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CHERN NUMBERS OF AMPLE VECTOR BUNDLES ON TORIC SURFACES

SANDRA DI ROCCO AND ANDREW J. SOMMESE

Abstract. This article shows a number of strong inequalities that hold for the Chern numbers c_1^2, c_2 of any ample vector bundle $\mathcal{E}$ of rank r on a smooth toric projective surface, S, whose topological Euler characteristic is $e(S)$. One general lower bound for c_1^2 proven in this article has leading term $(4r+2)e(S)\ln_2\left(\frac{e(S)}{12}\right)$. Using Bogomolov instability, strong lower bounds for c_2 are also given. Using the new inequalities, the exceptions to the lower bounds $c_1^2 > 4e(S)$ and $c_2 > e(S)$ are classified.

Introduction

Let $\mathcal{E}$ be an ample rank r bundle on a smooth toric projective surface S, whose topological Euler characteristic is $e(S)$. In this article, we prove a number of surprisingly strong lower bounds for $c_1(\mathcal{E})^2$ and $c_2(\mathcal{E})$.

First, we show Corollary 3.2, which says that, given S and $\mathcal{E}$ as above, if $e(S) \geq 5$, then $c_1(\mathcal{E})^2 \geq r^2 e(S)$. Though simple, this is much stronger than the known lower bounds over not necessarily toric surfaces. For example, see [BSS94, Lemma 2.2], where it is shown that there are many rank two ample vector bundles with $(c_1(\mathcal{E})^2, c_2(\mathcal{E})) = (4,1)$ on products of two smooth curves, at least one of which has positive genus.

We then prove an estimate, Theorem 3.6, which is quite strong for large $e(S)$ and r. As $e(S)$ goes to ∞ with r fixed, the leading term of this lower bound is $(4r+2)e(S)\ln_2(e(S)/12)$, while if e is fixed and r goes to ∞, the leading term of this lower bound is $3(e(S)-4)r^2$. For example, $c_1^2(\mathcal{E}) \geq 3r^2 e(S)$, for $r \leq 3$ if $e(S) \geq 13$, or for $r \leq 6$ if $e(S) \geq 19$, or for $r \leq 141$ if $e(S) \geq 100$. Or again, $c_1^2(\mathcal{E}) \geq 5r^2 e(S)$, for $r \leq 10$ if $e(S) \geq 100$. We include a three-line Maple program in Remark 3.7 for plotting the expression for the lower bound.

The strategy is to use the adjunction process to find lower bounds for $c_1(\mathcal{E})^2$. Toric geometry has two major implications for the adjunction process. First, given an ample rank r vector bundle $\mathcal{E}$ on a smooth toric surface S, there is the inequality $-\det\mathcal{E}\cdot K_S \geq e(S)(\text{rank }\mathcal{E})$. Adjunction theory yields the lower bound for $c_1(\mathcal{E})^2$ given in Corollary 3.2, which implies that $c_1(\mathcal{E})^2 > r^2 e(S)$ for $e(S) \geq 7$. The second important fact is that $h^0(tK_S + \det\mathcal{E}) > 0$ for integers t between 0 and at least rank $\mathcal{E} + \ln_2(e(S)/6)$. Adjunction theory yields the strong lower bound given in Theorem 3.6 for $c_1(\mathcal{E})^2$ when $e(S) \geq 7$.

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Using Bogomolov's instability theorem, we get the strong lower bound given in Theorem 3.9 for the second Chern class, $c_2(\mathcal{E})$, of a rank two ample vector bundle. Basically if $c_2(\mathcal{E})$ is less than one fourth the lower bound already derived for $c_1(\mathcal{E})^2$, then we have an unstable bundle, and Bogomolov's instability theorem combined with the Hodge index theorem gives strong enough conditions to get a contradiction. The short list of exceptions to the bound $c_2(\mathcal{E}) > e(S)$ are classified. Even assuming $\mathcal{E}$ very ample on a nontoric surface, the best general result [BSS96] shows only that $c_2(\mathcal{E}) \geq 1$ with equality for $\mathbb{P}^2$.

Inequalities derived from adjunction theory usually have the form, "some inequality is true if certain projective invariants are large enough." Typically, examples exist outside the range where the adjunction-theoretic method works. For rank two ample vector bundles $\mathcal{E}$ we use a variety of special methods, including adjunction theory and Bogomolov's instability theorem, to enumerate the exceptions to either the inequality $c_1(\mathcal{E})^2 \geq 4e(S)$ or the inequality $c_2(\mathcal{E}) \geq e(S)$ holding. The exceptions are collected in Table 1.

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1. Background material

In this paper we work over $\mathbb{C}$. By a variety we mean a complex analytic space, which might be neither reduced or irreducible.

A rank 2 vector bundle $\mathcal{E}$ on a nonsingular surface S is called *Bogomolov unstable* [R78], or *unstable* for short, if $c_1(\mathcal{E})^2 > 4c_2(\mathcal{E})$. When $\mathcal{E}$ is unstable there exist a line bundle $\mathcal{A}$ and a zero subscheme $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ fitting in the exact sequence

$$0 \to \mathcal{A} \to \mathcal{E} \to (\det \mathcal{E} - \mathcal{A}) \otimes \mathcal{I}_{\mathcal{Z}} \to 0, \tag{1}$$

with the property that for all ample line bundles $\mathcal{L}$ on S, $(2\mathcal{A} - \det \mathcal{E}) \cdot \mathcal{L} > 0$. The standard consequences of this result that we will often use in this article are:

(1) $(2\mathcal{A} - \det \mathcal{E}) \cdot (2\mathcal{A} - \det \mathcal{E}) > 0$, and $2\mathcal{A} - \det \mathcal{E}$ is $\mathbb{Q}$-effective; and
(2) for all nef and big line bundles $\mathcal{L}$ on S, $(2\mathcal{A} - \det \mathcal{E}) \cdot \mathcal{L} > 0$.

