the entire complex ζ-plane $\mathbf{C}$. This equation can also be written as the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial v}{\partial \bar\zeta} = \frac{f(\zeta)(2+|\zeta|^2)^{\lambda+1}}{2(1+|\zeta|^2)^{\frac{3-\lambda}{2}}}$$

for the function $v(\zeta) = (1+|\zeta|^2)^{\lambda/2}(2+|\zeta|^2)^{\lambda} w(\zeta)$, which grows at infinity $|\zeta| \to \infty$ for bounded w if $\operatorname{Re}\lambda > 0$ and vanishes at infinity if $\operatorname{Re}\lambda < 0$. Hence we can write

$$\begin{aligned} w(\zeta) &= \frac{P_{3m}(\zeta)}{(1+|\zeta|^2)^{\lambda/2}(2+|\zeta|^2)^{\lambda}} = \frac{1}{(1+|\zeta|^2)^{\lambda/2}(2+|\zeta|^2)^{\lambda}}\big(a_0 + a_1\zeta + \ldots + a_{3m}\zeta^{3m}\big) \\ &= \frac{1}{(2-|z|^2)^{\lambda}}\big(a_0(1-|z|^2)^{3\lambda/2} + a_1 z(1-|z|^2)^{\frac{3\lambda-1}{2}} + \ldots + a_{3m} z^{3m}(1-|z|^2)^{\frac{3(\lambda-m)}{2}}\big)\,, \end{aligned}$$

i.e., the homogeneous equation (1.12′) (having $f \equiv 0$) has exactly $3m+1$ linearly independent nontrivial solutions bounded in $|z| \le 1$ if $\operatorname{Re}\lambda > 0$, where $m = [\operatorname{Re}\lambda]$, and the inhomogeneous equation is solvable for any right-hand side, but if $\operatorname{Re}\lambda < -1$ then the homogeneous equation has no nonzero solutions and the inhomogeneous equation (1.12′) is solvable if and only if its right-hand side $f(z)$ satisfies a finite number of orthogonality conditions. Thus, for equations (1.12′) and (1.14′) in the disc $|z| < 1$ the kernel space as well as the co-kernel space are completely described without any boundary conditions.

1.3 For the systems (1.1)-(1.3) the boundary Γ of the domain G will be a characteristic set if the boundary values of the vector field $\big(a(x,y), b(x,y)\big)$ on Γ satisfy the identity

$$a(x,y)\cos(n,y) - b(x,y)\cos(n,x) \equiv 0\,, \quad (x,y) \in \Gamma\,, \tag{1.19}$$

where n denotes the unit outward normal to Γ. This condition means that the vector (a,b) at the boundary points on Γ is directed along the outward unit normal n. The Schwarz problem for the system (1.2) is to find a solution to this system in G, continuous up to the boundary, satisfying the condition

$$u(x,y) = \gamma(x,y)\,, \quad (x,y) \in \Gamma\,, \tag{1.20}$$

where $\gamma(x,y)$ is a continuous function given on Γ.

Theorem 1.1 *If the boundary Γ of the domain G is not characteristic for the system (1.2), i.e., if*

$$a(x,y)\cos(n,y) - b(x,y)\cos(n,x) \neq 0 \tag{1.21}$$

at any point $(x,y) \in \Gamma$, then the Schwarz problem (1.20) for this system is always solvable, and the corresponding homogeneous problem has the only solution $u = 0$, $v = \text{const}$.

Proof: From (1.12) it is easy to see that the function $u(x,y)$ satisfies the following second order equation

$$\Delta u - P^2 u - (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y})Pu = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \tag{1.22}$$

in G, where $\Delta :\equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, $P^2 = PP :\equiv (a\partial/\partial x + b\partial/\partial y)(a\partial/\partial x + b\partial/\partial y)$. Equation (1.22) is elliptic inside of G and degenerates on the boundary ∂G. The characteristic equation arises as a curve satisfying the characteristic equation

$$(1 - a^2(x,y))\Big(\frac{\partial \varphi}{\partial x}\Big)^2 - 2a(x,y)b(x,y)\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y} + (1 - b^2(x,y))\Big(\frac{\partial \varphi}{\partial y}\Big)^2 = 0$$

or

$$\Big(b(x,y)\frac{\partial \varphi}{\partial x} - a(x,y)\frac{\partial \varphi}{\partial y}\Big)^2 = 0\,, \tag{1.23}$$

because of the identity $a^2(x,y) + b^2(x,y) \equiv 1$ on Γ. But in view of (1.21) this means that no point on Γ has a tangent pointing in the characteristic direction. Therefore, any point of the boundary Γ is a regular point, that is, a barrier function exists and hence the Dirichlet problem (1.20) for equation (1.22) can be uniquely solved by the Perron method (see [2], [6]). However, if the condition (1.21) is violated on Γ (or on a part $\Gamma' \subset \Gamma$), then a barrier function may not exist (see [5]), i.e., the Dirichlet problem for equation (1.22) is not solvable.

