ALGEBRAIC ASPECTS OF THE DIRICHLET PROBLEM

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1. Introduction

Our aim in this paper is to make some remarks about certain algebraic aspects of the classical Dirichlet problem for the Laplace operator

$$\Delta := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, $n \geq 2$. Thus, letting $\mathbb{S}^{n-1} := \partial \mathbb{B}^n$ denote the unit sphere, we pose the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{B}^n \\ u = f, & \text{on } \mathbb{S}^{n-1}, \end{cases} \tag{1.1}$$

where the Dirichlet data f is a continuous function on the sphere $\mathbb{S}^{n-1}$. It will be convenient for us to assume, without loss of generality of course, that the data function f is defined in an open neighborhood of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The solution u is given by the Poisson integral

$$u(x) = \int_{\mathbb{S}^{n-1}} f(y) K_n(y, x) dS_y, \tag{1.2}$$

where dS_y is the surface area measure of the sphere and $K_n(x)$ is the Poisson kernel

$$K_n(x) = \omega_n \frac{1 - \|x\|^2}{\|y - x\|^n} \tag{1.3}$$

and ω_n is the reciprocal of the surface area of the sphere $\mathbb{S}^{n-1}$. Many classical analytic properties of the solution u (such as e.g. regularity up to the boundary) can be gleaned from this formula. In this paper, we shall address some questions of algebraic nature which do not seem to follow as easily from (1.2).

We take as our starting point the following well known result: *If $f(x)$ is a polynomial of degree m, then the solution $u(x)$ to the Dirichlet problem* (1.1) *is a polynomial of degree at most m.* The precise origin of this result is difficult to trace (see e.g. [F] for the result in $\mathbb{R}^3$), but a simple and elementary proof based on linear algebra can be found in e.g. [K], [KS], and [Sh] (where the conclusion is in fact proved for the Dirichlet problem in an arbitary ellipsoid). In light of this basic result, it seems natural to ask if the solution operator preserves wider classes of algebraic functions:

Question A. Assume that the data $f(x)$ is a *rational* function without poles on $\mathbb{S}^{n-1}$. Is then the solution $u(x)$ of the Dirichlet problem (1.1) necessarily rational?

Professor W. Ross' student Mr. T. Fergusson (private communication) has answered Question A in the affirmative for $n = 2$. (This also follows from a more general result due to the first author [E]; a

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very short and simple proof is given in section 3.) In the general form Question A was posed to us by Fergusson and Ross (written communication). More generally, one may ask the following.

Question B. Assume that the data $f(x)$ is an *algebraic* function which is analytic in a neighborhood of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Is then the solution $u(x)$ of the Dirichlet problem (1.1) necessarily algebraic?

Somewhat surprisingly perhaps, even though the answer to Question A is affirmative for $n = 2$, it is negative for all odd $n \geq 3$ and, at least, for all even n with $4 \leq n \leq 270$ (see Theorem 2.2 and the remark following it). The answer to Question B is negative in all dimensions. Our main results are examples of rational data functions f in $\mathbb{R}^n$, for all odd $n \geq 3$ (Theorem 2.1) and even $n \geq 4$ such that an additional condition is satisfied (Theorem 2.2), which yield non-rational—even non-algebraic—solutions to the Dirichlet problem (1.1), but also some positive results (Theorems 2.5 and 5.1) on Question A when f belongs to certain subclasses of rational functions in $\mathbb{R}^n$ with *n even.* We also give examples (Theorem 2.4) of algebraic data functions in $\mathbb{R}^2$ for which the solutions are non-algebraic.

The paper is organized as follows. In section 2, we state our main results more precisely. In section 3, we discuss the Dirichlet problem in two dimensions, its relation to the problem of Laurent decomposition, and give a proof of Theorem 2.4. The proofs of the remaining Theorems are then given in sections 4 and 5. As concluding remarks, we discuss in section 6 a connection with ultraspherical polynomials and the "Nehari transform".

2. Main results

Our first two results provide very simple rational functions $f(x)$ in $\mathbb{R}^n$, $n \geq 3$ (moreover if n is even the further arithmetical condition (2.2) is assumed), for which the solutions to (1.1) are not rational, or even algebraic, showing that the answers to both Questions A and B above are negative for these values of $n \geq 3$.

Theorem 2.1. *Let $|a| > 1$ and $n = 2k + 1 \geq 3$ be odd. Then, the solution $u(x)$ to the Dirichlet problem (1.1) in $\mathbb{R}^n$ with data*

$$f(x) = \frac{1}{(x_1 - a)^{2k-1}} \tag{2.1}$$

is not algebraic.

When $n = 4$, the solution to (1.1) with data given by (2.1) turns out to be rational (cf. Theorem 2.5 below). However, we have the following for the even dimensional Dirichlet problem.

Theorem 2.2. *Let $n = 2k \geq 4$ be even and assume that*

$$\sum_{\substack{0 \leq j \leq k-1 \\ k+j \text{odd}}} \sum_{l=j}^{k-1} \left(-\frac{1}{2}\right)^l \binom{k-1}{l} \binom{l+k-1}{l} \binom{l}{j} (-i)^j \neq 0, \tag{2.2}$$

Then, the solution $u(x)$ to the Dirichlet problem (1.1) in $\mathbb{R}^n$ with data

$$f(x) = \frac{1}{(1 + x_1^2 + x_2^2)^{k-1}} \tag{2.3}$$

is not algebraic.

Remark 2.3. The authors suspect, but have been unable to prove, that (2.2) holds for all $k \geq 2$. At least, we have verified, using the software package MatLab, that (2.2) holds for all $2 \leq k \leq 135$. The modulus of the left hand side grows steadily, although not quite monotonically, as k grows in this range. For $k = 2$, the left hand side is i, and for $k = 135$ it is of the order 10^{80}. Thus, the solution in $\mathbb{R}^n$ with data (2.3) is not algebraic, at least for all even n with $4 \leq n \leq 270$.

As mentioned above, the answer to Question A when $n = 2$ is affirmative (see e.g. section 3 below for a simple proof). However, the answer to Question B is negative, as is shown by the following result. We shall identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual way via $z = x_1 + ix_2$.

Theorem 2.4. *Let $w \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$. Then, the solution $u(x_1, x_2)$ to the Dirichlet problem* (1.1) *in $\mathbb{R}^2 \cong \mathbb{C}$ with data*

$$f(x_1, x_2) = |z - w|, \quad z = x_1 + ix_2, \tag{2.4}$$

is not algebraic.

Theorem 2.4 will follow from a more general result about Laurent decompositions in section 3.

We shall conclude this section by giving a positive result on Question A for axially symmetric rational data functions in $\mathbb{R}^4$. Recall that a function $f(x)$ in $\mathbb{R}^n$ is called *axially symmetric* if there is a unit vector $v \in \mathbb{R}^n$ such that

$$f(Ax) = f(x), \quad \forall A \in O_v(\mathbb{R}^n), \tag{2.5}$$

for all $x \in \mathbb{R}^n$ where both sides are defined. Here, $O_v(\mathbb{R}^n)$ denotes the subgroup of the orthogonal linear group $O(\mathbb{R}^n)$ which leaves the vector v invariant; the line spanned by v is called the axis of symmetry.

Theorem 2.5. *Consider the Dirichlet problem* (1.1) *in $\mathbb{R}^4$. If the data $f(x)$ is an axially symmetric rational function, then the solution is a rational (axially symmetric) function.*

Theorem 2.5 will follow from a more general, but slightly more technical result (Theorem 5.1 in section s-axsym below) for axially symmetric rational data functions in $\mathbb{R}^n$ for even n.

