1) $n=4: \; 1+2+3+4 = 10 = \frac{4\cdot 5}{2}$

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

2) $$\int_0^1 x^2 e^x\,dx = x^2 e^x\Big|_0^1 - \int_0^1 2x e^x\,dx$$

$$= e - \left(2xe^x\Big|_0^1 - \int_0^1 2e^x\,dx\right)$$

$$= e - \left(2e - 2e^x\Big|_0^1\right) = -e + (2e-2) = e-2$$

3) $$e^{3ix} = (\cos 3x + i \sin 3x)$$

$$\| $$

$$(\cos x + i\sin x)^3 = (\cos x)^3 + 3i(\cos x)^2\sin x - 3\cos x(\sin x)^2 - i(\sin x)^3$$

$$\Rightarrow \frac{\cos 3x}{\sin 3x} = \frac{(\cos x)^3 - 3(\cos x)(\sin x)^2}{3(\cos x)^2\sin x - (\sin x)^3} = \frac{\cos x}{\sin x}\left(\frac{(\cos x)^2 - 3(\sin x)^2}{3(\cos x)^2 - (\sin x)^2}\right)$$

4) $$1+2+4+\dots+2^{63} = \frac{1-2^{64}}{1-2} = 2^{64}-1$$

5) $$\sqrt{1+x} = f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + O(x^3) = 1 + \frac{x}{2} - \frac{x^2}{8} + O(x^3)$$

$$\left(f'(x) = \frac{1}{2\sqrt{1+x}},\; f''(x) = -\frac{1}{4}(1+x)^{-3/2}\right)$$

$$\Rightarrow \lim_{x\to 0} \frac{1+\frac{x}{2}-\sqrt{1+x}}{x^2} = \lim_{x\to 0}\left(\frac{\frac{x^2}{8}+O(x^3)}{x^2}\right) = \frac{1}{8}$$

6) $f(x) = x^2 - |3x-2| = \begin{cases} x^2-3x+2 & f.\ x \geq \frac{2}{3} \\ x^2+3x-2 & f.\ x \leq \frac{2}{3} \end{cases}$

$f'(x) = \left\{ \begin{array}{ll} 2x-3 & f.\ x > \frac{2}{3} \\ 2x+3 & f.\ x < \frac{2}{3} \end{array} \right\} \overset{!}{=} 0$

$\Rightarrow$ möjliga extrempunkter är $x_+ = \frac{3}{2}$, $x_- = -\frac{3}{2}$, och $x_0 = \frac{2}{3}$ (f är inte deriverbar i $x_0 = \frac{2}{3}$);

$f''(x) = +2$ f. $x \neq \frac{2}{3} \Rightarrow$ x_+ och x_- är lokala Minimipunkter;

$x_0 = \frac{2}{3}$ är en lokal Maximipunkt, eftersom

$\lim_{x \nearrow \frac{2}{3}} f'(x) \overset{!}{=} \frac{4}{3} + 3 > 0$, $\lim_{x \searrow \frac{2}{3}} f'(x) = \frac{4}{3} - 3 < 0$.

7) $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow y_h(x) = Ae^x + Be^{2x}$

$= (\lambda-2)(\lambda-1)$

$y_1'' - 3y_1' + 2y_1 \overset{!}{=} e^x$: $y_1(x) = C(x)e^x$ (Resonans)

$\|$ $\quad y_1' = (C+C')e^x$, $y_1'' = (C+2C'+C'')e^x$

$e^x(C+2C'+C''-3C-3C'+2C) \Rightarrow C'' - C' = 1 \Rightarrow C = -x$ + const.

$\underbrace{y_2'' - 3y_2' + 2y_2}_{} \overset{!}{=} 2x^2$: $y_2(x) = x^2+ax+c$, $y_2' = 2x+a$, $y_2'' = 2$

$= 2 - 3(2x+a) + 2(x^2+ax+c) \overset{!}{=} 2x^2$

$\Rightarrow -6x + 2ax = 0$, $-3a + 2c = 0 \Rightarrow a = 3$, $c = 9/2$

$\Rightarrow y(x) = Ae^x + Be^{2x} + (x^2 + 3x + 9/2) - xe^x$

8) $J = \int_{1/3}^{1} \frac{dr}{r^2\sqrt{3+\frac{2}{r}-\frac{1}{r^2}}} \overset{(u=\frac{1}{r})}{=} -\int_{3}^{1} \frac{du}{\sqrt{3+2u-u^2}} = +\int_{1}^{3} \frac{du}{\sqrt{3+2u-u^2}}$

$= \int_{1}^{3} \frac{du}{\sqrt{4-(u-1)^2}} \underset{(w=u-1)}{=} \int_{0}^{2} \frac{dw}{\sqrt{4-w^2}} \underset{(w=2\sin\theta)}{=} \int_{0}^{\pi/2} d\theta = \pi/2$

(OBS: $J = \lim_{\epsilon \to 0} \int_{1/3+\epsilon}^{1} \dots$ är en oegentlig integral eftersom $3+\frac{2}{r}-\frac{1}{r^2}=0$ f. $r=\frac{1}{3}$)

9) $x \neq 1$: $f(x) = x^2+x+1 \Rightarrow f'(x) = 2x+1$

$\left(f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2+(x+h)+1-x^2-x-1}{h}\right.$

$\left. = \lim_{h\to 0} \frac{2xh+h+h^2}{h} = 2x+1 \right)$

$x=1$: $\lim_{h\to 0}\left(\frac{(1+h)^2+(1+h)+1-3}{h}\right) = \lim_{h \to 0} \frac{3h+h^2}{h} = 3$

10) area $= 2(\pi r^2)+(2\pi r)h \overset{!}{=} A$; volym $= \pi r^2 h$

$\Rightarrow rh = -r^2 + \frac{A}{2\pi}$ (*)

$\Rightarrow V(r) = \pi r\left(\frac{A}{2\pi} - r^2\right)$. $V'(r) = \frac{A}{2} - 3\pi r^2 \overset{!}{=} 0$

$= \frac{rA}{2} - \pi r^3$ $\Rightarrow r^2 = \frac{A}{6\pi}$; $A = 6\pi R^2$

$RH \overset{(*)}{=} -R^2 + \frac{A}{2\pi} = -R^2+3R^2 = 2R^2 \Rightarrow H = 2R$

(OBS: man kan också minimera cylinderns area, då cylinderns volym är given, och få samma resultat)