Theorem 1.2 *If the boundary Γ of the domain G is characteristic, that is, if (1.19) holds at any point $(x,y) \in \Gamma$, then the homogeneous system corresponding to (1.2) has no non-constant continuous solutions in $\bar{G} = G + \Gamma$.*

Proof: Let (u,v) be a solution of system (1.2) with $f = g \equiv 0$, which is continuous in $\bar{G}$. Since $u(x,y)$ satisfies the homogeneous equation corresponding to (1.22), then according to (1.19) we have

$$0 = \int_\Gamma u\big[b(b\cos(n,x) - a\cos(n,y))\frac{\partial u}{\partial x} - a(b\cos(n,x) - a\cos(n,y))\frac{\partial u}{\partial y}\big]\,ds$$

$$- \int_G u(\Delta u - P^2 u - (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y})Pu)\,dxdy$$

$$= \int_\Gamma u\big[((1-a^2)\cos(n,x) - ab\cos(n,y))\frac{\partial u}{\partial x} + ((1-b^2)\cos(n,y) - ab\cos(n,x))\frac{\partial u}{\partial y})\big]\,ds$$

$$- \int_G u(\Delta u - P^2 u - (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y})Pu)\,dxdy$$

$$= \int_\Gamma \big[\frac{\partial}{\partial x}u(\frac{\partial u}{\partial x} - aPu) + \frac{\partial}{\partial x}u(\frac{\partial u}{\partial y} - bPu)\big]\,dxdy - \int_G u(\Delta u - P^2 u - (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y})Pu)\,dxdy$$

$$= \int_G \big[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 - (Pu)^2\big]\,dxdy$$

$$= \int_G \big[(\frac{\partial u}{\partial x} - \frac{a}{1+\sqrt{1-a^2-b^2}} Pu)^2 + (\frac{\partial u}{\partial y} - \frac{b}{1+\sqrt{1-a^2-b^2}} Pu)^2 \big] \, dxdy \,,$$

that is,

$$\Big(1 - \frac{a^2}{1+\sqrt{1-a^2-b^2}}\Big) \frac{\partial u}{\partial x} - \frac{ab}{1+\sqrt{1-a^2-b^2}} \frac{\partial u}{\partial y} = 0 \,,$$

$$\frac{-ab}{1+\sqrt{1-a^2-b^2}} \frac{\partial u}{\partial x} + \Big(1 - \frac{b^2}{1+\sqrt{1-a^2-b^2}}\Big) \frac{\partial u}{\partial y} = 0 \,,$$

and so $\partial u/\partial x = \partial u/\partial y = 0$, because $1 - (a^2+b^2)/(1+\sqrt{1-a^2-b^2}) = \sqrt{1-a^2-b^2} \neq 0$ in G. From this proof we see that the homogeneous Dirichlet problem for the homogeneous equation (1.22) has only the zero solution even in the case when (1.21) holds.

The Schwarz problem with the condition $v(x,y) = \gamma(x,y)$ on Γ for the system (1.3) is always solvable when (1.21) holds, and the corresponding homogeneous problem has the only solution $u = \text{const}, v = 0$. If Γ is characteristic for (1.3) then the homogeneous system (1.3) has no non-constant continuous solutions in $\bar{G}$. As an example we note that the systems (1.2) and (1.3) with $a(x,y) \equiv -y$ and $b(x,y) \equiv x$ satisfy the conditions of Theorem 1.1. in the unit disc $x^2 + y^2 < 1$.

2. Second order equations in a bounded domain of $\mathbf{R}^n$, which are elliptic inside the domain and degenerate on its boundary

Let $a(x) = \big(a_1(x), \ldots, a_n(x)\big)$ be a real vector field of class C^1 given in a bounded domain $\Omega \subset \mathbf{R}^n$. We consider the following second order equation

$$\Delta u - P^2 u + \sum b_k(x) \frac{\partial u}{\partial x_k} + b_0(x) u = f(x) \,, \quad x \in \Omega \,, \tag{2.1}$$

where

$$Pu = \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} \,. \tag{2.2}$$

The principal symbol of (2.1) is $|\xi|^2 - (a(x), \xi)^2$, with $|\xi| = (\sum_{k=1}^n \xi_k^2)^{1/2}$ and $(a(x), \xi) = \sum_{k=1}^n a_k(x) \xi_k$. It follows that equation (2.1) is elliptic in the interior of Ω and degenerate on its boundary $\partial\Omega$ if we assume that

$$|a(x)| < 1 \,, \quad x \in \Omega \,, \tag{2.3}$$

and

$$|a(x)| \equiv 1 \,, \quad x \in \partial\Omega \,, \tag{2.4}$$

where $|a(x)| = (\sum_{k=1}^n a_k^2(x))^{1/2}$. In view of (2.4) there are two possibilities: either

$$a_k(x) \cos(n, x_\ell) - a_\ell(x) \cos(n, x_k) \neq 0 \,, k \neq \ell \,, \quad x \in \partial\Omega \,, \tag{2.5}$$

or else

$$a_k(x) \cos(n, x_\ell) - a_\ell(x) \cos(n, x_k) = 0 \,, k \neq \ell \,, \quad x \in \partial\Omega \,. \tag{2.6}$$

In case (2.5) holds and the $b_j(x)$ are continuous in $\bar{\Omega}$, with $b_0(x) \leq 0$, the Dirichlet problem

$$u(x) = \gamma(x) \tag{2.7}$$

with given continuous data $\gamma(x)$ on $\partial\Omega$ has a unique solution (see [2], [6]), but in case (2.6) holds, there may not always exist a barrier function; this depends also on the lower terms (see [5]).