3. The two dimensional case

We identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way via $z = x_1 + ix_2$, so that $\Delta = 4\,\partial^2/\partial z \partial \bar{z}$. Hence, any harmonic (not necessarily real-valued) function $u(x) = u(z, \bar{z})$ in a simply connected domain $\Omega \subset \mathbb{C}$ is of the form $u(z, \bar{z}) = h(z) + \overline{g(z)}$, where $h, g \in \mathcal{O}(\Omega)$ and $\mathcal{O}(\Omega)$ denotes the space of analytic functions in Ω. If $f = f(z, \bar{z})$ is a (possibly complex-valued) real-analytic function in a neighborhood of $\mathbb{S}^1$ and u is the solution to the Dirichlet problem (1.1), with $n = 2$, then it is well known that u extends real-analytically to an open neighborhood of the closed unit disk $\overline{\mathbb{D}}$; we prefer to use the notation $\mathbb{D}$ rather than $\mathbb{B}^2$ for the open unit disk in $\mathbb{C}$. Thus, we have $u(z, \bar{z}) = h(z) + \overline{g(z)}$ for $h, g \in \mathcal{O}(\overline{\mathbb{D}})$ (i.e. h, g are analytic in some neighborhood of the closed unit disk $\overline{\mathbb{D}}$) and we have

$$f(z, \bar{z}) = h(z) + \overline{g(z)}, \quad z \in \mathbb{S}^1. \tag{3.1}$$

The functions h and g will be uniquely determined if we require e.g. $g(0) = 0$. Observe that on $\mathbb{S}^1$ we have $\bar{z} = 1/z$. Thus, if we write

$$F(z) := f(z, 1/z), \tag{3.2}$$

then $F(z)$ is analytic in some annular neighborhood of the unit circle $\mathbb{S}^1$ and, in view of (3.1), we have the identity

$$F(z) = h(z) + \bar{g}(1/z) \tag{3.3}$$

in some open neighborhood of $\mathbb{S}^1$, where $\bar{g}(w)$ is the analytic function

$$\bar{g}(w) := \overline{g(\bar{w})}.$$

This illustrates the connection between the Dirichlet problem in two dimensions and the problem of *Laurent splitting* of analytic functions. Recall that if $F(z)$ is an analytic function in a neighborhood of $\mathbb{S}^1$, then it has a unique series expansion

$$F(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \left(\frac{1}{z}\right)^k, \tag{3.4}$$

where the first series converges in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and the second series converges in a neighborhood of the closed complement $\mathbb{C} \setminus \mathbb{D}$ (with the limit 0 at ∞). Hence, if we set

$$h(z) := \sum_{k=0}^{\infty} a_k z^k, \quad g(w) = \sum_{k=1}^{\infty} \bar{b}_k w^k, \tag{3.5}$$

then we obtain a unique splitting (3.3) of $F(z)$, for z in some neighborhood of $\mathbb{S}^1$, with $h, g \in \mathcal{O}(\overline{\mathbb{D}})$ and $g(0) = 0$; we shall refer to (h, g) as the *Laurent pair* of F. Thus, we have proved the following.

Proposition 3.1. *Let $F(z)$ be an analytic function in some open neighborhood of the unit circle $\mathbb{S}^1$ and $f(z, \bar{z})$ a (possibly complex-valued) real-analytic function such that* (3.2) *holds. If $h, g \in \mathcal{O}(\overline{\mathbb{D}})$ with $g(0) = 0$, then (h, g) is the Laurent pair of F (i.e.* (3.3) *holds) if and only if*

$$u(z, \bar{z}) := h(z) + \bar{g}(\bar{z})$$

is the solution of the Dirichlet problem (1.1) *with $n = 2$ and data $f = f(z, \bar{z})$.*

Thus, we can, and we shall, reformulate and prove Theorem 2.4 regarding the two-dimensional Dirichlet problem in terms of Laurent splittings. First, however, let us state the following proposition which is equivalent to the statement in the introduction that the solution to the Dirichlet problem with rational data is rational. We shall also give a very short and simple proof. The proof by T. Fergusson, also not difficult, was based on the Cauchy residues theorem [Fe, written communication].

Proposition 3.2. *Let F be analytic in a neighborhood of $\mathbb{S}^1$ and let (h, g) the its Laurent pair (as defined above). If F is rational, then so are h and g.*

Proof. Since F is rational (or, equivalently, a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$) and g is analytic in a neighborhood of $\overline{\mathbb{D}}$, we can extend h, originally analytic in a neighborhood of $\overline{\mathbb{D}}$, as a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$ by defining it for $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by

$$h(z) = F(z) - \bar{g}(1/z),$$

where of course $z = \infty$ corresponds to $1/z = 0$. This implies that h is rational, and hence also g, being the difference between two rational functions, is rational. □

In order to prove Theorem 2.4, we observe that the data $f(z, \bar{z}) = |z - w| = \sqrt{(z - w)(\bar{z} - \bar{w})}$, with $w \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$, corresponds to

$$F(z) = f(z, 1/z) = \sqrt{\frac{(z - w)(1 - z\bar{w})}{z}}. \tag{3.6}$$

Thus, in view of Proposition 3.1, Theorem 2.4 follows from the following more general theorem about Laurent splittings.

Theorem 3.3. *Let F be analytic in a neighborhood of $\mathbb{S}^1$ and assume that F is of the form*

$$F(z) = \sqrt{R(z)}, \tag{3.7}$$

where R is a rational function with at least one simple zero in $\mathbb{D}$ and at least one simple zero in $\mathbb{C} \setminus \overline{\mathbb{D}}$. If (h, g) is the Laurent pair of F, then neither h nor g is algebraic.

Remark 3.4. Observe that the assumption that F is analytic in a neighborhood of $\mathbb{S}^1$ restricts the possible choices of rational functions R. For instance, R cannot have poles or simple zeros on $\mathbb{S}^1$. Moreover, R must have an even number of zeros and poles (counted with multiplicities) in $\mathbb{D}$.

Proof. Let $a \in \mathbb{D}$ and $b \in \mathbb{C} \setminus \overline{\mathbb{D}}$ be simple zeros of R, and let L be a smooth curve from a to b which does not intersect any other poles or zeros of R. We shall, as we may, take L to be the shortest arc of a circle with large radius through a and b. Let U be a small disk centered at the point of intersection between L and $\mathbb{S}^1$. We shall choose U so small that F, h, and g are all analytic in U. We shall also let $U_1 \subset \mathbb{D}$ and $U_2 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ be disks centered at a and b respectively, so small that R has no poles or zeros, except a and b, inside U_1 and U_2. Now define the closed curve γ to be the piece L' of L going from the point of intersection $L \cap \partial U_1$ to the point $L \cap \partial U_2$, around the circle ∂U_2 in the positive direction, back along $-L'$, and finally around the circle ∂U_1 in the positive direction. Observe that in a neighborhood of γ, by the construction, there is an analytic function $A(z)$ such that

$$F(z) = \sqrt{(z-a)(z-b)}A(z). \tag{3.8}$$

Let us start in U and continue the function $h(z)$ analytically along the portion of γ contained in $\mathbb{C} \setminus \overline{\mathbb{D}}$ until we return to U by using the identity $h(z) = F(z) - \bar{g}(1/z)$. Note that the analytic continuation of $F(z)$ changes sign as we go around b whereas $1/z$ stays inside $\overline{\mathbb{D}}$ where $\bar{g}$ is analytic. Thus, if we denote by $\tilde{h}(z)$ the analytic function in U obtained by this analytic continuation, then we conclude that $\tilde{h}(z) = -F(z) + \bar{g}(1/z)$. Hence, we have

$$\tilde{h}(z) = h(z) - 2F(z)$$

in U. If we use this identity to continue the new branch $\tilde{h}(z)$ in U along the remaining portion of γ inside $\mathbb{D}$ (where $h(z)$ is analytic) until we come back to U—and have completed a full tour of γ—then $F(z)$ changes sign again (since we go around a). The conclusion is that if we denote by $h_1(z)$ the analytic function in U obtained by analytically continuing $h(z)$ along the closed curve γ, then

$$h_1(z) = h(z) + 2F(z). \tag{3.9}$$

Also, as we have already seen, analytic continuation of the function $F(z)$ in U around γ leads to the same function $F(z)$ (since we change sign twice). Thus, it follows from (3.9) that the analytic continuation $h_k(z)$ of $h(z)$ around $k\gamma$ (i.e. k times around γ) satisfies

$$h_k(z) = h(z) + 2kF(z).$$

In particular, repeated analytic continuation of $h(z)$ yields an *infinite number* of different branches over U. This means, of course, that $h(z)$ cannot be an algebraic function. Consequently, g can also not be algebraic. This completes the proof. □

