Komplexa tal

$i = \sqrt{-1} \qquad i^2 = -1$

$a, b \in \mathbb{R}$, $a + ib$ komplexa tal

Om $b = 0$ har vi reella tal

Om $a = 0$ har vi imaginära tal.

addition:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

multiplikation:

$$(a+ib)(c+id)=ac+aid+$$

$$+ibc+i^2bd=ac+iad+$$

$$+ibc-bd=(ac-bd)+i(ad+bc)$$

Ex: $(1+i)(2-i)=(1\cdot 2-1(-1))+i(1\cdot(-1)+1\cdot 2)=$

$$=3+i$$

$$(-1)(a+ib) = -a-ib$$

$$(a+ib)+(-a-ib) = (a-a)+i(b-b) = 0$$

$$(a+ib)-(c+id) = (a-c)+i(b-d)$$

Real delen:

$$\mathrm{Re}(a+ib) = a$$

Imaginär delen:

$$\mathrm{Im}(a+ib) = b$$

Ex: Vad är reala och imaginär delarna till $3+i2$?

$$\mathrm{Re}(3+i2) = 3$$

$$\mathrm{Im}(3+i2) = 2$$

Konjugat:

$$\overline{a+ib} = a-ib$$

$$(a+ib)\left(\overline{a+ib}\right) = (a+ib)(a-ib) =$$

$$= a^2 + b^2 + i\left(\cancel{a(-b)+ba}\right)$$

$$= a^2 + b^2$$

Ex: $\dfrac{1}{1+i} = \left(\dfrac{1}{1+i}\right)\dfrac{(1-i)}{(1-i)} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1}{2} - i\dfrac{1}{2}$

Division:

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} =$$

$$= \frac{(ac+bd)+i(-ad+bc)}{c^2+d^2}$$

Ex: $\frac{2+i}{1-3i} = \frac{(2+i)(1+3i)}{(1-3i)(1+3i)} =$

$$= \frac{-1+i7}{10}$$

Belopp:

$$(a+ib)(a-ib) = a^2+b^2$$

Beloppet av $a+ib$ kallas uttrycket $\sqrt{a^2+b^2}$.

Viktigt att $a^2+b^2 \geq 0$ så att beloppet blir ett positivt reellt tal.