Let us next consider the homogeneous equation (2.1), i.e., $f(x) \equiv 0$, with $b_0(x) \equiv 0$ and

$$b_k(x) \equiv -a_k(x) \sum_{e=1}^{n} \frac{\partial a_e}{\partial x_e},$$

that is, the equation

$$\Delta u - P^2 u - \sum_{e=1}^{n} \frac{\partial a_e}{\partial x_e} P u = 0\,. \tag{2.8}$$

Theorem 2.1 *If the coefficients $a_k(x)$ of (2.8) satisfy the condition (2.3) in Ω and the conditions (2.4) and (2.6) on $\partial\Omega$, then this equation has no non-constant bounded solutions in $\bar{\Omega}$.*

Proof: Taking into account (2.8) and the equations

$$\sum_{\ell=1}^{n} \big(\cos(n, x_\ell) - a_\ell \sum_{k=1}^{n} a_k \cos(n, x_k)\big) = 0$$

on $\partial\Omega$ following from (2.4) and (2.6), we have

$$\sum_{\ell=1}^{n} \int_{\partial\Omega} u\,\big(\cos(n, x_\ell) - a_\ell \sum_{k=1}^{n} a_k \cos(n, x_k)\big) \frac{\partial u}{\partial x_\ell}\, ds - \int_{\Omega} u\,\big(\Delta u - P^2 u - \sum_{k=1}^{n} \frac{\partial a_k}{\partial x_k} P u\big)\, dx$$

$$= \sum_{k=1}^{n} \int_{\partial\Omega} u\,\big(\frac{\partial u}{\partial x_k} - a_k P u\big) \cos(n, x_k)\, ds - \int_{\Omega} u\,\big(\Delta u - P^2 u - \sum_{k=1}^{n} \frac{\partial a_k}{\partial x_k} P u\big)\, dx$$

$$= \sum_{k=1}^{n} \int_{\Omega} \Big[\frac{\partial}{\partial x_k}\big(u \frac{\partial u}{\partial x_k}\big) - u\, \frac{\partial^2 u}{\partial x_k^2}\Big]\, dx - \int_{\Omega} \Big[\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u\, a_k\, P u) - u\,(P^2 u + \sum_{k=1}^{n} \frac{\partial a_k}{\partial x_k} P u)\Big]\, dx$$

$$= \int_{\Omega} \Big[\sum_{k=1}^{n} (\frac{\partial u}{\partial x_k})^2 - (P u)^2\Big]\, dx \;=\; \sum_{k=1}^{n} \int_{\Omega} \Big(\frac{\partial u}{\partial x_k} - \frac{a_k(x)}{1 + \sqrt{1 - |a(x)|^2}} P u\Big)^2\, dx\,,$$

that is,

$$\sum_{\ell=1}^{n} \Big(\delta_{k\ell} - \frac{a_k(x) a_\ell(x)}{1 + \sqrt{1 - |a(x)|^2}}\Big) \frac{\partial u}{\partial x_\ell} = 0\,, \tag{2.9}$$

where $\delta_{k\ell}$ denotes the Kronecker delta. From (2.9) it now follows that $\partial u/\partial x_\ell = 0$, $1 \le \ell \le n$, because the determinant of the system (2.9) does not vanish in Ω:

$$1 - \frac{|a(x)|^2}{1+\sqrt{1-|a(x)|^2}} = \sqrt{1-|a(x)|^2} \neq 0\,, \quad x \in \Omega\,.$$

It follows from this proof that the homogeneous Dirichlet problem for the equation (2.8) has only the zero solution even in the case when (2.5) holds.

The particular equation

$$\Delta u - E^2 u + \lambda E u = 0 \tag{2.10}$$

in the unit ball $|x| < 1$ in $\mathbf{R}^n$, with $Eu = \sum_{k=1}^n x_k \partial/\partial x_k$ denoting the Euler operator and λ being a complex number, is obtained from (2.1) by putting $b_0(x) \equiv 0$, $b_k(x) = a_k(x) \equiv x_k$. The sphere $|x| = 1$ is a characteristic set for (2.10). The difference between equations for different λ can already be seen in case $n = 1$. For $\lambda = -1$ its general solution is

$$u(x) = c_1 + c_2 \log \sqrt{\frac{1+x}{1-x}}\,,$$

with c_1 and c_2 being constants, so that the only solutions that are bounded in the segment $[-1, 1]$ are the constant ones, and therefore the Dirichlet problem is not well posed. But for $\lambda = 0$ the Dirichlet problem with boundary conditions $u(-1) = a$, $u(1) = b$ has the unique solution

$$u(x) = \frac{a-b}{\pi} \arcsin x + \frac{3b-a}{2}$$

in the segment $[-1, 1]$. Similar conclusions are valid also for $n > 1$. Indeed, the equation (2.10) for $\lambda = -n$ is a particular case of (2.8), with $a_k(x) \equiv x_k$, $1 \le k \le n$, for which the condition (2.6) holds and therefore the constant is the only bounded solution in $|x| \le 1$. Seeking a solution of (2.10) of the form $u(x) = g(|x|^2)u^p(x)$, where $u^p(x)$ is a harmonic polynomial of degree p (so that $Pu^p = pu^p$), we obtain the Gauss hypergeometric equation for $g(t)$, $t = |x|^2$

$$t\,(1-t)\,g''(t) + \Big(\frac{n+2p}{2} - \big(p - \frac{\lambda}{2} + 1\big)\,t\Big)\,g'(t) - \frac{p(p-\lambda)}{4} g(t) = 0\,.$$

For $\lambda > -n$ the series for the hypergeometric function $F\big((n+2p)/2, p/2, (p-\lambda)/2; |x|^2\big)$ converges uniformly in the closed ball $|x| \le 1$, and the solution of the Dirichlet problem with boundary condition $u(x) = u^p(x)\,, |x| = 1$ is given by

$$u_p(x) = \frac{F\big((n+2p)/2, p/2, (p-\lambda)/2; |x|^2\big)}{F\big((n+2p)/2, p/2, (p-\lambda)/2; 1\big)}\, u^p(x) = a_p(|x|^2)\, u^p(x)\,.$$

The solution of the Dirichlet problem with an arbitrary boundary condition $u(x) = f(x)$, $|x| = 1$ can be obtained by superposition through the expansion of $f(x)$ into the series with respect to $u^p(x)$:

$$u(x) = \sum_p A_p(|x|^2)\, u^p(x)\,,$$

i.e., for $\lambda > -n$ the Dirichlet problem $u(x) = f(x)$, $|x| = 1$ for equation (2.10) is uniquely solvable in the unit ball of $\mathbf{R}^n$.