4. Proof of Theorems 2.1 and 2.2

The strategy in proving Theorems 2.1 and 2.2 will be to first show that the solutions to the corresponding Dirichlet problems, restricted to the line $x = (s, 0, \ldots, 0)$, can be expressed by integrals of a certain type. The proofs then reduce to showing that these integrals cannot be algebraic functions of s. Our starting point will be the Poisson integral (1.2) for the solution of (1.1) in $\mathbb{R}^n$. If we assume that the data function f is a function of x_1 alone, as in Theorem 2.1, then by using cylindrical coordinates $y = (t, r\xi)$, where $\xi \in \mathbb{S}^{n-2}$, $t \in [-1, 1]$ and $r = \sqrt{1-t^2}$, we obtain

$$u(x) = \omega_n(1 - \|x\|^2) \int_{-1}^{1} g(t) r^{n-3} \left(\int_{\mathbb{S}^{n-2}} \frac{dS_\xi}{((t-x_1)^2 + \sum_{j=2}^{n}(r\xi_j - x_j)^2)^{n/2}} \right) dt, \tag{4.1}$$

where $g = g(t)$ is the function of one variable such that $f(x) = g(x_1)$. As mentioned above, to prove Theorem 2.1 we shall show that the restriction of $u(x)$ to the axis of symmetry $x = (s, 0, \ldots, 0)$ is not algebraic. The function inside the integral over $\mathbb{S}^{n-2}$ then reduces to

$$\begin{aligned}\frac{1}{((t-s)^2 + \sum_{j=2}^n (r\xi_j)^2)^{n/2}} &= \frac{1}{((t-s)^2 + r^2)^{n/2}} \\ &= \frac{1}{(2s)^{n/2}} \frac{1}{(\phi(s) - t)^{n/2}},\end{aligned} \tag{4.2}$$

where in the first step we used that $\xi \in \mathbb{S}^{n-2}$, and in the second that $r^2 = 1 - t^2$; we also use the notation

$$\phi(s) = \frac{1}{2}\left(s + \frac{1}{s}\right). \tag{4.3}$$

Observe that the expression in (4.2), which appears as an integrand in (4.1), is in fact independent of ξ. Thus, we arrive at the following result.

Proposition 4.1. *Let $u(x)$ be the solution in $\mathbb{R}^n$ to the Dirichlet problem* (1.1) *in which the data function f is a function of x_1 alone, i.e. $f(x) = g(x_1)$. If we set $v(s) := u(s, 0, \ldots, 0)$, with $s \in (0, 1)$, then we have, for some constant c_n depending only on n,*

$$v(s) = c_n \frac{1 - s^2}{s^{n/2}} P_n(g)(\phi(s)), \tag{4.4}$$

where ϕ is given by (4.3) *and P_n is the integral operator*

$$P_n(g)(z) := \int_{-1}^{1} g(t)(1 - t^2)^{(n-3)/2} \frac{dt}{(z - t)^{n/2}}. \tag{4.5}$$

To show that $u(x)$ is not algebraic, it suffices to show that $P_n(g)(z)$ is not an algebraic function of z. Before proceeding to do this, we shall derive a similar, but slightly more complicated, integral formula for the solution $u(x)$ when the data function, as in Theorem 2.2, is a function of $x_1^2 + x_2^2$. Thus, assume that $f(x) = g(x_1^2 + x_2^2)$, where $g = g(u)$ is a function of one variable; we shall also assume here that $n \geq 4$. This time we use, in the Poisson integral (1.2), coordinates $y = (t_1, t_2, \rho\xi)$, where $(t_1, t_2) \in \mathbb{B}^2$, $\xi \in \mathbb{S}^{n-3}$, and $\rho = \sqrt{1 - (t_1^2 + t_2^2)}$. We then obtain

$$u(x) = \omega_n(1 - \|x\|^2) \int_{\mathbb{B}^2} g(t_1^2 + t_2^2)\rho^{n-4} \left(\int_{\mathbb{S}^{n-3}} \frac{dS_\xi}{((t_1 - x_1)^2 + (t_2 - x_2)^2 + \sum_{j=3}^n (\rho\xi_j - x_j)^2)^{n/2}} \right) dt, \tag{4.6}$$

where dt denotes $dt_1 dt_2$. Again restricting to the line $x = (s, 0, \ldots, 0)$ and essentially repeating the computation in (4.2), we obtain

$$u(s, 0 \ldots, 0) = c_n \frac{1 - s^2}{s^{n/2}} \int_{\mathbb{B}^2} g(t_1^2 + t_2^2)\rho^{n-4} \frac{1}{(\phi(s) - t_1)^{n/2}} dt, \tag{4.7}$$

where c_n is some constant depending only on n; in what follows, we shall, as is customary, use c_n to denote such a constant and the reader should be warned that the precise value of c_n may be different from formula to formula. By using polar coordinates $t_1 = r\cos\theta$, $t_2 = r\sin\theta$, we arrive at

$$u(s, 0 \ldots, 0) = c_n \frac{1 - s^2}{s^{n/2}} \int_0^1 g(r^2)(1 - r^2)^{(n-4)/2}\, r \left(\int_0^{2\pi} \frac{1}{(\phi(s) - r\cos\theta)^{n/2}} d\theta \right) dr. \tag{4.8}$$

We shall compute the inner integral over the circle for even $n = 2k$ with $k \geq 2$. We shall, initially, be interested in the solution $u(s, 0, \ldots, 0)$ with $s \in (0, 1)$. Observe that this corresponds to $z := \phi(s)$

real-valued with $z > 1$. Thus, in the computation that follows, we shall let z be real with $z > 1$ and $0 \le r \le 1$. We introduce $\zeta = e^{i\theta}$ and obtain

$$\int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} = -i\left(-\frac{2}{r}\right)^k \int_\gamma \frac{\zeta^{k-1} d\zeta}{(\zeta^2 - (2z/r)\zeta + 1)^k}, \tag{4.9}$$

where γ denotes the positively oriented unit circle. The latter integral can be computed by residues. If we denote by

$$h(\zeta) := \frac{\zeta^{k-1}}{(\zeta^2 - (2z/r)\zeta + 1)^k} \tag{4.10}$$

and observe that the denominator has two distinct real zeros (of multiplicity k) at the points

$$\zeta_0 := \frac{1}{r}(z - \sqrt{z^2 - r^2}), \quad \zeta_1 := \frac{1}{r}(z + \sqrt{z^2 - r^2}), \tag{4.11}$$

then we conclude, since $0 < \zeta_0 < 1 < \zeta_1$, that

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} &= 2\pi \left(-\frac{2}{r}\right)^k \operatorname{Res}_{\zeta_0} h \\ &= 2\pi \left(-\frac{2}{r}\right)^k \frac{1}{(k-1)!} \left[\left(\frac{d}{d\zeta}\right)^{k-1} \frac{\zeta^{k-1}}{(\zeta - \zeta_1)^k}\right]\Bigg|_{\zeta = \zeta_0}. \end{aligned} \tag{4.12}$$

