# On the faces problem for perfect codes

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**Abstract**

Perfect 1-error correcting codes $C$ in the hyper cube $Z_2^n$ are considered. The possibilities for the number $\gamma(C)$ of code words in a $k$-face $\gamma$ of the hyper cube are discussed. It is shown that the possibilities for the number $\gamma(C)$ depend on the dimension of the face $\gamma$, the rank of $C$ and the dimension of the kernel of $C$. Especially we get an answer to a question of Sergey V. Avgustinovich whether there is a perfect code with no full $(n-1)/2$-face or not.

## 1 Introduction

We consider the direct product $Z_2^n = Z_2 \times Z_2 \times ... \times Z_2$ of the field $Z_2 = \{0, 1\}$. The elements of this direct product will be called *words* of *length* $n$. The *weight* of a word $c$, $w(c)$, will be the number of non zero components of $c$. The *distance* between two words $c$ and $c'$, $d(c, c')$, will be the weight of the word $c - c'$.

A *perfect 1-error correcting binary code* of length $n$ is a subset $C$ of $Z_2^n$ satisfying the following condition:

$$\textit{To any } v \in Z_2^n \textit{ there is an unique } c \in C \textit{ with } d(c, v) \leq 1.$$

(By trivial counting arguments, the only possible values for the length of a perfect 1-error correcting binary code are $n = 2^m - 1$ where $m$ is an integer.)

A *k-face* of the n-cube $Z_2^n$ is the set of points

$$\Gamma^{i_1,\ldots,i_t}_{\sigma_{i_1},\ldots,\sigma_{i_t}} = \{x \in Z_2^n \mid x_{i_v} = \sigma_{i_v} \quad v = 1, 2, \ldots, t\},$$

where $t = n - k$.

A perfect code $C$ of length $n$ is said to be *full* on the $(n-1)/2$-face $\gamma$ if any point on $\gamma$ is at distance at most one from an unique word of

$$C \cap \gamma.$$

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Sergey V. Avgustinovich [1] proposed the problem whether or not there exists a perfect code $C$ which is not full on any $(n-1)/2$-face of the $n$-cube.

Below we give a formula that relates the number of full $(n-1)/2$-faces to the rank of the perfect code $C$. We also show that the possibilities for the number of words of a perfect code on a $(n-1)/2$-face are related to the size of the kernel of $C$. We remind on the definition of rank and kernel of a perfect code.

Consider the linear span $< C >$ of the words of $C$. For any code $C$, $< C >$ is a linear subspace of the vector space $Z_2^n$. The dimension of this subspace is the *rank* of $C$, $rank(C)$.

The *kernel* of a perfect code is the set

$$ker(C) = \{p \in Z_2^n \mid p + c \in C \quad \text{for all} \quad c \in C\}.$$

The kernel is a subspace of $Z_2^n$.

We show in Section 3

**Theorem 1** *The number of full $(n-1)/2$-faces of a perfect 1-error correcting binary perfect code $C$ of length $n$ is equal to*

$$(2^{n-rank(C)} - 1)2^{(n-1)/2}.$$

*Further the full $(n-1)/2$-faces may be divided into equivalence classes, such that each class consists of $2^{(n-1)/2}$ parallel faces.*

*In particular no full rank perfect code will have any full face.*

We also consider the orthogonal complement $\gamma^{\perp}$ of a full $(n-1)/2$-face $\gamma$. We get the following theorem.

**Theorem 2** *If $\gamma$ is a full $(n-1)/2$-face of a perfect 1-error correcting binary perfect code $C$ of length $n$ then $\gamma^{\perp} \cap C$ is isomorphic to an extended perfect 1-error correcting binary perfect code $C$ of length $n+1$.*

The proof technique of Theorem 1 also give the following theorem.

**Theorem 3** *For any perfect code $C$ of length $n$ and any $(n-1)/2$-face $\gamma$*

$$|C \cap \gamma| = \frac{t \cdot |ker(C)|}{2^{(n-1)/2}}$$

*for some integer $t$.*

To prove these theorems we use the technique with fourier coefficients, as described in [5] and summarized in the next section.