3. Second order equations and first order systems, which are elliptic in a bounded domain of the complex space and degenerate on its boundary

3.1. The following first order overdetermined systems

$$\frac{\partial u}{\partial \bar{z}_k} - z_k \mathcal{R} u + \lambda z_k u = f_k(z)\,, \quad 1 \le k \le n\,, \tag{3.1}$$

and

$$\frac{\partial u}{\partial \bar{z}_k} - \frac{z_k}{2-|z|^2} \mathcal{R} u + \frac{\lambda(2(n+1) - n|z|^2)}{(2-|z|)^2} z_k u = f_k(z)\,, \quad 1 \le k \le n\,, \tag{3.2}$$

with the radial operators $\mathcal{R} :\equiv \sum_{k=1}^n z_k \partial/\partial z_k$ and $\bar{\mathcal{R}} :\equiv \sum_{k=1}^n \bar{z}_k \partial/\partial \bar{z}_k$, are counterparts in the unit ball $|z| < 1$ of $\mathbf{C}^n$ to equations (1.14′) and (1.12′) respectively. We assume that the systems (3.1) and (3.2) are compatible, that is, the right-hand sides of (3.1) should satisfy the conditions

$$\frac{\partial f_k}{\partial \bar{z}_k} - \frac{\partial f_\ell}{\partial \bar{z}_k} = z_\ell \mathcal{R} f_k - z_k \mathcal{R} f_\ell\,, \quad k \ne \ell\,, \tag{3.3}$$

and the right-hand sides of (3.2) should satisfy the conditions

$$\frac{\partial f_k}{\partial \bar{z}_k} - \frac{\partial f_\ell}{\partial \bar{z}_k} = (z_\ell \mathcal{R} f_k - z_k \mathcal{R} f_\ell) \frac{1}{2-|z|^2}\,, \quad k \ne \ell\,. \tag{3.4}$$

Performing the change of variables

$$\zeta_k = \frac{z_k}{1-|z|^2}\,, \quad 1 \le k \le n\,,$$

we replace (3.1) by the system

$$\frac{\partial u}{\partial \zeta_k} + \frac{2\lambda \zeta_k}{\sqrt{1+4|\zeta|^2}(1+\sqrt{1+4|\zeta|^2}} u = \tilde{f}_k\,, \quad 1 \le k \le n\,,$$

with $\tilde{f}_k$ given by

$$\frac{2f_k}{1+\sqrt{1+4|z|^2}} - \frac{4\zeta_k \sum_{\ell=1}^n \bar{\zeta}_\ell f_\ell}{\sqrt{1+4|z|^2}(1+\sqrt{1+4|z|^2})^2}\,, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n\,.$$

This equation can also be written as the inhomogeneous Cauchy–Riemann system

$$\frac{\partial v}{\partial \bar{\zeta}_k} = \tilde{f}_k (1+\sqrt{1+4|z|^2})^\lambda\,, \quad 1 \le k \le n\,, \tag{3.5}$$

with respect to the function $v(\zeta) = (1+\sqrt{1+4|\zeta|^2})^\lambda u(\zeta)$. Since we are seeking a bounded solution u of (3.1) in the closed ball $|z| \le 1$, the system (3.5) for $\lambda = 0$ is always solvable and the corresponding homogeneous system has no other solutions except zero and the same is true for the system (3.1), but if $\operatorname{Re}\lambda > 0$, then (3.5) is solvable only under the conditions (3.3), and a solution of the corresponding homogeneous system is a polynomial of order $m = \bigl[\operatorname{Re}\lambda\bigr]$:

$$u(\zeta) = \frac{P_m(\zeta)}{(1+\sqrt{1+4|\zeta|^2})^\lambda} = \frac{1}{(1+\sqrt{1+4|\zeta|^2})^\lambda} \sum_{|\alpha|\le m} a_\alpha \zeta^\alpha \,,$$

so a solution of system (3.1) with $f_k \equiv 0$ is given by

$$u(z) = \sum_{|\alpha|\le m} a_\alpha z^\alpha \, (1-|z|^2)^{\lambda-|\alpha|} \,, \quad a_\alpha = \mathrm{const}\,. \tag{3.6}$$

If $\operatorname{Re}\lambda < -1$, then v decreases at infinity, so the system (3.5) (together with (3.1)) is solvable if its right-hand sides satisfy besides (3.3) also a finite number of integral conditions of orthogonality type. Turning now to the system (3.2), we make the change of variables

$$\zeta_k = \frac{z_k}{\sqrt{1-|z|^2}} \,, \quad 1 \le k \le n \,,$$

to get instead of (3.2) the system

$$\frac{\partial u}{\partial \bar\zeta_k} + \frac{\lambda \zeta_k \bigl(2(n+1)+(n+2)|\zeta|^2\bigr)}{2(1+|\zeta|^2)(2+|\zeta|^2)}\, u = \tilde f_k = \frac{f_k}{\sqrt{1+[\zeta|^2}} - \frac{\zeta_k \sum_{k=1}^n \bar\zeta_k f_k}{2(1+|\zeta|^2)^{3/2}}$$