If we write

$$\frac{\zeta^{k-1}}{(\zeta - \zeta_1)^k} = \frac{1}{\zeta - \zeta_1}\left(1 + \frac{\zeta_1}{\zeta - \zeta_1}\right)^{k-1} = \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{\zeta_1^l}{(\zeta - \zeta_1)^{l+1}},$$

then, by carrying out the differentiation in (4.12) term by term,

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} &= 2\pi \left(-\frac{2}{r}\right)^k \frac{1}{(k-1)!} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1}(k-1)! \binom{l+k-1}{l} \frac{\zeta_1^l}{(\zeta_0 - \zeta_1)^{l+k}} \\ &= -2\pi \left(\frac{2}{r}\right)^k \sum_{l=0}^{k-1} \binom{k-1}{l}\binom{l+k-1}{l} \frac{\zeta_1^l}{(\zeta_0 - \zeta_1)^{l+k}}. \end{aligned} \tag{4.13}$$

Now, $\zeta_0 - \zeta_1 = -2\sqrt{z^2 - r^2}/r$ and hence we can rewrite (4.13) as follows

$$\int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} = -2\pi \left(\frac{2}{r}\right)^k \sum_{l=0}^{k-1} \binom{k-1}{l}\binom{l+k-1}{l} \left(-\frac{1}{2}\right)^{l+k} \frac{r^k (z + \sqrt{z^2 - r^2})^l}{(z^2 - r^2)^{(l+k)/2}}. \tag{4.14}$$

By expanding $(z + \sqrt{z^2 - r^2})^l$ and simplifying, we obtain

$$\int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} = (-1)^{k+1} 2\pi \sum_{l=0}^{k-1}\sum_{j=0}^{l} \left(-\frac{1}{2}\right)^l \binom{k-1}{l}\binom{l+k-1}{l}\binom{l}{j} \frac{z^j}{(z^2 - r^2)^{(k+j)/2}}. \tag{4.15}$$

We can formulate this as follows.

Lemma 4.2. *For real z with $z > 1$, $0 \le r \le 1$, and $k \ge 2$, we have the following identity*

$$\int_0^{2\pi} \frac{d\theta}{(z - r\cos\theta)^k} = (-1)^{k+1} 2\pi \sum_{j=0}^{k-1} B_{jk} \frac{z^j}{(z^2 - r^2)^{(k+j)/2}}, \tag{4.16}$$

where

$$B_{jk} := \sum_{l=j}^{k-1} \left(-\frac{1}{2}\right)^l \binom{k-1}{l} \binom{l+k-1}{l} \binom{l}{j}. \tag{4.17}$$

Plugging this into (4.8), we obtain for the solution $u(x)$, with data $f(x) = g(x_1^2 + x_2^2)$ in $\mathbb{R}^n = \mathbb{R}^{2k}$,

$$u(s,0\ldots,0) = c_n \frac{1-s^2}{s^k} \sum_{j=0}^{k-1} B_{jk} z^j \int_0^1 g(r^2)(1-r^2)^{k-2} \frac{2rdr}{(z^2-r^2)^{(k+j)/2}}, \tag{4.18}$$

where B_{jk} is given by (4.17) above and $z = \phi(s)$ as given by (4.3). Thus, by making the substitution $u = r^2$, we obtain the following.

Proposition 4.3. *Let $u(x)$ be the solution in $\mathbb{R}^n$, with $n = 2k$ even and $k \geq 2$, to the Dirichlet problem* (1.1) *in which the data function f is a function of $x_1^2 + x_2^2$, i.e. $f(x) = g(x_1^2 + x_2^2)$. If we set $v(s) := u(s, 0, \ldots, 0)$, with $s \in (0, 1)$, then we have, for some constant c_n depending only on n,*

$$v(s) = c_n \frac{1-s^2}{s^k} \tilde{P}_k(g)(\phi(s)), \tag{4.19}$$

where ϕ is given by (4.3) *and $\tilde{P}_k$ is the operator*

$$\tilde{P}_k(g)(z) := \sum_{j=0}^{k-1} B_{jk} z^j \int_0^1 g(u)(1-u)^{k-2} \frac{du}{(z^2-u)^{(k+j)/2}} \tag{4.20}$$

and B_{jk} is given by (4.17).

Motivated by Propositions 4.1 and 4.3, we shall now proceed to study the integral, initially defined for real $w > 1$,

$$I(w) := \int_a^1 h(u) \frac{du}{(w-u)^{m/2}}, \tag{4.21}$$

in which m is an integer ≥ 2, a a real number ≤ 0, $h(u)$ is a rational function without poles on the real line segment $[a, \infty)$. We shall further assume that, as u goes to infinity in the complex plane, we have the following estimate

$$|h(u)| \leq C|u|^q, \tag{4.22}$$

for some constant $C > 0$ and some q with

$$q < \frac{m}{2} - 1. \tag{4.23}$$

Clearly, $I(w)$ extends as an analytic function in the region $\Omega := \mathbb{C} \setminus (-\infty, 1]$ by simply letting the square root in (4.24) be the principal branch. Moreover, $I(w)$ is analytically extendable across the segment $(-\infty, a)$, and the analytic continuation across this segment equals $(-1)^m I(w)$; in particular, if m is an even integer, we may continue $I(w)$ to an analytic function in the doubly connected region $\mathbb{C} \setminus [a, 1]$. Comparing the definition of $I(w)$ and the expressions for $v(s)$ in Propositions 4.1 and 4.3 in which w would correspond to $\phi(s)$ or $\phi(s)^2$, respectively, we realize that we should continue $I(w)$ analytically across the segment $[a, 1]$ (which corresponds to continuing our solution $v(s)$ across the unit circle in the s-plane) before attempting to detect any non-algebraic singularities at infinity. Let us denote by $I_+(x)$ and $I_-(x)$ the boundary value of I at x, for $x \in (a, 1)$, approaching from $\{\text{Im } w > 0\}$ and $\{\text{Im } w < 0\}$, respectively. Let Γ_+ and Γ_- denote smooth simple oriented curves from a to 1 such $\Gamma_\pm \setminus \{a, 1\}$ is contained in $\{\pm\text{Im } w > 0\}$ and such that $h(w)$ has no singularities in the region Ω bounded by the closed simple oriented curve $\Gamma := \Gamma_- - \Gamma_+$. By deforming the contour of integration in the definition of $I(w)$ into $\Gamma_\pm$, we conclude that

$$I_+(x) = \int_{\Gamma_-} h(u)\frac{du}{(w-u)^{m/2}}, \quad I_-(x) = \int_{\Gamma_+} h(u)\frac{du}{(w-u)^{m/2}}, \tag{4.24}$$

where the branch of the square root in each case (when m is odd) is the principal branch. Let us now denote by $\tilde{I}(w)$ the analytic continuation of $I_+(x)$ into the half plane $\{\mathrm{Im}\ w < 0\}$, or equivalently the analytic continuation of $I(w)$ across the segment $(a,1)$ from the upper half plane into the lower half plane. We then obtain, for $w = x - iy$ with $x \in (a,1)$ and $y > 0$, $(I_+ = I_- + I_+ - I_-)$

$$\tilde{I}(w) = I(w) + \int_{\Gamma_-} h(u)\frac{du}{(w-u)^{m/2}} - \int_{\Gamma_+} h(u)\frac{du}{(w-u)^{m/2}}, \tag{4.25}$$

where the square roots again are the principal branches. Note that when $m = 2p$ is even, we obtain

$$\tilde{I}(w) = I(w) + \int_{\Gamma} h(u)\frac{du}{(w-u)^{p}} = I(w) + H(w), \tag{4.26}$$

where again Γ is the closed simple contour $\Gamma_- - \Gamma_+$ and

$$H(w) = \frac{1}{(p-1)!}\left[\left(\frac{d}{du}\right)^{p-1} h(u)\right]\Bigg|_{u=w}. \tag{4.27}$$

