The Hierarchy of the Euclidean Non-linear Schrödinger Equation is a Harmonic Oscillator Containing KdV

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Abstract

It is proved that there are infinitely many commuting conservation laws I_n, $n = 0, 1, ...$ for the Euclidean non-linear Schrödinger equation, and that the KdV equation is contained in this hierarchy. Adding an extra conservation law one obtains an annihilation operator. The (stationary) passage from I_n to I_{n+1} is the creation operator. These two operators yield a quantised harmonic oscillator with ground state I_0.

1. Introduction, the free case. This article has developed from work on the symmetries and conserved quantities for certain diffusion processes appearing in 'Euclidean Quantum Mechanics'. In a sense, the essential (from Schrödinger) idea is to consider two heat equations (with linear interaction), for backward and forward motion. We refer to [2], [3], [5], [11], [12], [13], [17], [18], and further references therein, and start with a simpler, linear, model case, before considering the nonlinear ENLS equation in the next section.

(1.1) The backward and forward free heat equations (in $1 + 1$ dimensions) can be obtained from the Lagrangian ([2], [10])

$$L = \tfrac{1}{2}(u\dot{v} - \dot{u}v) + \tfrac{1}{2}u'v'.$$

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The vanishing of both variational derivatives of the Lagrangian, i.e. $\frac{\delta L}{\delta v} = \frac{\delta L}{\delta u} = 0$ yields the Euler-Lagrange equations. In this case,

$$\frac{\delta L}{\delta v} := \frac{\partial L}{\partial v} - \frac{d}{dt}\frac{\partial L}{\partial \dot{v}} - \frac{d}{dq}\frac{\partial I}{\partial v'},$$

with a similar expression for for $\delta L/\delta u$. The equations become

$$\dot{u} + \tfrac{1}{2}u'' = 0, \qquad -\dot{v} + \tfrac{1}{2}v'' = 0.$$

(1.2) The symmetry Lie algebra for the free heat system contains the vector fields (see [9] and [10])

$$v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial q}, \quad \frac{\partial}{\partial t}$$

corresponding to the conserved quantities (e.g. by Noether's theorem, [8], [9], [16])

$$I_0 = uv, \quad I_1 = \tfrac{1}{2}(u'v - uv'), \quad I_2 = \tfrac{1}{2}(u''v + uv'').$$

They belong to an infinite sequence of conserved quantities

$$I_n := \tfrac{1}{2}(u^{(n)}v + (-1)^n v^{(n)}), \quad n = 0, 1, 2, \ldots.$$

which, again by Noether's theorem, can be obtained from the generalised (evolutionary) vector fields ([8], [9], [16])

$$(-1)^n v^{(n)}\frac{\partial}{\partial v} - u^{(n)}\frac{\partial}{\partial u}, \quad n = 0, 1, 2, \ldots.$$

(1.3) Introduce the bracket (the integral means that calculations are made modulo total space derivatives) from [6]:

$$\{F, G\} := \int \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u} \right) dq.$$

Here F and G are *functionals* ([4], [6], [16]) or *differential functions* ([8], [9]), i.e. smooth functions of u, v and their space derivatives up to an arbitrary but finite order.

We refer to Dickey [4], Ch. 1.2, in which the algebraic construction behind the integral is treated in more detail.

(1.4) It is easily checked—just partially integrate—that all the I_n are in involution, i.e.

$$\{I_n, I_m\} = 0 \quad \text{for all } n,\, m.$$

For any functional F, $\{F, I_m\}$ is the derivative of F with respect to the 'time' t_m given by the hamiltonian I_m: $dF/dt_m = \{F, I_m\}$. Thus involutivity means that all I_n are conserved quantities w.r.t. d/dt_m for all m:

$$\frac{dI_n}{dt_m} = 0 \quad \text{for all } n,\, m.$$

(1.5) The *variational gradient* of a functional F is the vector δF with components $\delta F/\delta u$ and $\delta F/\delta v$. It is convenient to represent it (as in [6]) as the 2×2-matrix

$$\delta F := \begin{pmatrix} 0 & \delta F/\delta u \\ \delta F/\delta v & 0 \end{pmatrix}.$$