# 2 Preliminaries

## 2.1 Fourier coefficients

We consider a group algebra $R[x_1, x_2, ..., x_n]$. The elements of this group algebra are polynomials

$$r(x_1, x_2, ..., x_n) = \sum_{v \in Z_2^n} r_v \, x_1^{v_1} x_2^{v_2} ... x_n^{v_n} \qquad v = (v_1, v_2, ..., v_n) \tag{1}$$

where the coefficients $r_v$, $v \in Z_2^n$, belong to the set of real numbers $R$.

Let $y_t(x_1, x_2, ..., x_n)$, for $t \in Z_2^n$, denote the polynomial

$$y_t(x_1, x_2, ..., x_n) = \frac{1}{2^n} \prod_{i=1}^{n} (1 - x_i)^{t_i} (1 + x_i)^{1 - t_i} \qquad t = (t_1, t_2, ..., t_n).$$

It was proved in [5] that any polynomial $r(x_1, x_2, ..., x_n)$ of $R[x_1, x_2, ..., x_n]$ has an unique expansion

$$r(x_1, x_2, ..., x_n) = \sum_{t \in Z_2^n} A_t \, y_t(x_1, ..., x_n), \tag{2}$$

where $A_t \in R$ for $t \in Z_2^n$. The coefficients $A_t$, $t \in Z_2^n$, in the expansion (2) will be called the *fourier coefficients* of the polynomial $r(x_1, ..., x_n)$.

We note that the polynomials may be considered as polynomials in the ring $R[x_1, \ldots, x_n]$. We may hence make substitutions of $x_i$, $i = 1, 2, \ldots, n$ by real numbers, whereby equalities will remain true.

If we in the equality (2) substitute

$$x_i = \begin{cases} 1 & \text{if} \quad d_i = 0 \\ -1 & \text{if} \quad d_i = 1 \end{cases} \qquad d = (d_1, d_2, ..., d_n) \in Z_2^n, \tag{3}$$

then we get from the equations (1) and (2) that

$$A_d = \sum_{v \in Z_2^n} r_v (-1)^{v \cdot d}, \tag{4}$$

where

$$(v_1, v_2, ..., v_n) \cdot (d_1, d_2, ..., d_n) = v_1 d_1 + v_2 d_2 + ... + v_n d_n.$$

To a subset $C$ of $Z_2^n$ we associate the polynomial

$$C(x_1, x_2, ..., x_n) = \sum_{c \in C} x_1^{c_1} x_2^{c_2} ... x_n^{c_n} \qquad c = (c_1, c_2, ..., c_n).$$

We will say that the fourier coefficients of the polynomial $C(x_1, x_2, ..., x_n)$ are the *fourier coefficients of the set $C$*.

The following result was proved in [7], see also [5].

**Theorem 4** *If $C$ is a perfect 1-error correcting binary code of length $n$ then there are integers $A_0$ and $A_d$, $d \in D = \{t \in Z_2^n \mid w(t) = \frac{n+1}{2}\}$, such that*

$$C(x_1, ..., x_n) = \frac{A_0}{2^n} \prod_{i=1}^{n} (1 + x_i) \ + \sum_{d \in D} \frac{A_d}{2^n} \prod_{i=1}^{n} (1 + x_i)^{1-d_i} (1 - x_i)^{d_i}.$$

If we let $x_i = 1$ for $i = 1, 2, .., n$ in (2) and (3) we will get that

$$|C| = \sum_{c \in C} 1 = C(1, 1, ..., 1) = A_0.$$

Let $< d >^{\perp}$ denote the set of words that are orthogonal to the word $d = (d_1, d_2, ..., d_n)$ in $Z_2^n$, i.e.

$$< d >^{\perp} = \{(v_1, v_2, ...v_n) \mid d_1 v_1 + d_2 v_2 + ... + d_n v_n \equiv 0 \pmod 2\}.$$

We get from equation (4) that

$$A_d = 2 \mid < d >^{\perp} \cap C| - |C| \ . \tag{5}$$

Hergert [6] observed that if $d \neq 0$ is orthogonal to all words of $C$, then $w(d) = (n + 1)/2$. Hence, if $< C >$ denotes the linear span of the words of $C$, then we may conclude from (5) that

$$d \in < C >^{\perp} \quad \text{if and only if} \quad A_d = |C|. \tag{6}$$

## 2.2 Some notation

Below, a *perfect code* always will be a perfect 1-error correcting binary code in $Z_2^n$.