We define $\mathcal{H} := \det \mathcal{E}$. Note that

- $c_2(\mathcal{E}) = \mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) + \deg(\mathcal{Z})$, where $\deg(\mathcal{Z}) = h^0(\mathcal{O}_{\mathcal{Z}})$; and
- the line bundle $\mathcal{H} - \mathcal{A}$ is a quotient of $\mathcal{E}$ off a codimension two subset, and therefore it is ample when $\mathcal{E}$ is ample.

Using the Hodge inequality $(\mathcal{H}-\mathcal{A})^2\ (2\mathcal{A}-\mathcal{H})^2 \leq [(\mathcal{H} - \mathcal{A}) \cdot (2\mathcal{A} - \mathcal{H})]^2$, we obtain the following:

$$\mathcal{A} \cdot (\mathcal{H} - \mathcal{A}) \geq (\mathcal{H} - \mathcal{A})^2 + \sqrt{(\mathcal{H} - \mathcal{A})^2}. \tag{2}$$

A toric surface S is a surface containing a two-dimensional torus as Zariski open subset and such that the action of the torus on itself extends to S. All toric surfaces are normal. In this article we consider surfaces polarized by an ample vector bundle; therefore S will always denote a normal projective toric surface. For basic definitions on toric varieties we refer to [O88].

We recall that if $e := e(S)$ is the Euler characteristics of S, then $\operatorname{rank}(\operatorname{Pic}(S)) = e - 2$ and $K_S^2 = 12 - e$.

We need the following useful lemmas, which are probably well known.

Lemma 1.1. *Let $\mathcal{E}$ be a vector bundle over a normal n-dimensional toric variety. Assume $\mathbb{P}(\mathcal{E})$ is toric. Then $\mathcal{E} = \bigoplus L_i$, where the L_i are equivariant line bundles.*

Proof. Consider the bundle map $\mathbb{P}(\mathcal{E}) \to X$ with fiber $F = \mathbb{P}^{r-1}$, where $r := rank(\mathcal{E})$. Every fiber has r fixed points which define an unramified r-to-one cover of X, $p : Y \to X$. X being a normal toric variety, and thus simply connected, implies $Y = \bigcup X_i$ and $\mathcal{E} = \bigoplus L_i$. □

It is classical [L82], [RV] that a surjective morphism $p : X \to Y$, with connected fibers between normal projective varieties, induces a homomorphism from the connected component of the identity of the automorphism group of X to the connected component of the identity of the automorphism group of Y, with respect to which p is equivariant. Using this basic fact, we have the following lemma.

Lemma 1.2. *Let $p : X \to Y$ be a surjective morphism with connected fibers from a normal toric variety X onto a normal variety Y. Then Y admits the structure of a toric variety such that p becomes a toric morphism.*

Corollary 1.3. *Let L be an ample line bundle on a smooth projective toric surface S. If $f : S \to \mathbb{P}^1$ is a morphism with connected fibers, then the general fiber F is isomorphic to $\mathbb{P}^1$, there are at most two singular fibers, and $e(S) \leq 2 + 2L \cdot F$.*

Proof. Since the general fiber is toric, it is isomorphic to $\mathbb{P}^1$. From equivariance we see that any singular fiber must lie over the two fixed points of $\mathbb{P}^1$. Since there are at most $L \cdot F$ irreducible components in a fiber, and there are at most two singular fibers, the inequality follows by considering the cases of no, one, or two singular fibers. □

Corollary 1.4. *Let $f : S \to S'$ express a smooth toric surface S as the equivariant blowup of a smooth projective toric surface S' at a finite set B. Then $e(S) \leq 2e(S')$.*

Proof. Let $b := e(B)$, i.e., b equals the cardinality of the finite set B. Then we have $e(S) = e(S') + b$. Since S' is toric and the elements of B are fixed points of the toric action, we conclude that $e(B)$ is bounded by the cardinality of the set of toric fixed points on S', which is equal the Euler characteristic of S'. Thus we have $e(S) = e(S') + b \leq 2e(S')$. □

Let S be an irreducible toric surface. Then under the prescribed torus action there are $e := e(S)$ one-dimensional orbits. Denote their closures by D_i where $1 \leq i \leq e$. We have the fundamental fact that

$$-K_S = \sum_{i=1}^{e(S)} D_i. \tag{3}$$

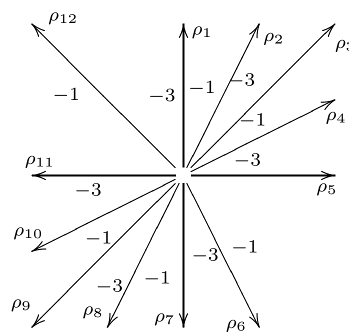
We begin with a very simple observation which is in fact an important tool in all our main results:

Lemma 1.5. *Let $\mathcal{E}$ be an ample rank r vector bundle on a projective normal toric surface S, and let $\mathcal{H}$ denote $\det \mathcal{E}$. Then $-K_S \cdot \mathcal{H} \geq re(S)$.*

Proof. Let $\mathcal{H} := \det \mathcal{E} = \sum_1^e a_i D_i$. By ampleness, $\mathcal{H} \cdot D_i \geq r$ for all $i = 1, \ldots, e$. Since $K_S = \sum_1^e (-D_i)$, we have $-K_S \cdot \mathcal{H} = \sum_1^e \mathcal{H} \cdot D_i \geq er$. □

Remark 1.6. In order to obtain the results in this paper we use the bound 1.5 for $-KL$. The following example shows that in general we cannot hope for a better bound.

Consider the toric surface given by the fan below, spanned by 12 edges $\{\rho_i\}$ and with 12 2-cones, i.e., 12 fixed points. The number before each edge indicates the self-intersection of the associated invariant divisor D_i.



This surface is the equivariant blowup of $\mathbb{P}^2$ in 9 points, and thus the Euler characteristic $e(S) = 12$. Consider the line bundle

$$\begin{aligned} L = 3D_1 + 5D_2 + 3D_3 + 5D_4 + 3D_5 + 5D_6 \\ + 3D_7 + 5D_8 + 3D_9 + 3D_{10} + 3D_{11} + 5D_{12}. \end{aligned}$$

It is ample, since $L \cdot D_i = 5 - 9 + 5 = 1$ for $i = 1, 3, 5, 7, 9, 11$ and $L \cdot D_i = 3 - 5 + 3 = 1$ for $i = 2, 4, 6, 8, 10, 12$. This also gives

$$-LK_S = \sum_1^{12} L \cdot D_i = 12 = e.$$

Clearly this example can be generalized to higher values of e.