(1.6) We see that

$$\delta I_{n+1} = C \delta I_n, \quad C := \begin{pmatrix} -D & 0 \\ 0 & D \end{pmatrix},$$

with $D = d/dq$. We shall also write this as $I_{n+1} = C I_n$. The transition in the sequence (I_n) is *stationary* in the sense that

$$I_n = C^n I_0, \quad n = 0,\, 1,\, 2, \ldots.$$

(1.7) Consider the functional

$$I := -qI_0 = -quv.$$

It is proved in the next section that

$$\{I, I_n\} = nI_{n-1}, \quad n = 0,\, 1,\, 2, \ldots.$$

(1.8) Write

$$A := \mathrm{Ad}_I = \{I, \cdot\}$$

and

$$\mathsf{H} := \tfrac{1}{2}[A, C]_+ = \tfrac{1}{2}(AC + CA).$$

A is the *annihilation* and C the *creation* operator. We have

$$\mathsf{H}I_n = (n + \tfrac{1}{2})I_n.$$

This is a representation of the one-dimensional quantised harmonic oscillator with hamiltonian $\mathsf{H} = \mathsf{N} + \frac{1}{2}$, where N denotes the number operator. It follows that

$$[A, C]I_n = I_n, \quad \text{i.e.} \quad [A, C] = \mathsf{1},$$

the identity.

Preliminary observations indicate that it is natural to look upon the I_n as unscaled Hermite polynomials (orthogonal w.r.t. a Gaussian probability measure, and) with norm $\sqrt{n!}$. With this convention, the number operator N is defined for all $F = \sum_0^\infty c_n I_n$ with $\sum_0^\infty n^2 |c_n|^2/n!$ finite.

(1.9) In general, if $AI_n = \lambda_n I_{n-1}$, and $CI_n = I_{n+1}$, we get $(AC + CA)I_n = (\lambda_{n+1} + \lambda_n)I_n$, and $(AC - CA)I_n = (\lambda_{n+1} - \lambda_n)I_n$. The latter is the identity operator (acting on I_n) if and only if (except for a trivial additive constant which we choose to be zero) $\lambda_n = n$. Then $(AC + CA)I_n = (2n + 1)I_n$, as above.

(1.10) I is *not* a conserved quantity, but is easily adjusted to become one. Fix I_k as our hamiltonian. Replacing I by

$$I_k^* := -(kt_k I_{k-1} + qI_0), \quad k = 1, 2, ...,$$

we have $dI_k^*/dt_k = 0$ and, since t_k is a parameter, the relation $\{I, I_n\} = nI_{n-1}$ is carried over to I_k^*:

$$\{I_k^*, I_n\} = -kt_k\{I_{k-1}, I_n\} - \{qI_0, I_n\} = -\{qI_0, I_n\} = \{I, I_n\}.$$

(1.11) For the heat system, $I^* := tI_1 - qI_0$ is a conserved quantity obtained from the vector field $\Lambda^* := t\frac{\partial}{\partial q} - q(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u})$. I_0, I_1, I^* form a Heisenberg algebra w.r.t. the bracket, as do the corresponding ordinary vector fields w.r.t the ordinary bracket. This vector field is often used as *recursion operator* to obtain an infinite number of conserved quantities for the heat equation. See Ibragimov

[8, 9] or Olver [16]. Λ^* plays a different role in our approach, and in a sense $\partial/\partial q$ is the recursion operator. This is expressed in the formula for C above. There is another way to represent it: with Hirota's bilinear derivative,

$$D(u \cdot v) := \frac{d}{d\varepsilon}(u(\cdot + \varepsilon)v(\cdot - \varepsilon))\big|_{\varepsilon=0} = u'v - uv',$$

we find

$$I_n = (\tfrac{1}{2}D)^n(u \cdot v), \qquad n = 0, 1, \ldots.$$

2. Euclidean NLS Equation.

(2.1) The euclidean non-linear Schrödinger equation is the following system of non-linear heat equations:

$$\dot{u} + \tfrac{1}{2}u'' - u^2 v = 0, \qquad -\dot{v} + \tfrac{1}{2}v'' - uv^2 = 0.$$

It can be found in the list of completely integrable time-symmetric heat equations in [14]. We may look upon these equations as a system

$$\dot{u} + \tfrac{1}{2}u'' - Vu = 0, \qquad -\dot{v} + \tfrac{1}{2}v'' - Vv = 0$$

with the 'non-linear potential' $V = uv$.

(2.2) The equations of motion result from calculus of variations with the Lagrangian

$$L = \tfrac{1}{2}(u\dot{v} - \dot{u}v) + \tfrac{1}{2}(u'v' + u^2v^2).$$

Then $\delta L/\delta v = -\dot{u} - \frac{1}{2}u + u^2 v$ and $\delta L/\delta u = \dot{v} - \frac{1}{2}u + uv^2$.