We will let $e_i$ denote a word of weight 1 with the single one in the i:th coordinate position.

Let $I$ be a subset of $\{1, 2, \ldots, n\}$ and let $g = \sum_{i \in I} e_i$. Define for any word $c \in Z_2^n$,

$$w_I(c) = c_1 g_1 + c_2 g_2 + \ldots + c_n g_n$$

where we do *not* count modulo 2. We define for any two words $c$ and $c'$ of $Z_2^n$,

$$d_I(c, c') = w_I(c - c').$$

We also need the usual so called *dot-product* in $Z_2^n$: If $c = (c_1, \ldots, c_n)$ and $v = (v_1, \ldots, v_n)$ then

$$c \cdot v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \pmod 2.$$

We will let *supp*$(t)$ denote the *support* of a word $t = (t_1, t_2, \ldots, t_n)$, i.e.

$$supp(t) = \{i \mid t_i \neq 0\}.$$

## 3 Proof of the theorems

From previous section we know that for any perfect code $C$ of length $n$

$$C(x_1,\ldots,x_n)=\sum_{t\in D} A_t y_t(x_1,\ldots,x_n) \tag{7}$$

where $D=\{t\in Z_2^n \mid w(t)=(n+1)/2\}$, and where

$$A_t=|\{c\in C \mid c\cdot t=0\}|-|\{c\in C \mid c\cdot t=1\}| \tag{8}$$

or equivalently

$$A_t=2|<t>^{\perp}\cap C|-|C|. \tag{8'}$$

We now consider a $(n-1)/2$-face $\gamma=\Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ of the $n$-cube $Z_2^n$. Below we will let $I=\{i_1,\ldots,i_s\}$, $\sigma=(\sigma_{i_1},\ldots,\sigma_{i_s})$ and $g=\sum_{i\in I} e_i$. To count the number of words of $C$ on $\gamma$ we make the substitution

$$x_i=1 \quad \text{if} \quad i\notin I.$$

With this substitution we get

$$C(x_1,\ldots,x_n)|_{x_i=1,\, i\notin I}=\sum_{\tau\in Z_2^s} h_\tau x_{i_1}^{\tau_{i_1}}\ldots x_{i_s}^{\tau_{i_s}} \qquad \tau=(\tau_{i_1},\ldots,\tau_{i_s}) \tag{9}$$

where $h_\tau$ *equals the number of words of* $C$ *in the face* $\Gamma^{i_1,\ldots,i_s}_{\tau_1,\ldots,\tau_s}$. The same substitution in the polynomials $y_t(x_1,\ldots,x_n)$ gives

$$y_t(x_1,\ldots,x_n)|_{x_i=1,\, i\notin I}=\begin{cases}\frac{2^{(n-1)/2}}{2^n}\Pi_{v=1}^s(1-x_{i_v}) & \text{if} \quad t=g\\ \frac{2^{(n-1)/2}}{2^n}\Pi_{v=1}^s(1+x_{i_v}) & \text{if} \quad t=0\\ 0 \quad \text{else}\end{cases}\cdot \tag{10}$$

Hence from (7), (9) and (10) we deduce that

$$\sum_{\tau\in Z_2^s} h_\tau x_{i_1}^{\tau_{i_1}}\ldots x_{i_s}^{\tau_{i_s}}=\frac{|C|}{2^n}2^{(n-1)/2}\Pi_{v=1}^s(1+x_{i_v})+\frac{A_g}{2^n}2^{(n-1)/2}\Pi_{v=1}^s(1-x_{i_v}).$$

Consequently

$$h_\tau=\begin{cases}(|C|+A_g)2^{-(n+1)/2} & \text{if} \quad w(\tau) \quad \text{is even}\\ (|C|-A_g)2^{-(n+1)/2} & \text{else.}\end{cases} \tag{11}$$