in the whole complex space $\mathbf{C}^n$ of the variable $\zeta = (\zeta_1, \dots, \zeta_n)$, and this equation can be written also as

$$\frac{\partial u}{\partial \bar\zeta_k} + \frac{\lambda\zeta_k}{2}\Bigl(\frac{n}{1+|\zeta|^2} + \frac{2}{2+|\zeta|^2}\Bigr) u = \tilde f_k \,,$$

or as the inhomogeneous Cauchy–Riemann system

$$\frac{\partial v}{\partial \bar\zeta_k} = \tilde f_k (1+|\zeta|^2)^{\lambda n/2} (2+|\zeta|^2)^\lambda \tag{3.7}$$

with respect to the function $v = (1+|\zeta|^2)^{\lambda n/2}(2+|\zeta|^2)^\lambda u$. If $\lambda = 0$, then it follows from (3.7) that inhomogeneous system (3.2) is solvable under the conditions (3.4) only, and the corresponding homogeneous system has no non-zero bounded solutions in the closed ball $|z| \le 1$. If $Re\lambda > 0$, then the inhomogeneous system (3.2) is solvable under conditions (3.4) only, and a solution of the corresponding homogeneous system is a polynomial of order $m = [Re\lambda(n+1)]$, but if $Re\lambda < 0$, then the system (3.2) is solvable if its right-hand sides, besides the compatibility conditions (3.4), satisfy also a finite number of integral conditions of orthogonality type.

3.2 Let $a(z)=\big(a_1(z),\ldots,a_n(z)\big)$ be a complex vector field of class C^1 given in a bounded domain Ω in the space $\mathbf{C}^n$ of variables $z=(z_1,\ldots,z_n)$, such that

$$|a(z)|=(\sum_{k=1}^{n}|a_k(z)|^2)^{1/2}<1 \tag{3.8}$$

in the interior of Ω, and

$$|a(z)|\equiv 1 \tag{3.9}$$

on the boundary $\partial\Omega$. We consider the following second order equation

$$\sum_{k=1}^{n}\frac{\partial^2 u}{\partial z_k\partial\bar z_k}-P\bar P u+\sum_{k=1}^{n}\Big(b_k(z)\frac{\partial u}{\partial z_k}+c_k(z)\frac{\partial u}{\partial\bar z_k}\Big)+b_0(z)u=f(z)\,,\quad z\in\Omega\,, \tag{3.10}$$

where $b_j(z)$, $c_k(z)$ and $f(z)$ are given continuous functions in $\bar\Omega$, and

$$P:\equiv\sum_{k=1}^{n}a_k(z)\frac{\partial}{\partial z_k}\,,\quad \bar P:\equiv\sum_{k=1}^{n}\overline{a_k(z)}\frac{\partial}{\partial\bar z_k}\,. \tag{3.11}$$

The principal symbol of (3.10) is

$$|\zeta|^2-|\langle a,\zeta\rangle|^2=|\zeta_1|^2+\ldots+|\zeta_n|^2-|a_1\bar\zeta_1+\ldots+a_n\bar\zeta_n|^2\,,$$

so that according to (3.8) and (3.9) the equation (3.10) is elliptic inside of Ω and degenerate on its boundary $\partial\Omega$. Due to condition (3.9) there are two possibilities: either

$$a_k(z)\frac{\partial\rho}{\partial\bar z_\ell}-a_\ell(z)\frac{\partial\rho}{\partial\bar z_k}\neq 0\,,\quad k\neq\ell \tag{3.12}$$

on the boundary $\partial\Omega$, or else

$$a_k(z)\frac{\partial\rho}{\partial\bar z_\ell}-a_\ell(z)\frac{\partial\rho}{\partial\bar z_k}\equiv 0 \tag{3.13}$$

on $\partial\Omega$, where $\rho(z)$ is the defining function for the domain Ω, i.e., $\Omega=\big\{z\in\mathbf{C}^n\,;\ \rho(z)<0\big\}$, $\partial\Omega=\big\{z\in\mathbf{C}^n\,;\ \rho(z)=0\big\}$, and $\operatorname{grad}\rho(z)\neq 0$ on $\partial\Omega$. In case (3.12) holds it can be shown as before that the Dirichlet problem $u(z)=\gamma(z)$, $z\in\partial\Omega$ with a given continuous $\gamma(z)$ on $\partial\Omega$, is well posed for (3.10), whereas in case of (3.13) the Dirichlet problem is in general not well posed for (3.10). Let us consider the homogeneous equation (3.10), $f\equiv 0$, first with the coefficients $b_j(z)\equiv 0$ and

$$c_k(z)\equiv-\overline{a_k(z)}\sum_{R=1}^{n}\frac{\partial a_\ell}{\partial z_\ell}\,,$$

that is, the equation

$$Hu = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - P\bar{P}u - \sum_{k=1}^{n} \frac{\partial a_k}{\partial z_k} \bar{P}u = 0\,, \tag{3.14}$$

and then with the coefficients $b_0(z) = c_k(z) \equiv 0$, and

$$b_k(z) \equiv -a_k(z) \sum_{\ell=1}^{n} \frac{\partial \bar{a}_\ell}{\partial \bar{z}_\ell}\,,$$

that is, the equation

$$\bar{H}u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - P\bar{P}u - \sum_{k=1}^{n} \frac{\partial \bar{a}_k}{\partial \bar{z}_k} Pu = 0\,. \tag{3.15}$$

Theorem 3.1 *If the coefficients $a_k(z)$ satisfy the condition (3.8) in Ω and the conditions (3.9) and (3.13) on $\partial\Omega$, then equation (3.14) has no other bounded solutions in $\bar{\Omega}$, except functions that are holomorphic in Ω; and equation (3.15) has no other bounded solutions in $\bar{\Omega}$, except functions that are antiholomorphic in Ω.*