We end with a simple corollary of Lemma 1.5.

Corollary 1.7. *Let $\mathcal{E}$ be an ample rank r vector bundle on a smooth projective toric surface S, and let $c_1^2 := c_1(\mathcal{E})^2$. If $c_1^2 \leq re(S)$, then $r \leq 3$ and either $g(\det \mathcal{E}) = 0$, and $(S, \mathcal{E})$ is*

(1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ *with* $(c_1^2, e) = (1, 3)$*; or*
(2) $(\mathbb{F}_0, aE + bf)$ *with* $1 \leq ab \leq 2$ *and* $(c_1^2, e) = (2ab, 4)$*; or*
(3) $(\mathbb{F}_1, E + 2f)$ *with* $(c_1^2, e) = (3, 4)$*; or*
(4) $(\mathbb{F}_2, E + 3f)$ *with* $(c_1^2, e) = (4, 4)$*; or*
(5) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ *with* $(c_1^2, e) = (4, 3)$*;*

or $g(\mathcal{H}) = 1$*, and* $(S, \mathcal{E})$ *is*

(1) $(S, -K_S)$ *with* $(c_1^2, e) = (6, 6)$*; or*
(2) $(\mathbb{F}_0, (E + f) \oplus (E + f))$ *with* $(c_1^2, e) = (8, 4)$*; or*
(3) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ *with* $(c_1^2, e) = (9, 3)$*.*

Proof. Let $\mathcal{H} := \det \mathcal{E}$. If $\mathcal{H}^2 \leq re$, then from $K_S \cdot \mathcal{H} \leq -re$ we conclude that $2g(\mathcal{H}) - 2 = \mathcal{H}^2 + K_S \cdot \mathcal{H} \leq 0$, and thus that $g(\mathcal{H}) \leq 1$.

If $g(\mathcal{H}) = 0$, we know from classification theory, e.g., [BS95], [F90], that S is $\mathbb{P}^2$ or $\mathbb{F}_r$. A simple calculation shows the listed examples are the only ones possible.

If $g(\mathcal{H}) = 1$, then from classification theory, e.g., [BS95], [F90], we know that $(S, \mathcal{H})$ is either a scroll over an elliptic curve or a del Pezzo surface with $\mathcal{H} = -K_S$. Since S is toric and therefore rational, S is del Pezzo. □

2. Vector bundles over $\mathbb{P}^2$ and $\mathbb{F}_\epsilon$

In this section we describe all pairs $(S, \mathcal{E})$ where $\mathcal{E}$ is an ample rank two bundle on a $\mathbb{P}^2$ or a Hirzebruch surface, with the property that either $c_1(\mathcal{E})^2 \leq 4e(S)$ or $c_2(\mathcal{E}) \leq e(S)$. Later in the paper it will be shown that these are all of the examples of rank 2 ample vector bundles $\mathcal{E}$ on smooth toric surfaces S with either $c_1(\mathcal{E})^2 \leq 4e(s)$ or $c_2(\mathcal{E}) \leq e(S)$. The following table includes the various cases. We give the Chern classes and indicate whether the bundle is Bogomolov unstable (U), stable (S), or a boundary case, i.e., $c_1^2 = 4c_2$, (B).

Table 1. All pairs $(S, \mathcal{E})$, with $\mathcal{E}$ an ample rank two vector bundle on a smooth toric projective surface S, and with either $c_1(\mathcal{E})^2 \leq 4e(S)$ or $c_2(\mathcal{E}) \leq e(S)$. The only class where we do not know existence and uniqueness is listed on the last line of the table.

S	$e(S)$	$\mathcal{E}$	$c_1(\mathcal{E})^2$	$c_2(\mathcal{E})$	$U/S/B$
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$	4	1	B
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$	9	2	U
$\mathbb{P}^2$	3	$T_{\mathbb{P}^2}$	9	3	S
$\mathbb{P}^2$	3	$\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$	16	3	U
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	8	2	B
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	12	3	B
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)) \otimes \xi$	16	4	B
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	16	4	B
$\mathbb{P}^1 \times \mathbb{P}^1$	4	$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)$	18	4	U
$\mathbb{F}_1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	12	3	B
$\mathbb{F}_1$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \xi$	16	4	B
$\mathbb{F}_2$	4	$p^*(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \xi$	16	4	B
del Pezzo	6	$(-K_S) \oplus (-K_S)$	24	6	B
del Pezzo	6	if any example exists, $\det \mathcal{E} = -2K_S$	24	≥ 7	S

Fix the notation $c_2 := c_2(\mathcal{E})$, $\mathcal{H} := c_1 = \det \mathcal{E}$, and $e := e(S)$. The strategy that we follow is to first classify the pairs with $c_1(\mathcal{E})^2 \leq 4e(S)$. Then any pair $(S, \mathcal{E})$ with $c_2 \leq e$ has already been enumerated, or we have $c_2 \leq e < 4c_1^2$. In the latter case the bundle is unstable and we use the extra relations arising from Bogomolov's instability theorem to classify the pair.

2.1. $\mathbb{P}^2$. Let $\mathcal{E}$ be a rank two ample vector bundle over $\mathbb{P}^2$. Since $\mathcal{H}$ is the determinant bundle of a rank two bundle, $\deg(\mathcal{H}|_\ell) \geq 2$ for every line $\ell \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. It follows that $\mathcal{H} = \mathcal{O}_{\mathbb{P}^2}(a)$ with $a \geq 2$. If $\mathcal{H}^2 \leq 4e = 12$, then $a = 2, 3$. In case $a = 2$, the restriction of $\mathcal{E}$ to each line ℓ is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, and thus by the classical results on uniform bundles [OSS80], $\mathcal{E} = \boxed{\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)}$. In case $a = 3$, the restriction of $\mathcal{E}$ to each line ℓ is $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)$, and thus by the classical results on uniform bundles [OSS80], $\mathcal{E}= \boxed{\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(2)}$ or $\mathcal{E}=\boxed{T_{\mathbb{P}^2}}$, the tangent bundle of $\mathbb{P}^2$.