From (8′) we thus have the following

**Proposition** *For the number of words* $h_\tau$ *in the* $(n-1)/2$*-face* $\Gamma^{i_1,\ldots,i_s}_{\tau_{i_1},\ldots,\tau_{i_s}}$ *the following formula is true:*

$$h_\tau=\begin{cases}2|<g>^{\perp}\cap C|2^{-(n+1)/2} & \textit{if} \quad w(\tau) \quad \textit{is even}\\ 2(|C|-|<g>^{\perp}\cap C|)2^{-(n+1)/2} & \textit{else.}\end{cases}$$

**Proof of Theorem 1:** Consider a face $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ where $s = (n+1)/2$. We assume that $w(\sigma)$ is an even number. The proof in the case $w(\sigma)$ is odd, is similar.

Assume $\gamma$ is a full face. Then, with $\sigma = (\sigma_1, \ldots, \sigma_s)$,

$$h_\sigma = 2^{\frac{n-1}{2} - \log\frac{n+1}{2}}. \tag{12}$$

By the previous proposition, as $|C| = 2^{n-\log(n+1)}$, a simple calculation shows that if (12) holds then, with $g$ as above, $|A_g|$ is maximal and $|<g>^\perp \cap C|$ equals $|C|$ or equivalently $g \in <C>^\perp$.

Assume that $g \in <C>^\perp$. Then $|<g>^\perp \cap C|$ equals $|C|$, $A_g$ is maximal and from the proposition above, (12) will be true. This implies that $\gamma$ is a full face.

We have proved that $\gamma$ is a full face if and only if $g \in <C>^\perp$. We also get from the previous paragraph that if $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$, where $s = (n+1)/2$, is a full $(n-1)/2$-face, then any $(n-1)/2$-face $\gamma = \Gamma^{i_1,\ldots,i_s}_{\tau_{i_1},\ldots,\tau_{i_s}}$, where $\tau = (\tau_{i_1}, \ldots, \tau_{i_s})$ has an even weight, is a full face. Theorem 1 is proved.

**Remark 1** It is a triviality to show that if a perfect code has a full $(n-1)/2$-face then it also must have empty faces.

**Definition** To faces in $n$-cube $Z_2^n$, $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ and $\gamma' = \Gamma^{j_1,\ldots,j_{n-s}}_{\tau_{j_1},\ldots,\tau_{j_{n-s}}}$ are said to be orthogonal to each other if

$$\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_{n-s}\} = \emptyset.$$

We say that $\gamma'$ is an *orthogonal complement* of $\gamma$ and write $\gamma' = \gamma^\perp$.

**Proof of Theorem 2:** Assume $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$, $s = (n+1)/2$, is a full face of the perfect code $C$. Let as above $g = e_{i_1} + \ldots + e_{i_s}$. As $\gamma$ is a full face we adopt from the previous proof that the fourier coefficient $A_g$ equals $|C|$.

We now substitute $x_i$ by $-1$ if $i \in \{i_1, \ldots, i_s\}$ in equation (7). As in the proof of the proposition we get

$$\sum_{\sigma \in Z_2^{n-s}} h_\sigma x_{j_1}^{\sigma_{j_1}} \ldots x_{j_{n-s}}^{\sigma_{j_{n-s}}} = \frac{A_g}{2^n} 2^{(n+1)/2} \Pi_{v=1}^{n-s}(1 + x_{j_v}).$$

As $A_g = |C| = 2^{n-\log(n+1)}$ we deduce that

$$h_\sigma = 2^{\frac{n-1}{2} - \log\frac{n+1}{2}} \tag{13}$$

for any $\sigma = (\sigma_{j_1}, \ldots, \sigma_{j_{n-s}}) \in Z_2^{(n-1)/2}$.

As $g \in < C >^{\perp}$, every word $c \in C$ will satisfy $g \cdot c \equiv 0 \; mod\,(2)$ or equivalently

$$w_I(c) \equiv 0 \mod (2) \quad \text{where} \quad I = \{i_1, \ldots, i_s\}. \tag{14}$$

As the difference between any two words of even weight is an even number we get that

$$d_I(c, c') \geq 4 \tag{15}$$

for any two words $c$ and $c'$ of $C$. The theorem is now proved by (13), (14) and (15).