Proof: Taking into account (3.14) and the equalities

$$\frac{\partial \rho}{\partial z_\ell} - \overline{a_\ell(z)} \sum_{k=1}^{n} a_k(z) \frac{\partial \rho}{\partial z_k} = 0$$

on $\partial\Omega$, which follow from (3.13) and (3.9), we have

$$0 = \sum_{\ell=1}^{n} \int_{\partial\Omega} \bar{u} \Big(\frac{\partial \rho}{\partial z_\ell} - a_\ell(z) \sum_{k=1}^{n} a_k(z) \frac{\partial \rho}{\partial z_k} \Big) \frac{\partial u}{\partial \bar{z}_\ell}\, ds - \int_\Omega \bar{u}\, Hu\, d\Omega$$

$$= \int_{\partial\Omega} \bar{u} \sum_{k=1}^{n} \big(\frac{\partial u}{\partial \bar{z}_k} - a_k(z)\bar{P}u \big) \frac{\partial \rho}{\partial z_k} ds - \int_\Omega \bar{u} \sum_{k=1}^{n} \big(\frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \frac{\partial}{\partial z_k}(a_k(z)\bar{P}u) \big)\, d\Omega$$

$$= -\sum_{k=1}^{n} \int_\Omega \bar{u} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \sum_{k=1}^{n} \int_\Omega a_k(z) \frac{\partial \bar{u}}{\partial z_k} \bar{P}u\, d\Omega + \sum_{k=1}^{n} \int_\Omega \frac{\partial}{\partial z_k} \big(\bar{u} \frac{\partial u}{\partial \bar{z}_k} \big)\, d\Omega$$

$$= \sum_{k=1}^{n} \int_\Omega \Big| \frac{\partial u}{\partial \bar{z}_k} \Big|^2 d\Omega - \int_\Omega \big| \bar{P}u \big|^2 d\Omega$$

$$= \sum_{k=1}^{n} \int_\Omega \Big| \frac{\partial u}{\partial \bar{z}_k} \Big|^2 d\Omega - \int_\Omega \big(2 - \frac{|a(z)|^2}{1+\sqrt{1-|a(z)|^2}}\big) \frac{1}{1+\sqrt{1-|a(z)|^2}} \big| \bar{P}u \big|^2 d\Omega$$

$$= \sum_{k=1}^{n} \int_\Omega \big(\Big| \frac{\partial u}{\partial \bar{z}_k} \Big|^2 - 2\,\mathrm{Re}\, \frac{a_k(z)}{1+\sqrt{1-|a(z)|^2}} \frac{\partial \bar{u}}{\partial z_k} \bar{P}u + \frac{|a_k(z)|^2}{(1+\sqrt{1-|a(z)|^2}} \big)^2 \big| \bar{P}u \big|^2 \big)\, d\Omega$$

$$= \sum_{k=1}^{n} \int_{\Omega} \Big| \frac{\partial u}{\partial \bar z_k} - \frac{a_k(z)}{1+\sqrt{1-|a(z)|^2}} \bar P u \Big|^2 \, d\Omega \,,$$

that is,

$$\sum_{\ell=1}^{n} \Big(\delta_{k\ell} - \frac{a_k(z)\overline{a_\ell(z)}}{1+\sqrt{1-|a(z)|^2}} \Big) \frac{\partial u}{\partial \bar z_k} = 0 \,,$$

and hence $\partial u / \partial \bar z_k = 0$, $1 \le k \le n$, because the determinant of this last system does not vanish in Ω:

$$1 - \frac{|a(z)|^2}{1+\sqrt{1-|a(z)|^2}} = \sqrt{1-|a(z)|^2} \neq 0 \,, \quad z \in \Omega \,.$$

The first assertion of the Theorem is thereby proved. The second assertion is proved in the same way. The particular case of equation (3.10) with $a_k(z) \equiv z_k$, $b_k(z) \equiv \alpha z_k$, $c_k(z) \equiv \beta z_k$, $1 \le k \le n$, $b_0(z) = \alpha\beta$, $f \equiv 0$, $\alpha, \beta = \text{const}$, that is,

$$\sum_{k,\ell}^{n} \big(\delta_{k\ell} - z_k \bar z_\ell \big) \frac{\partial^2 u}{\partial z_k \partial \bar z_k} + \alpha \mathcal{R} u + \beta \bar{\mathcal{R}} u - \alpha\beta\, u = 0 \tag{3.16}$$

in the unit ball $|z| < 1$, where $\mathcal{R} :\equiv \sum_{k=1}^{n} z_k \partial / \partial z_k$ and $\bar{\mathcal{R}} :\equiv \sum_{k=1}^{n} \bar z_k \partial / \partial \bar z_k$ are radial operators, appears in [1]. In this case the conditions (3.13) are obviously fulfilled, i.e., the sphere $|z| = 1$ is the characteristic set for (3.16). As is shown in [1], if $\operatorname{Re}(n + \alpha + \beta) > 0$ and neither $n + \alpha$ nor $n + \beta$ is zero, then the Dirichlet problem for (3.16) is uniquely solvable, though the solution is not as smooth as in the case of the Dirichlet problem for equations that are elliptic in the whole domain $\bar\Omega$, but if one of the above conditions is violated, then the Dirichlet problem is not well posed: the function $u(z) = \big(1 - |z|^2\big)^{\lambda}$ for example is a solution of the equations

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar z_k} - \mathcal{R}\bar{\mathcal{R}} u + \lambda\, \mathcal{R} u - n\, \bar{\mathcal{R}} u + \lambda\, n\, u = 0 \,,$$
$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar z_k} - \mathcal{R}\bar{\mathcal{R}} u - n\, \mathcal{R} u + \lambda\, \bar{\mathcal{R}} u + \lambda\, n\, u = 0 \,,$$

in the unit ball $|z| < 1$ vanishing ($\operatorname{Re}\lambda > 0$) on the sphere $|z| = 1$.