Now assume that $c_2(\mathcal{E})\le 3$, but $c_1^2>4e=12$. Thus it follows that $\mathcal{H}=\mathcal{O}_{\mathbb{P}^2}(a)$ with $a\ge 4$. Since $\mathcal{E}$ is unstable, we have a sequence as in (1) where $\mathcal{H}-\mathcal{A}=\mathcal{O}_{\mathbb{P}^2}(x)$ and $\mathcal{A}=\mathcal{O}_{\mathbb{P}^2}(x+b)$ for $x,b>0$. The inequalities $3\ge c_2(\mathcal{E})=x(x+b)+\deg(\mathcal{Z})$ and $a=2x+b\ge 4$ yield the only numerical possibility: $(x,b+x)=(1,3)$ and $\deg(\mathcal{Z})=0$. Since $H^1(\mathbb{P}^2,2\mathcal{A}-\mathcal{H})=0$, we conclude that the exact sequence splits, and it follows that $\mathcal{E}= \boxed{\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(3)}$.

2.2. **The Hirzebruch surfaces $\mathbb{F}_\epsilon$.** Let $\mathbb{F}_\epsilon=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(\epsilon))$ be the Hirzebruch surface of degree ϵ. Denote by $p:\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\otimes\mathcal{O}_{\mathbb{P}^1}(\epsilon))\to\mathbb{P}^1$ the projection map, and let F denote a fiber of p. Let $\xi_{\mathcal{E}}$ denote the tautological line bundle on $\mathbb{F}_\epsilon$, so that $p_*\xi_{\mathcal{E}}\cong\mathcal{E}$. Recall that $\mathrm{Pic}(\mathbb{F}_\epsilon)=\mathbb{Z}F\oplus\mathbb{Z}E$, where E is the section corresponding to the surjection $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(\epsilon)\to\mathcal{O}_{\mathbb{P}^1}$. Note that $E^2=-\epsilon$.

The following is useful.

Lemma 2.1. *Let $\mathcal{E}$ be a rank r ample vector bundle on $\mathbb{F}_\epsilon$. Then $\det\mathcal{E}\cdot F\ge r$, with equality if and only if $\mathcal{E}\cong p^*V\otimes\xi_{\mathcal{E}}$, where $V\cong\mathcal{E}_E$. In particular, in this case*

$$c_1(\mathcal{E})^2=r^2\epsilon+2r\det\mathcal{E}\cdot E\ge r^2(2+\epsilon),$$

and

$$c_2(\mathcal{E})=\binom{r}{2}\epsilon+(r-1)\det\mathcal{E}\cdot E\ge\binom{r}{2}(2+\epsilon).$$

Proof. Since $\mathcal{E}$ is a rank r ample vector bundle, and F is a smooth rational curve, we conclude that $\det\mathcal{E}\cdot F\ge r$, with equality if and only if $\mathcal{E}_F\cong\mathcal{O}_{\mathbb{P}^1}(1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(1)$. In this case we have that $\mathcal{E}\otimes\xi_{\mathcal{E}}^*$ is trivial on every fiber, and thus $\mathcal{E}\otimes\xi_{\mathcal{E}}^*\cong p^*V$ for some rank r vector bundle on $\mathbb{P}^1$. Finally, note that $V\cong(p^*V)_E\cong\mathcal{E}_E$. The rest of the lemma is a straightforward calculation. □

We record one simple corollary of the above lemma.

Corollary 2.2. *Let $\mathcal{E}$ be a rank r ample vector bundle on $\mathbb{F}_\epsilon$. If $\epsilon\ge 2$ and $c_1(\mathcal{E})^2\le 4r^2$, then $\epsilon=2$ and $\mathcal{E}\cong p^*(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(1))\otimes\xi_{\mathcal{E}}$. In this case $c_1(\mathcal{E})^2=4r^2$ and $c_2(\mathcal{E})=2r(r-1)$.*

Proof. Let $\mathcal{H}:=\det\mathcal{E}=aE+bF$. Using Lemma 2.1, we only need to show that $a=\mathcal{H}\cdot F=r$. Assume therefore that $a\ge r+1$. Then we have $\mathcal{H}^2\ge a(2b-a\epsilon)\ge(r+1)(2r+(r+1)\epsilon)>4r^2$.

Now assume that $c_2\le e=4$ or $c_1^2\le 4e=16$, and $\mathcal{E}|_F=\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(b)$ with $a,b>0$.

Case I: First consider the case when $(a,b)=(1,1)$. We are in the situation of Lemma 2.1. Letting $V=\mathcal{O}_{\mathbb{P}^1}(\alpha)\oplus\mathcal{O}_{\mathbb{P}^1}(\beta)$, we get

$$4\ge c_2(\mathcal{E})=c_2(p^*(V)\otimes\xi)=\xi^2+\alpha+\beta=\epsilon+\alpha+\beta,$$

or

$$12\ge c_1^2=c_1(p^*(V)\otimes\xi)^2=4\xi^2+4\alpha+4\beta=4(\epsilon+\alpha+\beta);$$

The only possible numerical possibilities are $\mathcal{E}=\boxed{p^*(\mathcal{O}_{\mathbb{P}^1}(\alpha)\oplus\mathcal{O}_{\mathbb{P}^1}(\beta))\otimes\xi}$ with $(\epsilon,\alpha,\beta)=(0,1,1),(0,1,2),(0,1,3),(0,2,2),(1,1,1),(1,1,2),(2,1,1)$.