In particular, if m is even, then $\tilde{I}(w)$ is equal to $I(w)$ modulo a rational function. When m is odd, we can compute the integrals over $\Gamma_\pm$ in (4.25), for $w \in (a,1)$, as follows. For $R > 0$ large, let γ_R be the closed oriented curve consisting of $-\Gamma_+$, followed by Γ_-, the line segment $I_R := [1,R]$, the circle $C_R := \{|w| = R\}$ in the negative direction, and finally the line segment $-I_R$. For fixed $w \in (a,1)$, we continue the square root $\sqrt{w-u}$ to a continuous function of u along γ_R, starting from the principal branch along $-\Gamma_+ + \Gamma_-$. This defines an analytic branch of this square root, as a function of u, in the open domain V bounded by the closed curve γ_R. The orientation of γ_R, however, is opposite the positively oriented boundary ∂V. Thus, by the Residue Theorem, we have, for R large enough,

$$\int_{\gamma_R} h(u)\frac{du}{(w-u)^{m/2}} = -2\pi i \sum_{l=1}^{q} \mathrm{Res}_{a_l}\left[\frac{h(u)}{(w-u)^{m/2}}\right]. \tag{4.28}$$

where $a_1, \ldots, a_q$ denote the poles of h and the residue is taken with respect to the variable u. (Recall that h was assumed not to have any poles on $[a,\infty)$ or inside the curve Γ.) The integral over the circle C_R in (4.28) tends to 0 as $R \to \infty$ in view of (4.22) and (4.23). Observe also that the integrals over the two oppositely oriented line segments I_R and $-I_R$ do not cancel due to the fact that the branches of the square root $\sqrt{w-u}$ differ by a sign in each integral. Thus, by letting $R \to \infty$, we conclude that

$$\int_{\Gamma_-} h(u)\frac{du}{(w-u)^{m/2}} - \int_{\Gamma_+} h(u)\frac{du}{(w-u)^{m/2}} + 2\int_1^\infty h(u)\frac{du}{(w-u)^{m/2}} = -2\pi i \sum_{l=1}^{q} \mathrm{Res}_{a_l}\left[\frac{h(u)}{(w-u)^{m/2}}\right], \tag{4.29}$$

where the square root of the negative quantity $w-u$ for $u \in [1,\infty)$ is $i\sqrt{u-w}$. By using (4.25), we conclude that for the analytic continuation $\tilde{I}(w)$ of $I(w)$ across $(a,1)$ from the upper half plane into the lower half plane, we have

$$\tilde{I}(w) = I(w) - 2i\int_1^\infty h(u)\frac{du}{(u-w)^{m/2}} - 2\pi i \sum_{l=1}^{q} \mathrm{Res}_{a_l}\left[\frac{h(u)}{(w-u)^{m/2}}\right]. \tag{4.30}$$

Proof of Theorem 2.1. Recall that $n = 2k+1$ with $k \geq 1$. As mentioned above, it suffices to show that the function $J(w) := P_n(g)(w)$, given by (4.5), is not algebraic with $g(t) = 1/(t+2)^{2k-1}$. Thus, to reach a contradiction, we shall assume that $J(w)$ is algebraic. Observe that $J(w)$ is equal to the integral $I(w)$ above with $m = 2k+1$, $a = -1$, and

$$h(u) = \frac{(1-u^2)^{k-1}}{(u+2)^{2k-1}}. \tag{4.31}$$

Hence by (4.30), if we denote by $\tilde{J}(w)$ the analytic continuation of $J(w)$ across $(-1, 1)$ from the upper half plane to the lower half plane, then we have

$$\tilde{J}(w) = J(w) - 2i \int_1^\infty h(u) \frac{du}{(u-w)^{k+1/2}} - 2\pi i \operatorname{Res}_{-2}\left[\frac{h(u)}{(w-u)^{k+1/2}}\right], \tag{4.32}$$

where h is as in (4.31). Observe that the function

$$H(w) := -2\pi i \operatorname{Res}_{-2}\left[\frac{h(u)}{(w-u)^{k+1/2}}\right]$$

is algebraic, since it is a finite linear combination of derivatives, with respect to u, of $(w-u)^{-k-1/2}$ evaluated at $u = -2$. Now, if $J(w)$ were algebraic, then of course $\tilde{J}(w)$ would be algebraic. Thus, we conclude that if $J(w)$ were algebraic, then the integral

$$K(w) := \int_1^\infty \frac{(1-u^2)^{k-1}}{(u+2)^{2k-1}} \frac{du}{(u-w)^{k+1/2}} \tag{4.33}$$

would be an algebraic function of w. Observe that we may rewrite this integral as follows

$$K(w) = \int_1^\infty \frac{(1-u^2)^{k-1}}{(u+2)^{2k-1}} \frac{du}{(u-w)^{k+1/2}} = \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} \int_1^\infty \frac{u^{2l}}{(u+2)^{2k-1}} \frac{du}{(u-w)^{k+1/2}} \tag{4.34}$$

To obtain a contradiction, we shall need the following lemma.

Lemma 4.4. *Let $b > -1$ and $p + \alpha > 1$. Then, as $x \to \infty$, we have*

$$\int_1^\infty \frac{du}{(u+b)^p (u+x)^\alpha} = \begin{cases} O(x^{-\alpha}), & \text{if } p > 1 \\ x^{-\alpha} \ln x + O(x^{-\alpha}), & \text{if } p = 1. \end{cases} \tag{4.35}$$

Proof. The statement for $p > 1$ is an immediate consequence of Lebesgue's dominated convergence theorem. (Simply multiply the integral by x^α and take the limit as $x \to \infty$.) Let us therefore assume that $p = 1$. Making the change of variables $u + b = t(x-b)$, we obtain, with $y = x - b$,

$$\begin{aligned} \int_1^\infty \frac{du}{(u+b)(u+x)^\alpha} &= y^{-\alpha} \int_{(1+b)/y}^\infty \frac{dt}{t(t+1)^\alpha} \\ &= y^{-\alpha} \left(\int_{(1+b)/y}^1 \frac{dt}{t(t+1)^\alpha} + \int_1^\infty \frac{dt}{t(t+1)^\alpha} \right). \end{aligned} \tag{4.36}$$

Observe that the function $1/(t(t+1)^\alpha)$ is integrable on $[1, \infty]$ by the assumption that $1 + \alpha > 1$. Thus, to prove the lemma it suffices to show that

$$\int_{(1+b)/y}^\infty \frac{dt}{t(t+1)^\alpha} = \ln y + O(1). \tag{4.37}$$

An integration by parts yields

$$\int_{(1+b)/y}^{1} \frac{dt}{t(t+1)^\alpha} = \frac{1}{(1+(1+b)/y)^\alpha} \ln(y/(1+b)) + \alpha \int_{(1+b)/y}^{1} \frac{\ln t\, dt}{(t+1)^{\alpha+1}}. \tag{4.38}$$

Since the function $f(t) = \ln t/(t+1)^{\alpha+1}$ is integrable on $[0, 1]$, the conclusion of Lemma 4.4 now follows. □