(2.3) The 'usual' non-linear Schrödinger equation, say

$$\mathrm{i}\dot{\psi} = -\tfrac{1}{2}\psi'' + |\psi|^2\psi,$$

results from the Lagrangian

$$L = \frac{\mathrm{i}}{2}(\psi\overline{\dot{\psi}} - \dot{\psi}\overline{\psi}) + \frac{1}{2}(\psi\overline{\psi'} + \psi^2\overline{\psi}^2).$$

It is extensively treated in Faddeev and Takhtajan [6]. Our key operator C below is a slight adaption of one of their many ways to approach this equation. See [6], Ch. III.5 and V.4.

(2.4) For the ENLS equations, the infinitesimal symmetries

$$v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial q}, \quad \frac{\partial}{\partial t}$$

yield the conserved quantities

$$I_0 = uv, \quad I_1 = \tfrac{1}{2}(u'v - uv'), \quad I_2 = \tfrac{1}{2}(u''v + uv'' - 2u^2v^2).$$

The first one is a basic density, the *vacuum vector* or *ground state* for the oscillator to be constructed below. The second one is the *momentum density* and the third is the *energy density.* The densities are calculated in the *state* (u, v).

(2.5) We will need the explicit form $\delta I_0 = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$.

Write $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let F be a functional. We find

$$[\delta I_0, \delta F] = \left(v\frac{\delta F}{\delta v} - u\frac{\delta F}{\delta u}\right)\sigma_3.$$

Hence

$$\{I_0, F\} = \frac{1}{2}\int \mathrm{tr}\left(\sigma_3[\delta I_0, \delta F]\right) dq,$$

so that $\{I_0, F\} = 0$ if and only if there is a functional a_F such that

$$v\frac{\delta F}{\delta v} - u\frac{\delta F}{\delta u} = Da_F.$$

(2.6) For such an F we may form

$$U_0\delta F := [\delta I_0, D^{-1}[\delta I_0, \delta F]] = -2a_F \begin{pmatrix} 0 & v \\ -u & 0 \end{pmatrix},$$

and

$$C\delta F := -\sigma_3(D - U_0)\delta F = \begin{pmatrix} 0 & -\left(D\frac{\delta F}{\delta u} + 2a_F v\right) \\ D\frac{\delta F}{\delta v} - 2a_F u & 0 \end{pmatrix}.$$

Starting from I_0, one finds $C\delta I_0 = I_1$, $C\delta I_1 = I_2$ and
$C\delta I_2 = I_3 := \frac{1}{2}(u'''v - uv''') - \frac{3}{2}uv(u'v - uv')$.

(2.7) Definition. We *define* I_n by

$$\delta I_n := C^n \delta I_0.$$

The next few I_n are

$$\begin{aligned}
I_4 &= \tfrac{1}{2}(u^{(4)}v + uv^{(4)}) - uv(u''v + uv'') + 4uu'vv' + 2u^3v^3,\\
I_5 &= \tfrac{1}{2}(u^{(5)}v - uv^{(5)}) + 5uv(u''v' - u'v'') + 5u^2v^2(u'v - uv'),\\
I_6 &= \tfrac{1}{2}(u^{(6)}v + uv^{(6)}) - 3(uu''v'^2 + u'^2vv'') - 12uu''vv'' + 5u'^2v'^2\\
&\quad -(u''^2v^2 + u^2v''^2) - 50u^2u'v^2v' - 10uv((u'^2v^2 + u^2v'^2) - 5u^4v^4.
\end{aligned}$$

We can now prove

(2.8) Theorem. *All I_n are in involution:*

$$\{I_n, I_m\} = 0 \qquad \text{for} \quad n,\, m = 0,\, 1, \ldots.$$

Proof. Suppose I_0, ..., I_n all Poisson commute, and form I_{n+1}. Then

$$\{I_{n+1}, I_0\} = \{CI_n, I_0\} = \{I_n, CI_0\} = \{I_n, I_1\} = 0,$$

as one easily sees that C is symmetric w.r.t. the bracket. Repeating this argument one finds $\{I_{n+1}, I_j\} = 0$ for $j < n$. In general, using that $Da_F = v\frac{\delta F}{\delta v} - u\frac{\delta F}{\delta u}$, we get

$$\begin{aligned}
&-\{CF, F\}\\
&= \int \left(\left(D\frac{\delta F}{\delta u} + 2a_F v\right)\frac{\delta F}{\delta v} + \left(D\frac{\delta F}{\delta v} - 2a_F u\right)\frac{\delta F}{\delta u} \right) dq\\
&= \int D\left(\frac{\delta F}{\delta u}\frac{\delta F}{\delta v} + 2a_F^2\right) dq = 0.
\end{aligned}$$

The claim follows upon letting $F = I_n$.