Case II: Assume now that $(a,b)\ne(1,1)$. First, let us consider the case $\epsilon=0$. Then $\mathcal{H}_F=\det(\mathcal{E})|_F=\mathcal{O}_{\mathbb{P}^1}(a+b)$ implies $c_1^2\ge 18>4e(S)$. Thus if $c_2\le e=4$, then $c_1^2 \geq 4c_2(\mathcal{E})$, which means $\mathcal{E}$ is unstable. Consider the exact sequence (1). We have that $\mathcal{H}-\mathcal{A} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(x,y)$ for some $x>0$, $y>0$, and $\mathcal{A} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(x+t,y+l)$ for some $t>0$, $l>0$. The inequality $4 \geq c_2(\mathcal{E}) = x(y+l)+y(x+t)+\deg(\mathcal{Z})$ yields $\deg(\mathcal{Z}) = 0$ and $(x,y,x+t,y+l) = (1,1,2,2)$. Since $\deg(\mathcal{Z}) = 0$ and $H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(t,l)) = 0$, we conclude that $\mathcal{E} = \boxed{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)}$.

Now assume that $\epsilon \geq 1$, and let $\mathcal{H} = yF + xE$ with $x = a+b \geq 3$ and $\mathcal{H}\cdot E = -x\epsilon + y \geq 2$ (since $\mathcal{H}$ is the determinant of a rank 2 ample vector bundle). It follows that $\mathcal{H}^2 = a(2b-a\epsilon) \geq a(4+a\epsilon) \geq 3(4+a\epsilon) \geq 21 > 4e(S)$. Thus, if $c_2 \leq e = 4$, then $c_1^2 > 4c_2(\mathcal{E})$, and thus $\mathcal{E}$ is unstable. Let $\mathcal{A} := \alpha E_0 + \beta F$ be the line bundle in the sequence (1). We have the following straightforward inequalities:

(1) $x \geq 3$, $\epsilon \geq 1$, $y \geq x\epsilon + 2 \geq 5$;
(2) $x-\alpha > 0$, $y-\beta > 0$, $y \geq \beta + (x-\alpha)\epsilon + 1$;
(3) $2\alpha > x$, $2\beta > y$;
(4) $0 < (2\mathcal{A}-\mathcal{H})^2 = (2\alpha - x)(4\beta - 2y - (2\alpha - x)\epsilon) > 0$, and in particular $4\beta + x\epsilon > 2y + 2\alpha\epsilon$; and
(5) $\mathcal{A}\cdot(\mathcal{H}-\mathcal{A}) \leq c_2(\mathcal{E}) \leq 4$, which gives $-\alpha(x-\alpha)\epsilon + \beta(x-\alpha) + \alpha(y-\beta) \leq 4$.

Note that inequality (5) of the list can be written as

$$\alpha(\alpha\epsilon - x - \beta + y) + \beta(x-\alpha) \leq 4.$$

Using inequality (2) from the list, $y - \beta \geq (x-\alpha)\epsilon + 1$, we get

$$\begin{aligned} 4 &\geq \alpha(\alpha\epsilon - x + (x-\alpha)\epsilon + 1) + \beta(x-\alpha) \\ &\geq \alpha x(\epsilon - 1) + \alpha + \beta(x-\alpha). \end{aligned}$$

Now using equations (3) and (2) from the list we get the absurdity

$$4 \geq \alpha x(\epsilon - 1) + \frac{x+1}{2} + \frac{y+1}{2} \geq 0 + \frac{4}{2} + \frac{5+1}{2} \geq 5.$$

□

3. Lower bounds for the Chern numbers of $\mathcal{E}$

In this section we obtain a number of lower bounds for $c_1(\mathcal{E})^2$ for a rank r ample vector bundle on a smooth toric surface. Our main tool is adjunction theory: good references for the standard adjunction results that we use are [BS95, Ch. 10, 11] and [F90]. The following is a restatement, taking into account the geometry of toric surfaces, of the main result for the adjunction theory for surfaces. Recall that on a toric surface, a line bundle is ample if and only if it is very ample.

Theorem 3.1. *Let L be an ample line bundle on a smooth projective toric surface S.*

(1) *If $e = e(S) \geq 5$, then $K_S + L$ is spanned by global sections.*
(2) *If $e = e(S) \geq 7$, then S is the equivariant blowup $\pi : S \to S_1$ of a smooth toric projective surface S_1 at a finite set B, such that $L = \pi^*L' - \pi^{-1}(B)$, where $K_S + L \cong \pi^*(K_{S_1} + L')$, and both L' and $L_1 := K_{S_1} + L'$ are very ample.*

Proof. Using [BS95, 9.2.2], note that the exceptions to $K_S + L$ being spanned by global sections are all ruled out by $e(S) \geq 5$. The associated map p_{K_S+L} has a Remmert-Stein factorization $p = s \circ \pi$, where $\pi : S \to S_1$ has connected fibers. By

Lemma 1.5, we see that $e \geq 7$ rules out $\dim S_1 = 0$. If $\dim S_1 = 1$, then we have that $L \cdot F = 2$ for a general fiber of r, but this and $e \geq 7$ contradict Corollary 1.3.

Since $\dim S_1 = 2$, it follows from adjunction theory that $\pi : S \to S_1$ is the blowup of a smooth toric projective surface S_1 at a finite set B, such that $L = \pi^* L' - \pi^{-1}(B)$, where $K_S + L \cong \pi^*(K_{S_1} + L')$, and both L' and $L_1 := K_{S_1} + L'$ are ample. The very ampleness of the last two bundles follows from the fact that ample line bundles are very ample on toric varieties. □

Corollary 3.2. *Let $\mathcal{E}$ be an ample rank r vector bundle on a nonsingular toric surface S. If $e(S) \geq 5$, then*

$$c_1(\mathcal{E})^2 \geq r^2 e(S)$$

with equality only if $\det \mathcal{E} = -rK_S$ and $e(S) = 6$.