Now, observe that, for $u \in [1, \infty)$, we have $u < u + 2 < 4u$. Thus, if we set $w = -x$ in (4.34) with $x > 0$, then we conclude from (4.34) and Lemma 4.4 (with $b = 0$) that, as $x \to \infty$,

$$Ax^{-k-1/2} \ln x \le |K(-x)| \le Bx^{-k-1/2} \ln x, \tag{4.39}$$

for some constants $0 < A < B$. If $K(w)$ were algebraic, then $K(w)$ would have a Puiseux expansion at infinity, i.e. it would be given for large w by a convergent series

$$K(w) = \sum_{l=-r}^{\infty} b_l \left(\frac{1}{w}\right)^{l/p},$$

for some integers p and r. This clearly contradicts the estimate (4.39) and, hence, completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. The proof of Theorem 2.2 proceeds along the same lines as that of Theorem 2.1. This time, we have $n = 2k$ with $k \ge 2$, and we shall show, as is sufficient to prove Theorem 2.2, that $J(z) := \tilde{P}_k(g)(z)$, given by (4.20) with $g(u) = 1/(1+u)^{k-1}$, is not algebraic. Observe that we can write

$$J(z) = \sum_{j=0}^{k-1} B_{jk} z^j I_j(w),$$

where B_{jk} is given by (4.17), $w = z^2$, and $I_j(w)$ is as in (4.21) with $a = 0$, $m = k + j$, and

$$h(u) = \frac{(1-u)^{k-2}}{(1+u)^{k-1}}. \tag{4.40}$$

Thus, if we denote by $\tilde{J}(z)$ the analytic continuation of $J(z)$ across the segment $(0, 1)$ from the upper half plane into the lower half plane, we obtain, by the arguments preceeding the proof of Theorem 2.1 above (since $w = z^2$ then also crosses $(0, 1)$ from the upper half plane into the lower half plane),

$$\begin{aligned}\tilde{J}(z) = J(z) + \sum_{\substack{0\le j\le k-1\\ k+j\,\text{even}}} z^j H_j(w) - \\ \sum_{\substack{0\le j\le k-1\\ k+j\,\text{odd}}} B_{jk} z^j \left(2i \int_1^\infty h(u) \frac{du}{(u-w)^{(k+j)/2}} + 2\pi i \operatorname{Res}_{-1}\left[\frac{h(u)}{(w-u)^{(k+j)/2}}\right]\right),\end{aligned} \tag{4.41}$$

where $H_j(w)$ is the rational function given by (4.27) with h is as in (4.40) and $p = (k + j)/2$. Hence, if we assume, in order to reach a contradiction, that $J(z)$ is algebraic, then we conclude (cf. the proof of Theorem 2.1 above) that the function

$$K(z) = \sum_{\substack{0\le j\le k-1\\ k+j\,\text{odd}}} B_{jk} z^j \int_1^\infty \frac{(1-u)^{k-2}}{(1+u)^{k-1}} \frac{du}{(u-w)^{(k+j)/2}} \tag{4.42}$$

must be algebraic. We write

$$\frac{(1-u)^{k-2}}{(1+u)^{k-1}} = \frac{1}{1+u}\left(-1+\frac{2}{1+u}\right)^{k-2}$$

and expand the right hand side to obtain

$$\begin{aligned} K(z) &= \sum_{\substack{0\le j\le k-1\\ k+j\,\mathrm{odd}}} B_{jk} z^j \int_1^\infty \frac{(1-u)^{k-2}}{(1+u)^{k-1}} \frac{du}{(u-w)^{(k+j)/2}} \\ &= \sum_{\substack{0\le j\le k-1\\ k+j\,\mathrm{odd}}} \sum_{l=0}^{k-2} (-1)^{k-2-l}\binom{k-2}{l} 2^l\, B_{jk} z^j \int_1^\infty \frac{1}{(1+u)^{l+1}} \frac{du}{(u-w)^{(k+j)/2}}. \end{aligned} \tag{4.43}$$

We set $z=-iy$, with $y>0$, in $K(z)$ and let $y\to\infty$. By Lemma 4.4 (with $b=1$), we then obtain

$$\begin{aligned} K(-iy) &= (-1)^{k-2} \sum_{\substack{0\le j\le k-1\\ k+j\,\mathrm{odd}}} B_{jk}(-iy)^j \left(y^{-(k+j)}\ln y + O(y^{-(k+j)})\right) \\ &= My^{-k}\ln y + O(y^{-k}), \end{aligned} \tag{4.44}$$

where

$$M = (-1)^{k-2} \sum_{\substack{0\le j\le k-1\\ k+j\,\mathrm{odd}}} B_{jk}(-i)^j. \tag{4.45}$$

By using the definition (4.17) of B_{jk}, we obtain

$$M = (-1)^{k-2} \sum_{\substack{0\le j\le k-1\\ k+j\,\mathrm{odd}}} \sum_{l=j}^{k-1} \left(-\frac{1}{2}\right)^l \binom{k-1}{l}\binom{l+k-1}{l}\binom{l}{j}(-i)^j, \tag{4.46}$$

Since $M\neq 0$ by the assumption (2.2), the estimate (4.44) contradicts the hypothesis that $K(w)$ is algebraic in precisely the same way as in the proof of Theorem 2.1 above (by considering the Puiseux expansion of K at infinity). □

5. Axially symmetric data in even dimensions

Let $f(x)$ be an axially symmetric rational function without singularities on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We shall assume that $n=2k$ is even and $k\ge 2$. Since the Laplace operator commutes with rotations, we may assume that the axis of symmetry is the x_1-axis. If we write $r=\sqrt{x_2^2+\ldots+x_n^2}$, then f is a function of x_1 and r. Since f is rational, f must in fact be a function of x_1 and r^2. Now, on $\mathbb{S}^{n-1}$ we have $x_1^2+\ldots+x_n^2=1$, or equivalently $r^2=1-x_1^2$. It follows that there is a rational function of one variable $g(t)$ such that $f(x)=g(x_1)$ on the sphere $\mathbb{S}^{n-1}$. Since f has no poles on $\mathbb{S}^{n-1}$, we deduce that g cannot have any poles on the segment $[-1,1]$. Thus, we shall consider the Dirichlet problem (1.1) with data $f(x)=g(x_1)$, where $g(t)$ is a rational function without poles in $[-1,1]$. We have the following result which contains, as a special case, Theorem 2.5.

Theorem 5.1. *Let $n=2k\ge 4$ and $f(x)=g(x_1)$, where g is a rational function of one variable. Assume in addition, if $k\ge 3$, that g has $k-2$ rational integrals; i.e. there is a rational functions $G(t)$ such that $d^{k-2}G/dt^{k-2}=g$. Then, the solution $u(x)$ to* (1.1) *is an axially symmetric rational function.*

Proof. First, we observe that, for even $n = 2k \geq 4$ and $y \in \mathbb{R}^n \setminus \overline{\mathbb{B}^n}$, the harmonic potential

$$U_y(x) = \frac{1}{\|x-y\|^{n-2}} = \left(\sum_{j=1}^n (x_j - y_j)^2\right)^{-(k-1)} \tag{5.1}$$

is rational without poles in a neighborhood of $\overline{\mathbb{B}^n}$. Let us take $y = y(s) := (s, 0, \ldots, 0)$ with $|s| > 1$. Then, for x on the sphere $\mathbb{S}^{n-1}$, we have (cf. (4.2))

$$\begin{aligned} U_{y(s)}(x) &= \left((x_1 - s)^2 + \sum_{j=2}^n x_j^2\right)^{-(k-1)} \\ &= \frac{1}{(-2s)^{k-1}} \frac{1}{(x_1 - \phi(s))^{k-1}}, \end{aligned} \tag{5.2}$$

where $\phi(s)$ is given in (4.3). Let us for simplicity denote $U_{y(s)}$ by U_s. Observe that ϕ is an analytic 2-to-1 mapping of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C}$ which sends both the punctured unit disk $\mathbb{D} \setminus \{0\}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$ conformally onto $\mathbb{C} \setminus [-1, 1]$. Hence, for $x \in S^{n-1}$, we may continue $U_s(x)$ as an analytic function of s to $\mathbb{C} \setminus \overline{\mathbb{D}}$ and, for any $w \in \mathbb{C} \setminus [-1, 1]$, there is a unique $s = s(w)$ with $s \in \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $w = \phi(s)$ and hence

$$u(x) = (-2s)^{k-1} U_s(x)$$

is the solution to the Dirichlet problem (1.1) with

$$f(x) = \frac{1}{(x_1 - w)^{k-1}}.$$

Also, note that for any integer $m \geq 1$, the function

$$U_s^m(x) := \frac{\partial^m}{\partial s^m} U_s(x) \tag{5.3}$$

is harmonic away from the point $y = (s, 0, \ldots, 0)$ and satisfies, for x on the sphere $\mathbb{S}^{n-1}$,