3.3 Let $\big(a(z), b(z)\big)$ be a complex valued vector field of class C^1 given in a bounded domain Ω of $\mathbf{C}^2$ such that

$$\big|a(z)\big|^2 + \big|b(z)\big|^2 < 1 \tag{3.17}$$

inside Ω and

$$\big|a(z)\big|^2 + \big|b(z)\big|^2 \equiv 1 \tag{3.18}$$

on the boundary $\partial\Omega$. The two first order systems

$$\frac{\partial u}{\partial \bar z_1} - a(z)\, \bar P u - \frac{\partial v}{\partial z_2} = f(z) \,, \quad \frac{\partial u}{\partial \bar z_2} - b(z)\, \bar P u + \frac{\partial v}{\partial z_1} = g(z) \,, \tag{3.19}$$

and

$$\frac{\partial u}{\partial \bar z_1} - \frac{\partial v}{\partial z_2} + \overline{b(z)} P v = f(z)\,, \quad \frac{\partial u}{\partial \bar z_2} + \frac{\partial v}{\partial z_1} - \overline{a(z)} P v = g(z)\,, \tag{3.20}$$

where

$$P :\equiv a(z)\,\frac{\partial}{\partial z_1} + b(z)\,\frac{\partial}{\partial z_2}\,, \quad \bar P :\equiv \overline{a(z)}\,\frac{\partial}{\partial \bar z_1} + \overline{b(z)}\,\frac{\partial}{\partial \bar z_2}\,,$$

both have the same principal symbol

$$|\zeta_1|^2 + |\zeta_2|^2 - |a(z)\bar\zeta_1 + b(z)\bar\zeta_2|^2\,,$$

so that both systems are elliptic in Ω and degenerate on $\partial\Omega$. From (3.17) it follows that the function $u(z)$ satisfies the second order equation

$$\frac{\partial^2 u}{\partial z_1 \partial z_1} + \frac{\partial^2 u}{\partial z_2 \partial \bar z_2} - P\bar P u - \Big(\frac{\partial a}{\partial z_1} + \frac{\partial b}{\partial z_2}\Big)\bar P u = \frac{\partial f}{\partial z_1} + \frac{\partial g}{\partial z_2} \tag{3.21}$$

and from (3.18) it follows that the function $v(z)$ satisfies the second order equation

$$\frac{\partial^2 v}{\partial z_1 \partial \bar z_1} + \frac{\partial^2 v}{\partial z_2 \partial \bar z_2} - P\bar P v - \Big(\frac{\partial \bar a}{\partial \bar z_1} + \frac{\partial \bar b}{\partial \bar z_2}\Big) P v = \frac{\partial g}{\partial \bar z_1} - \frac{\partial f}{\partial \bar z_2}\,. \tag{3.22}$$

Theorem 3.2 *If the vector field $\big(a(z), b(z)\big)$ satisfies (3.17) in Ω, together with the boundary conditions (3.18) and $a(z)\,\partial\rho/\partial\bar z_2 - b(z)\,\partial\rho/\partial\bar z_1 \neq 0$ on $\partial\Omega$, then there exists a unique solution (u,v) of the system (3.19) in Ω, satisfying the condition $u(z) = \gamma(z)$ on $\partial\Omega$ and with $v(z)$ lying in the orthogonal complement of the subspace of functions consisting of functions that are antiholomorphic in Ω. There also exists a unique solution (u,v) of the system (3.20) such that $v(z) = \gamma(z)$ on $\partial\Omega$ and with $u(z)$ lying in the orthogonal complement of the subspace consisting those functions that are holomorphic in Ω.*

Proof: Equation (3.21) and the condition $a(z)\,\partial\rho/\partial\bar z_2 - b(z)\,\partial\rho/\partial\bar z_1 = 0$ coincide with (3.14) and (3.13) for $n = 2$, while (3.22) and the condition $a(z)\,\partial\rho/\partial\bar z_2 - b(z)\,\partial\rho/\partial\bar z_1 \neq 0$ coincide with (3.15) and (3.12) for $n = 2$. Therefore the functions $u(z)$ and $v(z)$ are uniquely determined as solutions of the Dirichlet problem. Then the function $v(z)$ is determined from the anti Cauchy–Riemann system following from (3.19) by the condition $v(z) \perp \bar H(\Omega)$ and the function $u(z)$ is determined from the Cauchy–Riemann system following from (3.20) by the condition $u(z) \perp H(\Omega)$, where $H(\Omega)$ and $\bar H(\Omega)$ denote the subspaces of functions that are holomorphic respectively antiholomorphic in Ω (see [3]).