Proof. Let $\mathcal{H} := \det \mathcal{E}$. Let t be the smallest positive integer for which $tK_S + \mathcal{H}$ is not ample. Since $e(S) \geq 5$, we have $E \cdot (tK_S + \mathcal{H}) = 0$ for a smooth rational curve E with self-intersection -1. Thus we have

$$-t + E \cdot \mathcal{H} = E \cdot (tK_S + \mathcal{H}) = 0.$$

Since $\mathcal{E}$ has rank r, we have that $r \leq \mathcal{H} \cdot E = t$. Thus $rK_S + \mathcal{H}$ is spanned. Using Lemma 1.5, we have

$$\mathcal{H}^2 \geq -\mathcal{H} \cdot rK_S \geq r^2 e(S).$$

Moreover, since $\mathcal{H}$ is ample, we have equality only if $\mathcal{H} \cong -rK_S$. In this case we have $r^2 K_S^2 = \mathcal{H}^2 = r^2 e(S)$, or $K_S^2 = e(S)$. Since $K_S^2 + e(S) = 12$, we conclude that $K_S^2 = 6$. □

Lemma 3.3. *Let $\mathcal{E}$ be an ample rank two vector bundle on a nonsingular toric surface S. If $\det \mathcal{E} = -2K_S$, $e(S) = 6$, and $c_2(\mathcal{E}) \leq 6$, then $\mathcal{E} := -K_S \oplus -K_S$.*

Proof. A simple computation shows that the Chern character of $\mathcal{E} \otimes K_S$ is $2 + (K_S^2 - c_2(\mathcal{E})) = 2$. Thus $\chi(\mathcal{E} \otimes K_S) = 2$. Since $H^2(\mathcal{E} \otimes K_S) = H^0(\mathcal{E}^*) = 0$, we conclude that $\dim H^0(\mathcal{E} \otimes K_S) \geq 2$. Choose linearly independent $s_1, s_2 \in H^0(\mathcal{E} \otimes K_S)$.

If $s_1 \wedge s_2 \neq 0$, then, since $\det(\mathcal{E} \otimes K_S) = \mathcal{O}_S$, we conclude that $\mathcal{E} \otimes K_S = \mathcal{O}_S \oplus \mathcal{O}_S$, i.e., $\mathcal{E} \cong -K_S \oplus -K_S$.

Thus we can assume without loss of generality that $s_1 \wedge s_2 = 0$. The saturation $\mathcal{A}$ of the images of $\mathcal{O}_S$ in $\mathcal{E}$, under the two maps $g \to g \cdot s_i$, are equal. $\mathcal{A}$ is invertible, and tensoring with $-K_S$ we have an exact sequence

$$0 \to \mathcal{A} - K_S \to \mathcal{E} \to \mathcal{Q} \otimes \mathcal{I}_{\mathcal{Z}} \to 0,$$

with $\mathcal{Z}$ a 0-dimensional subscheme of S. Note that $\mathcal{Q}$ is ample, and therefore, since S is toric, very ample. Since $e(S) = 6$, we know that S is not $\mathbb{P}^2$ or a quadric, and thus

$$\mathcal{Q}^2 \geq 3. \tag{4}$$

Thus the Hodge index theorem gives $(\mathcal{Q} \cdot (-K_S))^2 \geq \mathcal{Q}^2 (-K_S)^2 \geq 18$, which implies that

$$\mathcal{Q} \cdot (-K_S) \geq 5. \tag{5}$$

Since $h^0(\mathcal{A}) \geq 2$, we have $\mathcal{Q} \cdot \mathcal{A} \geq 1$. Using this, and equations (4) and (5), we have

$$6 = c_2(\mathcal{E}) = (\mathcal{A} - K_S) \cdot \mathcal{Q} + \deg \mathcal{Z} \geq 1 + 5 + \deg \mathcal{Z}.$$

Thus $\deg \mathcal{Z} = 0$ and $\mathcal{A} \cdot \mathcal{Q} = 1$. The exact sequence

$$0 \to \mathcal{A} - K_S \to \mathcal{E} \to \mathcal{Q} \to 0$$

gives $-2K_S = c_1(\mathcal{E}) = \mathcal{A} + \mathcal{Q} - K_S$ and $K_S + \mathcal{A} + \mathcal{Q} = \mathcal{O}$. Thus $(K_S + \mathcal{Q}) \cdot \mathcal{Q} = -\mathcal{A} \cdot \mathcal{Q} = -1$. This is absurd, since on any smooth surface S, the parity of $(K_S + L) \cdot L$ is even for any line bundle L. □

Remark 3.4. We do not know if there are any examples of $\mathcal{E}$ satisfying all the hypotheses of Lemma 3.3, except that $c_2(\mathcal{E}) > 6$.

Remark 3.5. The only smooth toric surfaces S with $e(S) \leq 4$ are $\mathbb{P}^2$ or Hirzebruch surfaces. Corollary 1.7 classifies the exceptions to $c_1^2(S) > r^2 e(S)$ for $r = 1$, and §2 classifies the exceptions for $r = 2$ and $e(S) \leq 4$. They are contained in Table 1. For $\mathbb{P}^2$ it seems difficult to classify the exceptions when $r \geq 3$. For the Hirzebruch surfaces $\mathbb{F}_\epsilon$, Corollary 2.2 classifies the exceptions if $\epsilon \geq 2$.

If $e(S_1) \geq 7$, we can repeat the procedure in Theorem 3.1, using L_1 on S_1 in the same way we used L on S, and get (S_2, L_2). We say the procedure has terminated when we reach the first integer b with $e(S_b) \leq 6$. (See [BL89] for a further study of the adjunction process.) We call the sequence $(S, L), \ldots, (S_b, L_b)$ the iterated adjunction sequence, and b the adjunction length of S.

Notice that in the iterated adjunction sequence, at every step we contract down (-1)-lines in S_i with respect to the polarization $K_{S_i} + L_i$. This implies by Corollary 1.4 that $e(S_{i+1}) \geq \lfloor \frac{e(S_i)}{2} \rfloor$. If we assume, to start with, that the surface S has $e(S) \geq 2^{b-1} \cdot 6 + 1$, then the adjunction length is at least b.

We have the following strong bound.