(2.9) Corollary. *For any n, $m = 0, 1, \ldots$*

$$\frac{dI_n}{dt_m} = 0.$$

In particular, each I_n is a conservation law for the euclidean non-linear Schrödinger equations

$$\dot{u} + \tfrac{1}{2}u'' - u^2v = 0, \qquad -\dot{v} + \tfrac{1}{2}v'' - uv^2 = 0.$$

(2.10) Remark on KdV. The Lagrangian using I_3 as Hamiltonian, i.e.

$$L_3 = \tfrac{1}{2}(u\dot{v} - \dot{u}v) + \tfrac{1}{2}(u'''v - uv''') - \tfrac{3}{2}uv(u'v - uv')$$

produces the (Euler-Lagrange) equations of motion

$$\dot{u} = u''' - 6uu'v, \qquad \dot{v} = v''' - 6uvv'.$$

Of course, any I_n is a conservation law for these equations too. Upon choosing $v \equiv 1$ we obtain the Koorteweg-deVries equation for u:

$$\dot{u} = u''' - 6uu'.$$

In this sense, the ENLS hierarchy contains KdV.

As in §1, we introduce

$$I := -qI_0 = -quv.$$

Our next result is

(2.11) Theorem. *For all* $n \geq 0$,

$$\{I, I_n\} = nI_{n-1}.$$

Proof: Write $a_n := a_{I_n}$. In general,

$$\{I, I_n\} = \int -q\left(v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u}\right) dq = \int -qa_n'\, dq = \int a_n\, dq.$$

Hence, we want to prove that

$$a_n = nI_{n-1}.$$

Use of the (creation) operator C and partial integration, leads to the relation

$$\begin{aligned} a_{n+1}' =& u^{(n+1)}v + (-1)^n uv^{(n+1)} \\ & -2\Big((a_1u)^{(n-1)}v + (-1)^n u(a_1v)^{(n-1)} + \ldots \\ & \quad + (a_{n-2}u)''v - u(a_{n-2}v)'' + (a_{n-1}u)'v + u(a_{n-1}v)'\Big). \end{aligned}$$

Assuming $a_k = kI_{k-1}$ for all $k \leq n$, we may write

$$a'_{n+1} = A_{n+1} + A_{n-1} + \ldots..,$$

where the index on the right refers to the total number of derivatives. The terms of lowest order will come from

$$-2\big((a_{n-1}u)'v + u(a_{n-1}v)'\big)$$

if n is even, and from

$$-2\big((a_{n-2}u)''v - u(a_{n-2}v)'' + (a_{n-1}u)'v + u(a_{n-1}v)'\big)$$

if n is odd.

In the former case, the hypothesis yields

$$a'_{n+1} = -2(n-1)((I_{n-2}uv)' + I'_{n-2}uv) + \text{ h. o. t.}$$

In general,

$$I_{2m} = c_{2m}(uv)^{m+1} + \text{ h. o. t.}$$

where the coefficient is (if $(-1)!! = 1$)

$$c_{2m} = (-2)^m \frac{(2m-1)!!}{(m+1)!}, \qquad m = 0, 1, 2, \ldots$$

This expression can be found by repeated use of the following formulae for C^2:

$$\begin{aligned}\frac{\delta I_{k+2}}{\delta u} &= D^2 \frac{\delta I_k}{\delta u} + 2(a_k v)' - 2a_{k+1}v,\\ \frac{\delta I_{k+2}}{\delta v} &= D^2 \frac{\delta I_k}{\delta v} - 2(a_k u)' - 2a_{k+1}u.\end{aligned}$$

With $n = 2m$, and writing $s := uv$, the terms of lowest order are

$$\begin{aligned}&-2(2m-1)c_{2(m-1)}\big((s^{m+1})' + (s^m)'s\big)\\ =&-2(2m-1)c_{2(m-1)}(2m+1)s^m s'\\ =&(2m+1)(-2)^m \frac{(2m-1)!!}{m!}\frac{(s^{m+1})'}{m+1} = (2m+1)(c_{2m}s^{m+1})',\end{aligned}$$

which proves the assertion in this case.