$$U_s^m(x) = \sum_{j=0}^m \psi_j(s) \frac{1}{(x_1 - \phi(s))^{k-1+j}}, \tag{5.4}$$

where $\psi_1(s), \ldots, \psi_m(s)$ are rational functions of s and

$$\psi_m(s) = \frac{(k-2+m)!}{(k-2)!} \frac{(-\phi'(s))^m}{(-2s)^{k-1}}.$$

In particular, for any $w \in \mathbb{C} \setminus [-1, 1]$, we have $\psi_m(s) \neq 0$ for $s = s(w)$ with $s \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\phi(s) = w$. We conclude that there are coefficients $c_j \in \mathbb{C}$, $j = 0, \ldots, m$, such that

$$u(x) := \sum_{j=0}^m c_j U_s^j(x), \tag{5.5}$$

where we use the notation $U_s^0 := U_s$, solves the Dirichlet problem (1.1) with data $f(x) = 1/(x_1 - w)^{k-1+m}$. In particular, the solution u to (1.1) with data $f(x) = 1/(x_1 - w)^l$, where $l \geq k - 1$, is rational. Recall that any rational function of one variable can be decomposed as a finite sum of singular parts

$$g(t) = p(t) + \sum_{j=1}^p \sum_{l=1}^{q_j} a_{jl} \frac{1}{(t - w_j)^l}, \tag{5.6}$$

where $p(t)$ is a polynomial, $w_1, \ldots, w_p$ denote the poles of g in $\mathbb{C}$, and $q_1, \ldots, q_p$ their orders. If we also recall that the solution of the Dirichlet problem (1.1) with polynomial data is polynomial, then it follows from the above discussion that the solution $u(x)$ to (1.1) with axially symmetric data $f(x) = g(x_1)$, where g is given by (5.6), is rational provided that the coefficients a_{jl} satisfy

$$a_{jl} = 0, \quad \forall\, l < k-1 \text{ and } j = 1, \ldots, p. \tag{5.7}$$

Clearly, this condition is vacuous when $k = 2$, and it is not difficult to see that for $k \geq 3$ the condition (5.7) is equivalent to $g(t)$ having $k-2$ rational integrals. Since, as is well known, the solution to the Dirichlet problem with axially symmetric data is axially symmetric, the proof of Theorem 5.1 is complete. □

We should point out that we do not know if the condition that $g(t)$ has $k-2$ rational integrals is necessary for the solution $u(x)$ to be rational in Theorem 5.1. In particular, we do not know if e.g. the solution in, say, $\mathbb{R}^6$ with data $f(x) = 1/(x_1 - 2)$ is rational. However, we can say that the solution will be rational on the axis of symmetry. Indeed, by Proposition 4.1, the solution $u(x)$ to (1.1) with $f(x) = g(x_1)$, restricted to the axis of symmetry $x = (s, 0, \ldots, 0)$, is given by (4.4) where $P_n(g)(z)$ is given by (4.5). By an integration by parts, we have, using the notation $n = 2k$,

$$\begin{aligned}
P_n(g)(z) &= \int_{-1}^{1} g(t)(1-t^2)^{k-3/2} \frac{dt}{(z-t)^k} \\
&= \left[\frac{1}{k-1} \frac{g(t)(1-t^2)^{k-3/2}}{(z-t)^{k-1}}\right]_{-1}^{1} - \frac{1}{k-1} \int_{-1}^{1} \frac{d}{dt}(g(t)(1-t^2)^{k-3/2}) \frac{dt}{(z-t)^{k-1}} \\
&= -\frac{1}{k-1} \int_{-1}^{1} g'(t)(1-t^2)^{k-3/2} \frac{dt}{(z-t)^{k-1}} + \frac{2k-3}{k-1} \int_{-1}^{1} g(t)t(1-t^2)^{k-5/2} \frac{dt}{(z-t)^{k-1}} \\
&= -\frac{1}{k-1} P_{n-2}(T_1 g)(z) + \frac{2k-3}{k-1} P_{n-2}(T_2 g)(z) \\
&= P_{n-2}(V_k g)(z),
\end{aligned} \tag{5.8}$$

where

$$V_k g(t) = -\frac{1}{k-1} T_1 g(t) + \frac{2k-3}{k-1} T_2 g(t) \tag{5.9}$$

and

$$T_1 g(t) := g'(t)(1-t^2), \quad T_2 g(t) := t g(t). \tag{5.10}$$

Observe that if g is rational, then $T_1 g$ and $T_2 g$, and hence also $V_k g$, are rational. We conclude, by induction, that for any rational function $g(t)$ and $n = 2k \geq 6$, we have

$$P_n(g) = \sum_{l=1}^{2^{k-2}} a_l P_4(g_l), \tag{5.11}$$

where $a_l \in \mathbb{R}$ and g_l, for $l = 1, \ldots 2^{k-2}$, are rational functions. By Proposition 4.1, the solution to (1.1) restricted to $x = (s, \ldots, 0)$ is given by

$$v(s) = \sum_{l=1}^{2^{k-2}} a'_l s^{k-2} v_l(s), \tag{5.12}$$

where $a'_l \in \mathbb{R}$ and $v_l(s)$, for $l = 1, \ldots, 2^{k-2}$, is the restriction to the axis of symmetry of the solution to the Dirichlet problem (1.1) in $\mathbb{R}^4$ with data $f(x) = g_l(x_1)$. Hence, by Theorem 2.5, $v(s)$ is rational. We may formulate this as follows.

Theorem 5.2. *Let $n = 2k \geq 4$, $f(x) = g(x_1)$, where $g(t)$ is a rational function of one variable. Let $u(x)$ be the solution to the Dirichlet problem* (1.1)*, and let $v(s) = u(s, 0 \ldots, 0)$ be its restriction to the x_1-axis. Then, $v(s)$ is rational.*

6. Ultraspherical polynomials and The Nehari transform

We shall conclude this paper by exploring a relationship between the Dirichlet problem and series of ultraspherical polynomials. We shall follow closely the notations in [SW] and [S]. Thus, $P_m^{(\lambda)}(t)$ will denote the ultraspherical (or Gegenbauer) polynomial of degree m with parameter λ. Recall that the $P_m^{(\lambda)}(t)$ are given by the generating function

$$\frac{1}{(1 - 2tw + w^2)^\lambda} = \sum_{m=0}^{\infty} P_m^{(\lambda)}(t) w^m, \tag{6.1}$$

from which one easily deduces the recursion formula

$$\frac{dP_m^{(\lambda)}}{dt}(t) = 2\lambda P_{m-1}^{(\lambda+1)}(t), \quad m \geq 1. \tag{6.2}$$

We mention here that the ultraspherical polynomials are special cases of the Jacobi polynomials $P_m^{(\alpha,\beta)}(t)$ in the sense that $P_m^{(\lambda)}(t) = c(m, \lambda) P_m^{(\alpha,\beta)}(t)$ where $\alpha = \beta = \lambda - 1/2$ and $c(m, \lambda)$ is a normalization factor depending on m and λ. Furthermore, as special cases of the ultraspherical polynomials we have the Legendre polynomials $P_m(t)$ (normalized by $P_m(1) = 1$) via $P_m^{(1/2)}((t) = P_m(t)$, and the Chebyshev polynomials of the second kind $U_m(t)$ via $P_m^{(1)}(t) = U_m(t)$ (see e.g. [S]). For our purposes, however, a more relevant connection is to the spherical harmonics in $\mathbb{R}^n$. For any $y \in S^{n-1} \subset \mathbb{R}^n$, the restriction of the function

$$f(x) := P_m^{(n-2)/2}(\langle x, y \rangle),$$

where

$$\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j,$$

to the sphere S^{n-1} is a spherical harmonic of order m, i.e. the restriction to S^{n-1} of a homogeneous harmonic polynomial of degree m. More precisely, the function $H_m(x)$ defined in $\mathbb{R}^n$ by

$$H_m(x) := \|x\|^m P_m^{(n-2)/2} \left(\frac{\langle x, y \rangle}{\|x\|} \right) \tag{6.3}$$

is a homogeneous harmonic polynomial of degree m. As a consequence of Theorem 5.2, we obtain the following result.