Theorem 3.6. *Let S be a nonsingular toric surface with $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$ for some integer $b \geq 0$ and $e := e(S)$. Let $\mathcal{E}$ be an ample rank r vector bundle on S. Then*

$$c_1(\mathcal{E})^2 \geq e(3r^2 + 2r + 4br + 2b - 2) - 12(b+1)(b+2r) - 12r(r-1) + \frac{e}{2^{b-1}} - 2.$$

Proof. Since $\mathcal{H} := \det(\mathcal{E})$ is the determinant of a rank r ample vector bundle, there are no smooth rational curves C on the polarized surface $(S, \mathcal{H})$ with $\mathcal{H} \cdot C \leq r - 1$. Therefore by Theorem 3.1, $L := K + (r-1)\mathcal{H}$ ample. Using Lemma 3, we have the bound

$$-K\mathcal{H} \geq re. \tag{6}$$

The assumption $e(S) \geq 2^b \cdot 6 + 1$ implies that we have the adjunction sequence $(S, L), \ldots, (S_b, L_b), (S_{b+1}, L_{b+1})$ with L_{b+1} very ample. It follows that the sectional genus $g(L_{b+1}) = g(K_{S_b} + L_b) \geq 0$, i.e., $(K_{S_b} + L_b) \cdot (K_{S_b} + K_{S_b} + L_b) \geq -2$.

Let $S \to S_1 \to \ldots \to S_b$ be the sequence of contractions and let π_i denote the i-th contraction map. For simplicity let us set $K_i := (\pi \circ \pi_1 \ldots \circ \pi_i)^*(K_{S_i})$, $K_0 := K_S$, and $S := S_0$. We have

$$\begin{aligned}(K_{S_b} + L_b) &\cdot (K_{S_b} + K_{S_b} + L_b) \\ &= (K_b + K_{b-1} + \ldots + K_1 + K_0 + L) \cdot (K_b + K_b + K_{b-1} + \ldots + K_1 + K_0 + L).\end{aligned}$$

We can further decompose

$$\begin{aligned}
&K_b \cdot (K_b + K_{b-1} + \ldots + K_1 + K_0 + L)\\
&\quad = K_b^2 + K_b \cdot (K_{b-1} + K_{b-2} + \ldots + K_1 + K_0 + L)\\
&\quad = K_b^2 + K_{b-1} \cdot (K_{b-1} + K_{b-2} + \ldots + K_1 + K_0 + L)\\
&\quad = K_b^2 + K_{b-1}^2 + K_{b-1} \cdot (K_{b-2} + \ldots + K_1 + K_0 + L)\\
&\quad\ \ \vdots\ \ \vdots\\
&\quad = K_b^2 + K_{b-1}^2 + K_{b-2}^2 + \ldots + K_1^2 + K_0^2 + K_0 \cdot L\\
&(K_b + K_{b-1} + K_{b-2} + \ldots + K_1 + K_0 + L)^2\\
&\quad = K_b^2 + 2K_b \cdot (K_{b-1} + \ldots + K_1 + K_0 + L)\\
&\qquad + (K_{b-1} + \ldots + K_1 + K_0 + L)^2\\
&\quad = K_b^2 + 2(K_{b-1}^2 + K_{b-2}^2 + \ldots + K_1^2 + K_0^2 + K_0 \cdot L)\\
&\qquad + K_{b-1}^2 + 2K_{b-1} \cdot (K_{b-2} + \ldots + K_1 + K_0 + L)\\
&\qquad + (K_{b-2} + \ldots + K_1 + K_0 + L)^2\\
&\quad = K_b^2 + 3K_{b-1}^2 + 5K_{b-2}^2 + 7K_{b-3}^2\\
&\qquad + \ldots + (2b-1)K_1^2 + (2b+1)K_0^2 + (2b+2)K_0 \cdot L + L^2
\end{aligned}$$

Then

(7)

$$\begin{aligned}
&(K_{S_b} + L_b) \cdot (K_{S_b} + K_{S_b} + L_b)\\
&\quad = 2K_b^2 + 4K_{b-1}^2 + 6K_{b-2}^2 + \ldots + 2bK_1^2 + (2b+2)K_0^2 + (2b+3)K_0 \cdot L + L^2\\
&\quad \geq -2.
\end{aligned}$$

Recall that $K_i^2 = 12 - e(S_i)$ and $e(S_i) \geq (\frac{e}{2^i})$. Then

$$\begin{aligned}
L^2 + (2b+3)K_0 \cdot L \quad &\geq \quad -2 - 2\left(12 - \frac{e}{2^b}\right) - 4\left(12 - \frac{e}{2^{b-1}}\right)\\
&\qquad - \ldots - (2b+2)(12-e) + (2b+3)e\\
&\geq \quad -2 - 12(b+1)(b+2) + \frac{2e}{2^b}\sum_{j=0}^{b}((j+1)2^j).
\end{aligned}$$

Using $\sum_{j=0}^{b}((j+1)2^j) = 2^{b+1}b + 1$, we have

$$L^2 + (2b+3)K_0 \cdot L \quad \geq \quad -2 - 12(b+1)(b+2) + 4eb + \frac{e}{2^{b-1}}.$$

Recalling equation (6) and the fact that $L = (r-1)K_0 + \mathcal{H}$, we get

$$\begin{aligned}
\mathcal{H}^2 \ &\geq \ -2 - 12(b+1)(b+2) + 4eb + \frac{e}{2^{b-1}} + 2(r-1)re + (r-1)^2(e-12)\\
&\quad + \ (2b+3)re + (2b+3)(r-1)(e-12)\\
&= \ e(3r^2 + 2r + 4br + 2b - 2) - 12(b+1)(b+2r) - 12r(r-1) + \frac{e}{2^{b-1}} - 2. \ \square
\end{aligned}$$

Remark 3.7. To get a global feel for the bound, we have found it helpful to graph the expression. We include a short Maple V Release 5.1 program to plot the expression divided by part of the leading term. Varying the range of the rank r and the Euler characteristic e, and of the exact variant of `lowerBound`, the scaled expression for the lower bound is useful.

```
b := floor(ln[2]((e-1)/6));
lowerBound := (r,e) -> e*(3*r^2+2*r+4*b*r+2*b-2)-12*(b+1)*(b+2*r)
                              -12*r*(r-1)+e/2^(b-1)-2;
plot3d(lowerBound(r,e)/(r*e*(3*r+4*b)),r=1..20,
                        e=13..100,style=PATCH,axes=BOXED);
```

Remark 3.8. It is easily checked that the expression in e and r occurring in the lower bound is an increasing function of e and r for $e \geq 7$, $r \geq 1$. It is also easy to check using the above bound that $c_1(\mathcal{E})^2 \geq 2r^2 e(S)$ if $e(S) \geq 12$, and $c_1(\mathcal{E})^2 \geq 3r^2 e(S)$ if $e(S) \geq 6r + 7$.