In the case when n is *odd*, $n = 2m+1$, the lowest order terms for a'_{2m+2} are obtained from

$$\begin{aligned}
-2\big(2m(I_{2m-1}s)' \\
+2mI'_{2m-1}s + (2m-1)I_{2m-2}a \\
+(2m-1)(I_{2m-2}a)'\big),
\end{aligned}$$

where, in addition to $s = uv$, we have written $a := u'v - uv'$. In general,

$$I_{2m+1} = c_{2m+1}s^m a + \text{ h. o. t.}$$

for some constant c_{2m+1}. Hence the middle terms above are

$$\begin{aligned}
& 2mc_{2m-1}(s^{m-1}a)'s + (2m-1)c_{2m-2}(s^m)'a \\
= \ & (2mc_{2m-1}(m-1) + (2m-1)mc_{2m-2})s^{m-1}s'a + 2mc_{2m-1}s^m a' \\
= \ & (2(m-1)c_{2m-1} + (2m-1)c_{2m-2})(s^m)'a + 2mc_{2m-1}s^m a' \\
= \ & 2mc_{2m-1}(s^m a)',
\end{aligned}$$

provided $2(m-1)c_{2m-1} + (2m-1)c_{2m-2} = 2mc_{2m-1}$, i.e.

$$c_{2m-1} = \frac{2m-1}{2}c_{2(m-1)}.$$

One may deduce this formula from the formula for c_{2m} together with the formulae for C^2 displayed above.

The lowest order terms become

$$-2(2 \cdot 2mc_{2m-1} + (2m-1)c_{2m-2})(s^m a)'.$$

The coefficient can be written

$$\begin{aligned}
& -2(2m-1)(2m+1)c_{2m-2} \\
& = 2(m+1) \cdot \frac{2m+1}{2}(-2)^m \frac{(2m-1)!!}{(m+1)!} = 2(m+1)c_{2m+1},
\end{aligned}$$

which proves our claim

$$a'_{2(m+1)} = 2(m+1)c_{2m+1}(s^m a)' + \text{ h. o. t.}$$

By induction, we may assume that all terms of order strictly less than the highest order, viz. $n+1$, satisfy the corresponding identity. It remains to prove that the $J_n := \frac{1}{2}(u^{(n)}v + (-1)^n uv^{(n)})$ fulfil

$$\{I, J_n\} = nJ_{n-1} \qquad \text{for all} \quad n.$$

This is the relation $\{I, I_n\} = nI_{n-1}$ in the free case, referred to in §1. We must show that $-q(u^{(n)}v - (-1)^n uv^{(n)}) \simeq nJ_{n-1}$, where '$\simeq$' signifies equivalence modulo total (space) derivatives. The left hand-side is equivalent to

$$\begin{aligned} &u^{(n-1)}v + (-1)^{n-1}uv^{(n-1)} + q(u^{(n)}v' - (-1)^n u'v^{(n)}) \\ &\quad = 2J_{n-1} + q(u^{(n)}v' - (-1)^n u'v^{(n)}). \end{aligned}$$

If $n = 2m+1$, repetition of this leads to

$$\begin{aligned} &\simeq 2mJ_{2m} - (-1)^m q(u^{(m+1)}v^{(m)} + u^{(m)}v^{(m+1)}) \\ &\simeq 2mJ_{2m} + (-1)^m u^{(m)}v^{(m)} \simeq (2m+1)J_{2m}. \end{aligned}$$

A similar, slightly longer, calculation yields the result for even n.
The theorem follows.

We have the following version of Newton's free equations:

(2.12) Corollary. Write $\overline{q} := \int q\, uv\, dq$ for the expectation value of the position. Then

$$\frac{d^2\overline{q}}{dt_n^2} = 0, \quad n = 0,\ 1,\ 2,\ .....$$

Proof: The left hand-side is $-\{\{I, I_n\}, I_n\} = -n\{I_{n-1}, I_n\} = 0$.

(2.13) Remarks: (i) As in the introduction, we may now use A and C to get an oscillator with Hamiltonian $\frac{1}{2}(AC + CA)$.

(ii) For the ENLS equation $I^* = tI_1 - qI_0$ is a conservation law, just as in the free case. Also, we may, for any $k > 0$, replace I by I_k^*, which is a conservation law using I_k as Hamiltonian in the Lagrangian $L = L_k := \frac{1}{2}(u\dot{v} - \dot{u}v) - I_k$.

(iii) The Lagrangian integrals used here are in a sense generalisations to infinite dimension of the Hilbert integral and Poincaré-Cartan invariant $\int p\, dq - H(t, q, p)\, dt$ from classical mechanics, see e.g., Arnold [1]. The first fundamental form pdq may be replaced by $\frac{1}{2}(pdq - qdp) = \frac{1}{2}(p\dot{q} - q\dot{p})\, dt$ along solutions.

The 'extra variable trick', adding v to u, in classical mechanics can be found in the book of Morse and Feshbach, [15]. It is mentioned in passing in Goldstein [7], that it yields the Schrödinger equations from a variational principle. See also Brandão and Kolsrud [3] and Ibragimov and Kolsrud [10], where it is used to obtain conservation laws via Noether's theorem for several known evolution equations.

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