Theorem 3.3 *If the vector field $\big(a(z), b(z)\big)$ satisfies (3.17) in Ω, together with the boundary conditions (3.18) and $a(z)\,\partial\rho/\partial\bar z_2 - b(z)\,\partial\rho/\partial\bar z_1 = 0$ on $\partial\Omega$, then the homogeneous systems (3.19), (3.20) $f = g \equiv 0$ have no other continuous solutions in $\bar\Omega$ except $\big(\varphi(z), \bar\psi(z)\big)$, where φ and ψ are arbitrary holomorphic functions in Ω.*

Proof: Since equation (3.21) and condition $a(z)\,\partial\rho/\partial\bar z_2 - b(z)\,\partial\rho/\partial\bar z_1 = 0$ on $\partial\Omega$ coincide with equation (3.14) and conditions (3.13) for $n = 2$ and equation (3.22) and above condition coincides with equation (3.15) and conditions (3.13) for $n = 2$, it follows that the result is a consequense of Theorem 3.1. As an example note that the systems (3.19) and

(3.20) with $a(z) \equiv z_1$, $b(z) \equiv z_2$ in the ball $|z_1|^2 + |z_2|^2 < 1$ of $\mathbf{C}^2$ satisfy the conditions of Theorem 3.3, while the same systems with $a(z) \equiv \bar{z}_2$, $b(z) \equiv -\bar{z}_1$ satisfy the conditions of Theorem 3.2.

3.4 Let us consider now two more first order systems in $\mathbf{C}^3$ which can be treated in the same way. Let $a(z) = \big(a_1(z), a_2(z), a_3(z)\big)$ be a complex valued vector field of class C^1 given in a bounded domain Ω of $\mathbf{C}^3$ such that

$$|a(z)| = \big(|a_1(z)|^2 + |a_2(z)|^2 + |a_3(z)|^2\big)^{1/2} < 1 \tag{3.23}$$

in the interior of Ω and $|a(z)| \equiv 1$ on the boundary $\partial\Omega$. We introduce the operators

$$\bar{\partial}_k :\equiv \frac{\partial}{\partial \bar{z}_k} - a_k(z)\,\bar{P}\,, \quad \partial_k :\equiv \frac{\partial}{\partial z_k} - \overline{a_k(z)}\,P\,, \quad k = 1, 2, 3\,,$$

where P and $\bar{P}$ are defined by (3.11) for $n = 3$. The first system

$$\begin{aligned} \operatorname{div}_{\bar{z}} u = 0\,, \\ \bar{\partial} u_0 + \operatorname{rot}_z u = 0\,, \end{aligned} \tag{3.24}$$

where $\operatorname{div}_{\bar{z}} u = \sum_{k=1}^{3} \partial u_k / \partial \bar{z}_k$, $\bar{\partial} :\equiv (\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3)$ and

$$\operatorname{rot}_z u = \Big(\frac{\partial u_3}{\partial z_2} - \frac{\partial u_2}{\partial z_3}, \frac{\partial u_1}{\partial z_3} - \frac{\partial u_3}{\partial z_1}, \frac{\partial u_2}{\partial z_1} - \frac{\partial u_1}{\partial z_2}\Big)$$

has principal symbol $-\big(|\zeta|^2 - |\langle a, \zeta\rangle|^2\big)\,|\zeta|^2$ and is therefore elliptic inside of Ω and degenerates on $\partial\Omega$. The second system

$$\begin{aligned} \operatorname{div}_{\bar{z}} u = 0\,, \\ \operatorname{grad}_{\bar{z}} u_0 + \big[\partial \times u\big] = 0\,, \end{aligned} \tag{3.25}$$

where $\operatorname{grad}_{\bar{z}} u_0 = \big(\partial u_0/\partial \bar{z}_1, \partial u_0/\partial \bar{z}_1, \partial u_0/\partial \bar{z}_1\big)$ and

$$\big[\partial \times u\big] = \big(\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1\big)$$

has principal symbol $-\big(|\zeta|^2 - |\langle a, \zeta\rangle|^2\big)^2$ and therefore is also elliptic inside of Ω and degenerates on $\partial\Omega$. The function $u_0(z) = \varphi(z)$ and the vector-function $u(z) = \big(u_1(z), u_2(z), u_3(z)\big)$ with φ being an arbitrary holomorphic function in Ω and u an arbitrary solution of the elliptic system

$$\operatorname{div}_{\bar{z}} u = 0\,, \quad \operatorname{rot}_z u = 0 \tag{3.26}$$

will evidently satisfy the system (3.24). Conversely, we have the following result.

Theorem 3.4 *If the vector field $a(z)$ satisfies (3.23) in Ω, together with the conditions $|a(z)| \equiv 1$ and $\big[a(z) \times \operatorname{grad}_{\bar{z}} \rho(z)\big] = 0$ on $\partial\Omega$, then any continuous solution (u_0, u) of (3.24)* *in $\bar{\Omega}$ is of the form $\big(\varphi(z)\, u(z)\big)$, where $\varphi(z)$ is an arbitrary holomorphic function in Ω and $u(z)$ is an arbitrary solution of the system (3.26).*

Proof: Applying $\partial/\partial z_1$ to the second line, $\partial/\partial z_2$ to the third line, and $\partial/\partial z_3$ to fourth line of (3.24) we get

$$\Big(\frac{\partial}{\partial z_1 \partial \bar{z}_1} + \frac{\partial}{\partial z_2 \partial \bar{z}_2} + \frac{\partial}{\partial z_3 \partial \bar{z}_3}\Big) u_0 - P\bar{P}u_0 - \Big(\frac{\partial a_1}{\partial z_1} + \frac{\partial a_2}{\partial z_2} + \frac{\partial a_3}{\partial z_3}\Big)\bar{P}u_0 = 0$$

Since $\big[a(z) \times \mathrm{grad}_{\bar{z}}\rho(z)\big] = 0$ on $\partial\Omega$, then by Theorem 3.1 for the case $n = 3$, we have $u_0(z) \equiv \varphi(z)$ and then (3.24) is reduced to (3.26). Let us finally remark that it is also interesting to study properties of system (3.5).

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