Theorem 3.6 gives a strong asymptotic lower bound for c_1^2 as e goes to ∞. For any fixed $c > 0$, there will only be a finite number of possible pairs (c_1^2, e) of numerical invariants for ample vector bundles $\mathcal{E}$ on smooth toric surfaces S with $L^2 \leq ce$. For example, $c_1^2 \geq 2r^2 e(S)$ as soon as $e(S) \geq 13$. This suggests that enumerating the pairs $(S, \mathcal{E})$ with $\mathcal{H}^2 \leq cre(S)$, where $\mathcal{E}$ is an ample vector bundle on a smooth toric surface S, and small $c > 1$ should be a tractable classification problem with a nice answer.

Theorem 3.9. *Let $\mathcal{E}$ be an ample rank two vector bundle on a nonsingular toric variety S with $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$ for some integer $b \geq 0$ and $e := e(S)$. Then*

$$c_2(\mathcal{E}) \geq -3(b+2)(b+3) + \frac{5b+7}{2} e + \frac{e}{2^{b+1}} - \frac{1}{2}.$$

Proof. If the inequality is not satisfied, then, using Theorem 3.6, $c_1(\mathcal{E})^2 > 4c_2(\mathcal{E})$, and thus the bundle would be unstable. The exact sequence (1) and the inequality (2) give

$$c_2(\mathcal{E}) \geq (\mathcal{H} - \mathcal{A})^2 + \sqrt{(\mathcal{H} - \mathcal{A})^2};$$

the divisor $\mathcal{H} - \mathcal{A}$ is ample, and thus by Theorem (3.6)

$$\begin{aligned} &- 3(b+2)(b+3) + \frac{(5b+7)}{2} e + \frac{e}{2^{b+1}} - \frac{1}{2} \\ &\qquad > c_2(\mathcal{E}) \geq e(6b+3) - 12(b+1)(b+2) + \frac{e}{2^{b-1}} - 2 + 1, \end{aligned}$$

which is equivalent to $18b^2 + 42b + 13 - 7eb + e - 3e/2^b > 0$, which is impossible. □

Remark 3.10. We expect that a generalization of Theorem 3.9 to ample vector bundles of arbitrary rank r is true. Based on a strong dose of optimism, we conjecture that if $\mathcal{E}$ is an ample rank r vector bundle on a smooth toric projective surface S with $2^b \cdot 12 \geq e(S) \geq 2^b \cdot 6 + 1$ for some integer $b \geq 0$, then

$$\begin{aligned} c_2(\mathcal{E}) \geq \frac{r-1}{2r} [&e(S)(3r^2 + 2r + 4br + 2b - 2) \\ &- 12(b+1)(b+2r) - 12r(r-1) + \frac{e}{2^{b-1}} - 2]. \end{aligned}$$

We now turn to the special case of rank two bundles where the inequality $c_2(\mathcal{E}) > e(S)$ fails to be true.

Lemma 3.11. *Let $\mathcal{E}$ be an unstable ample rank two vector bundle on a smooth toric projective surface S. If $\mathcal{E}$ is Bogomolov unstable and $c_2(\mathcal{E}) \leq e(S) + \sqrt{e(S)}$, then S is either $\mathbb{P}^2$ or $\mathbb{F}_\epsilon$ with $\epsilon \leq 2$.*

Proof. Assume that $\mathcal{E}$ is Bogomolov unstable. Consider the sequence (1) and the inequality

$$e(S)+\sqrt{e(S)} \geq c_2(\mathcal{E}) = \mathcal{A}\cdot(\mathcal{H}-\mathcal{A})+\deg(\mathcal{Z}) \geq (\mathcal{H}-\mathcal{A})^2+\sqrt{(\mathcal{H}-\mathcal{A})^2}.$$

We can then assume $(\mathcal{H}-\mathcal{A})^2 \leq e$. We now apply Theorem 1.7 to the ample line bundle $\mathcal{H}-\mathcal{A}$. □

Remark 3.12. Let $\delta := \min\{L^2 | L$ an ample line bundle on $S\}$. The above argument implies that any ample vector bundle $\mathcal{E}$ with $c_2(\mathcal{E}) < \delta + \sqrt{\delta}$ is Bogomolov stable.

Corollary 3.13. *Let $\mathcal{E}$ be an ample rank two vector bundle on a smooth toric projective surface S. Assume that $c_2(\mathcal{E}) \leq e(S)$. If $\mathcal{E}$ is not Bogomolov stable, then $(S,\mathcal{E})$ is contained in Table 1.*

Proof. Simply use Lemma 3.11 and the results for $\mathbb{P}^2$ and the Hirzebruch surfaces from §2. □

Proposition 3.14. *Let $\mathcal{E}$ be an ample rank two vector bundle on a smooth projective toric surface S. If either $c_1(\mathcal{E})^2 \leq 4e(S)$ or $c_2(\mathcal{E}) \leq e(S)$, then $(S,\mathcal{E})$ is in Table 1.*

Proof. We can also assume that S is neither $\mathbb{P}^2$ or a Hirzebruch surface by using the results of §2. Thus $e(S) \geq 4$. Using Corollary 3.2 and Lemma 3.3, we can assume without loss of generality that $c_1(\mathcal{E})^2 > 4e(S)$. If $c_2(\mathcal{E}) \leq e$, then we are in the situation of Lemma 3.13. □

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Department of Mathematics, KTH, S-100 44 Stockholm, Sweden
E-mail address: sandra@math.kth.se

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
E-mail address: sommese@nd.edu