Theorem 6.1. *Let $n \geq 4$ be even and $\{c_m\}_{m=0}^{\infty}$ a sequence of complex numbers such that*

$$\limsup_{m \to \infty} |c_m|^{1/m} < 1. \tag{6.4}$$

Then, the series

$$g(z) := \sum_{m=0}^{\infty} c_m P_m^{(n-2)/2}(z) \tag{6.5}$$

converges to a holomorphic function in an open neighborhood of the segment $[-1,1]$ *in the complex plane. If* g *is rational, then the function* $\tilde{g}(w)$ *defined by the Taylor series*

$$\tilde{g}(w) := \sum_{m=0}^{\infty} c_m P_m^{(n-2)/2}(1) w^n \tag{6.6}$$

is also rational.

Remark 6.2. The numbers $P_m^{\lambda}(1)$, which appear in the Taylor series (6.6) with $\lambda = (n-2)/2$, can be easily computed from the generating formula (6.1). One obtains

$$P_m^{\lambda}(1) = \frac{2\lambda(2\lambda+1)\dots(2\lambda+m-1)}{m!}. \tag{6.7}$$

For instance, if $n = 6$, so that $\lambda = 2$, then

$$P_m^{(2)}(1) = \binom{m-3}{m} = \frac{(m+1)(m+2)(m+3)}{3!}.$$

Thus, the conclusion of Theorem 6.1 with $n = 6$ implies that the Taylor series

$$\sum_{m=0}^{\infty} c_m w^{m+3}$$

has a rational third derivative provided that the corresponding ultraspherical series (6.5) is rational. Similarly, the conclusion of the theorem in general can be formulated as saying that the derivative of order $n-3$ of the Taylor series

$$\sum_{m=0}^{\infty} c_m w^{m+n-3}$$

is rational when the series (6.5) is.

Proof. The convergence of the series (6.5) defining g, subject to the condition (6.4), follows from well known estimates for Jacobi polynomials. If we now consider the Dirichlet problem (1.1) in $\mathbb{R}^n$ with data $f(x) = g(x_1)$, where g is assumed to be rational, then the solution $u(x)$ restricted to the x_1-axis is rational by Theorem 5.2. On the other hand, the solution $u(x)$ can be given explicitly by

$$u(x) = \sum_{m=0}^{\infty} c_m \|x\|^m P_m^{(n-2)/2}\left(\frac{x_1}{\|x\|}\right), \tag{6.8}$$

and hence the restriction of u to the x_1-axis is precisely $\tilde{g}(x_1)$. □

The transformation of a sum of Jacobi polynomials to a Taylor series with the same coefficients was apparently first studied by Nehari [N] who demonstrated, under the assumption (6.4), a correspondence between the singular points of the series of Jacobi polynomials on its ellipse of convergence, and those of the associated Taylor series on its circle of convergence. However, results of the precision of Theorem 6.1 do not follow from Nehari's theorem. For further developments of Nehari's results, see also [EKS].

The conclusion of Theorem 6.1 is in general false for odd n. Let us demonstrate this for $n = 3$. The corresponding value of $\lambda = (n-2)/2$ is then $1/2$, and hence the relevant ultraspherical polynomials $P_m^{(1/2)}(t)$ are just the classical Legendre polynomials, which we denote, as is customary, by $P_m(t)$. (The integral operator P_n defined in (4.5) will not be used in this section, so there should be no risk of confusing the Legendre polynomials with these integral operators.) Since $P_m(1) = 1$, what we have to exhibit is a sequence $\{c_m\}$, subject to the condition (6.4), for which the series

$$\sum_{m=0}^{\infty} c_m P_m(t), \tag{6.9}$$

is a rational function, but for which

$$\sum_{m=0}^{\infty} c_m w^m \tag{6.10}$$

is not. For this we invoke the classical formula of Heine (c.f. [WW], p. 322)

$$\frac{1}{t-z} = \sum_{m=0}^{\infty} (2m+1) Q_m(t) P_m(z), \tag{6.11}$$

where $Q_m(t)$ denotes the mth order Legendre function of the second kind. For fixed complex t not on the segment $[-1, 1]$, the numbers $c_m := (2m+1)Q_m(t)$ satisfy (6.4), and the series on the right in (6.11) converges uniformly for z in any compact subset of the domain bounded by the ellipse with foci at -1 and 1 and passing through t. The Nehari transform of the series in (6.11), i.e. the result of replacing $P_m(z)$ by w^m, is

$$F_t(w) = \sum_{m=0}^{\infty} (2m+1) Q_m(t) w^m \tag{6.12}$$

and we claim that for fixed complex t not in the segment $[-1, 1]$, the function $F_t(w)$, given by (6.12), is not a rational function of w. Indeed, from the identity ([WW], p. 321, Example 20)

$$\sum_{m=0}^{\infty} Q_m(t) w^m = \frac{1}{(1 - 2tw + w^2)^{1/2}} \cosh^{-1}\left(\frac{w-t}{(t^2-1)^{1/2}}\right), \tag{6.13}$$

we can easily derive a closed expression for $F_t(w)$. If we denote the function in (6.13) by $G_t(w)$, then we have

$$F_t(w^2) = \frac{d}{dw}(w G_t(w)). \tag{6.14}$$

Thus, if F_t were rational, then $d/dw(wG_t(w^2)$ would be as well. But, it is not; indeed, it is not even algebraic, and the same is therefore true of $F_t(w)$. We do not insist on all details, because the same conclusion follows from our earlier results: We have shown (cf. the proof of Theorem (2.1)) that the solution to the Dirichlet problem (1.1) in $\mathbb{R}^3$ with data $f(x) = 1/(2 - x_1)$ (and the same could be easily be shown with data $f_t(x) = 1/(t - x_1)$ and t not in the segment $[-1, 1]$) restricted to the x_1-axis is not algebraic. By using the Heine expansion (6.11) and solving the Dirichlet problem as in the proof of Theorem 6.1, we conclude that the Nehari transform $F_t(w)$ (given by (6.12)) is not algebraic. The argument sketched in this section yields an alternative proof, and interpretation, of Theorem 2.1, and we have given it here to show the link with ultraspherical polynomials and the Nehari transform. From a purely technical point of view, it appears that our earlier method, based on the Poisson integral, is simpler and also does not rely on the knowledge of Heine's expansion and (6.13).

One could also, as a tour de force, solve the axially symmetric Dirichlet problem (1.1) in $\mathbb{R}^n$, for all odd n greater than 1, with the special data

$$\frac{1}{(t - x_1)^{(n-1)/2}}$$

by an explicit series of Legendre functions as follows. By differentiating (6.11) r times with respect to z, we obtain an expansion of the function

$$\frac{1}{(t-z)^{r+1}}$$

in terms of rth order derivatives of Legendre polynomials P_n, which in view of the recursion formula (6.2) are ultraspherical polynomials with parameter $\lambda = r + 1/2$ and that correspond to zonal harmonics in the ball in dimension $2r+3$. Observe that for odd n larger than 3, the exponent $(n-1)/2$ is smaller than that, $2k-1 = n-2$, appearing as the exponent in Theorem 2.1. Thus, for odd $n \geq 5$, it may seem at first glance that we do not get an alternative approach to Theorem 2.1 using the expansion (6.11). But, in fact, we can explicitly obtain the expansion of $1/(t-z)^s$ for every integer $s \geq (n-2)/2$ in terms of ultraspherical polynomials with parameter $\lambda = (n-2)/2$ if we differentiate (6.11) $(n-3)/2$ times with respect to z and $s-(n-3)/2$ times with respect to t. Presumably this could lead to an alternative proof of Theorem (2.1) and also further generalizations, but we shall not pursue that here.

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