**Proof of Theorem 3:** Let $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ be any $(n-1)/2$-face of the hypercube. From (11) follows that the number of words of $C \cap \gamma$ depends on the fourier coefficient $A_g$ where

$$g = e_{i_1} + \ldots + e_{i_s}.$$

Consider the kernel of $C$. The perfect code $C$ is the disjoint union of cosets of this kernel:

$$C = ker(C) \cup (a_1 + ker(C)) \cup (a_2 + ker(C)) \cup \ldots \cup (a_k + ker(C)).$$

For any $p \in ker(C)$ and for any $g$ with $A_g \neq 0$, $p \cdot g = 0$, see [5]. Hence, for any $c \in a_i + ker(C)$,

$$c \cdot g = a_i \cdot g.$$

From (8) we thus get that $A_g$ is a multiple of the of the number of words of the kernel of $C$. Let $t(g)$ denote the number of words $a_i$, $i = 1, 2, \ldots, k$, with $a_i \cdot g = 0$. Then, by (8'),

$$A_g = 2 \cdot |ker(C)|(1 + t(g)) - |C|$$

and hence from (11) we get that the number of words in the $(n-1)/2$-face $\gamma$ is

$$\frac{|C| + (-1)^{w(\sigma_{i_1},\ldots,\sigma_{i_s})}(2 \cdot |ker(C)|(1 + t(g)) - |C|)}{2^{(n+1)/2}}.$$

As $|C|$ is a multiple of $|ker(C)|$, the theorem is proved.

**Remark** Any perfect code has as many full 3-faces as there are words of weight 3. No full rank perfect code of length $n$ has any full $(n-1)/2$-face by Theorem 1. It would be interesting to decide if, for some $k$ with $2 \leq k \leq \log(n+1) - 3$, there are full rank perfect codes of length $n$ with a full $((n+1)/2^k - 1)$-faces.

# 4 Some results for d-faces

We first consider $((n+1)/2^k-1)$-faces. We need a notation.

Let $N_k(n,2)$ denote the number of subspaces of dimension $k$ of a vector space of dimension $n$ over the finite field $Z_2$. By [3]

$$N_k(n,2)=\prod_{i=0}^{k-1}\frac{2^{n-i}-1}{2^{i+1}-1}$$

**Theorem 5** *Let $C$ be a perfect code of length $n$. If the rank of $C$ equals $r$ then, for any integer $k$ in the interval $1\le k\le n-r$, there are at least $N_k(n-r,2)$ different equivalence classes of full $((n+1)/2^k-1)$-faces. Each such equivalence class contains $2^{n-k+1-2^{-k}(n+1)}$ full and mutually parallel $((n+1)/2^k-1)$-faces.*

**Proof:** We consider the dual space $C^\perp$ of $C$. The dimension of $C^\perp$ equals $n-r$ and any word of $C^\perp$ has weight $(n+1)/2$, see [6]. As $C^\perp$ is a simplex code, [6], it follows that to any subspace $L$ of dimension $k$ of $C^\perp$ there is exactly one subset $J=\{j_1,j_2,\ldots,j_\mu\}$, $\mu=(n+1)/2^k-1$, of $\{1,2,\ldots,n\}$ such that the support of any of the words in $L$ has an empty intersection with the set $J$.

We now proceed as in the proof of Theorem 1. In equation (7) we perform the substitution

$$x_j=1 \quad \text{if} \quad j\in J.$$

We know from (5), that for any word $g$ of $C^\perp$, $A_g=|C|$. By trivial counting arguments, as in the proof of Theorem 1, we get that the face

$$\Gamma^{i_1,\ldots,i_t}_{\sigma_{i_1},\ldots,\sigma_{i_t}}=\{x\in Z_2^n \mid x_{i_v}=\sigma_{i_v} \quad v=1,2,\ldots,t\},$$

where

$$\{i_1,i_2,\ldots,i_t\}=\{1,2,\ldots,n\}\setminus J,$$

is a full $((n+1)/2^k-1)$-face if and only if the word

$$g=e_{\sigma_{i_1}}+e_{\sigma_{i_2}}+\ldots+e_{\sigma_{i_t}}$$

belongs to $L^\perp$. As the dimension of $L^\perp$ equals $n-k$, the number of such words $g$ will be

$$\frac{2^{n-k}}{2^{(n+1)/2^k-1}}$$

(where $2^{(n+1)/2^k-1}$ simply is the number of words of length $(n+1)/2^k-1$).

Finally we consider the most general case. We give a formula for the number of words of a perfect code on a $d$-face $\gamma$, for any integer $d$. The derivation of this formula, which is very similar to the proof of Theorem 3, will be omitted.

**Lemma** *Let $C$ be a perfect code of length $n$ and assume*

$$C = ker(C) \cup (a_1 + ker(C)) \cup (a_2 + ker(C)) \cup \ldots \cup (a_k + ker(C)).$$

*Let for any $g \in Z_2^n$, $t(g)$ denote the number of words in the set $\{a_1, a_2, \ldots, a_k\}$ that are orthogonal to the word $g$. The number of words of $C$ on a $d$-face $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ equals*

$$|C \cap \gamma| = 2^{d-\log(n+1)} + \sum_{\substack{g, \\ supp(g) \subseteq \{i_1,\ldots,i_s\} \\ w(g)=(n+1)/2}} 2^{d-n}(-1)^{\sigma \cdot g}(2|ker(C)|(1 + t(g)) - |C|)$$

*where $\sigma = (\sigma_{i_1}, \ldots, \sigma_{i_s})$.*

We give two corollaries of this Lemma. The first shows that the number of words on a face is related to the size of the kernel.

**Corollary 1** *Let $C$ be any perfect code of length $n$. The number of words on a $d$-face $\gamma$ will be equal to*

$$t \cdot 2^{d-n}|ker(C)|$$

*for some integer $t$.*

**Corollary 2** *Let $C$ be a perfect code of length $n$. The number of words of $C$ on a $d$-face $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$, $s = n - d$, equals*

$$2^{d-\log(n+1)}$$

*if and only if $t(g) = (k-1)/2$ for all $g$ with $w(g) = (n+1)/2$ and $supp(g) \subseteq \{i_1, \ldots, i_s\}$.*

By Theorem 4, if $A_g \neq 0$ then $w(g) = (n + 1)/2$. Hence, as $A_0 = |C|$ we get the following Corollary already proved by Avgustinovich and Vasilieva [2].

**Corollary 3** (Avgustinovich-Vasilieva) *Let $C$ be a perfect code of length $n$. For any $d > (n - 1)/2$, the number of words on a $d$-face will be*

$$2^{d-\log(n+1)}.$$

**Example** We consider the case $d = \log(n+1)$. We get from the Corollary 2 that if $C$ is a perfect code of length $n$ and $\{i_1, \ldots, i_s\}$, $s = n - d$, a subset of the set $\{1, 2, \ldots, n\}$ such that $t(g) = (k-1)/2$, (where $k$ is as in the Lemma) for any word $g$ with $w(g) = (n+1)/2$ and $supp(g) \subseteq \{i_1, \ldots, i_s\}$, then the number of words of $C$ on the $d$-face $\gamma = \Gamma^{i_1,\ldots,i_s}_{\sigma_{i_1},\ldots,\sigma_{i_s}}$ for any word $(\sigma_{i_1}, \ldots, \sigma_{i_s})$, will be equal to one. This means that the perfect code $C$ is *systematic*. The converse statement will also be true.

**Remark 1** Theorem 1 can rather easily be proved also by using the Corollary 3.

**Remark 2** We also very much would like to mention that Theorem 1, Theorem 2, Corollary 2 and Corollary 3 are easy and immediate consequences of the results of [4]. Just put $x_i = 1$, in the equation (7), for $i \in J$ where $J$ is a suitable chosen subset of the set of positions $\{1, 2, \ldots, n\}$.

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