Låt

$$z = a+ib \quad , \quad \bar{z} = a-ib$$

det ger

$$z + \bar{z} = a+ib + a-ib = 2a$$

dvs $\quad \mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$

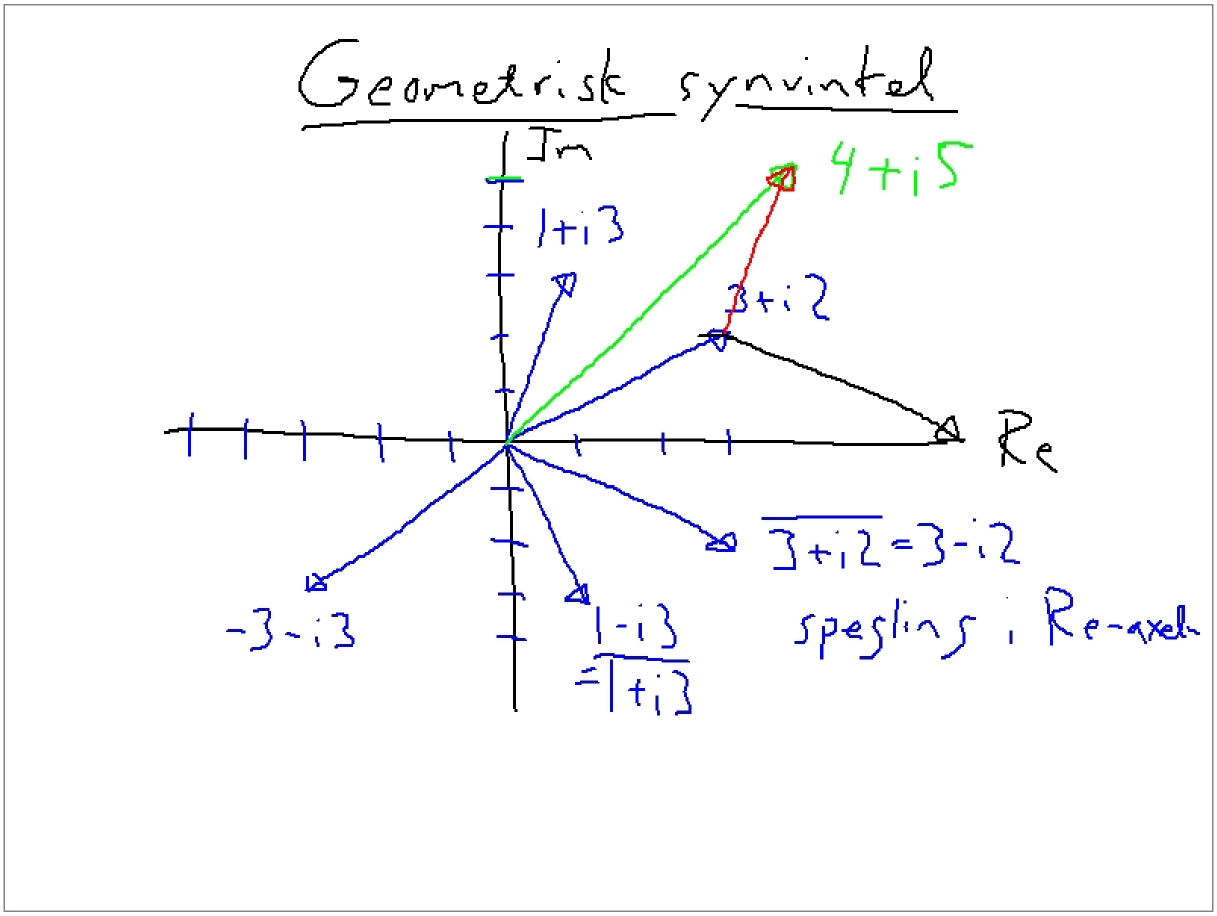
på samma sätt

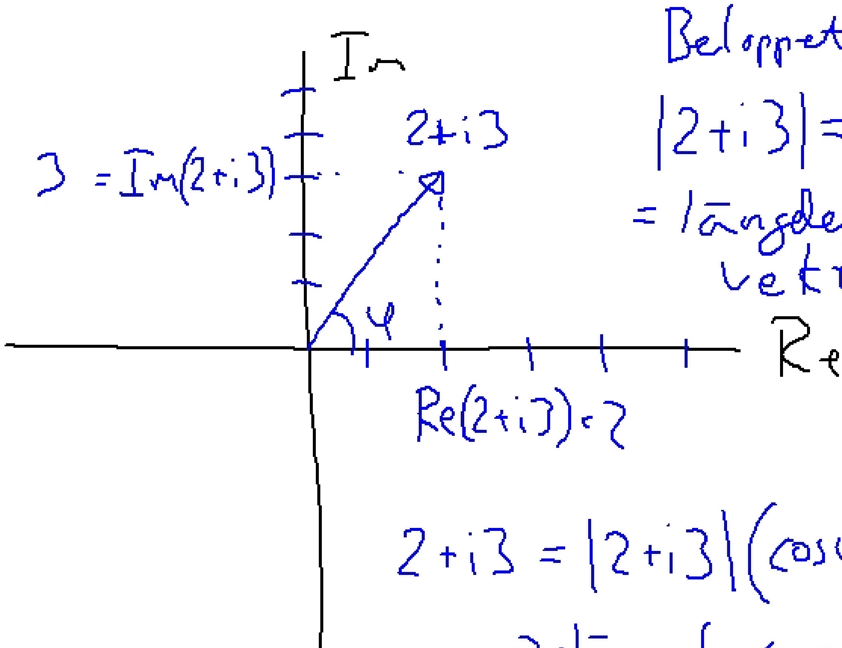
$$z - \bar{z} = a+ib - (a-ib) = i2b$$

dvs $\quad \mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$

$$\begin{cases} z = \mathrm{Re}\,z + i\,\mathrm{Im}\,z \\ \bar{z} = \mathrm{Re}\,z - i\,\mathrm{Im}\,z \end{cases}$$

Föreläsning 9, sid 8





Beloppet

$$|2+i3| = \sqrt{2^2+3^2}$$

= längden på vektorn

$$2+i3 = |2+i3|(\cos\varphi + i\sin\varphi)$$

polär form

Polär form

$$z = re^{i\varphi}$$

φ kallas argument

där $r = |z|$ och

(Eulers formel) $e^{i\varphi} = \cos\varphi + i\sin\varphi$

Ex: Skriv $1+i$ på polär form.

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

det ser

$$\cos\varphi = \frac{1}{\sqrt{2}}, \quad \sin\varphi = \frac{1}{\sqrt{2}}$$

dvs $\varphi = \frac{\pi}{4} + 2\pi n$

$$re^{i\varphi}\cdot se^{i\psi} = rs(\cos\varphi + i\sin\varphi)\cdot(\cos\psi + i\sin\psi) = rs\big((\cos\varphi\cos\psi - \sin\varphi\sin\psi) + i(\cos\varphi\sin\psi + \sin\varphi\cos\psi)\big) =$$

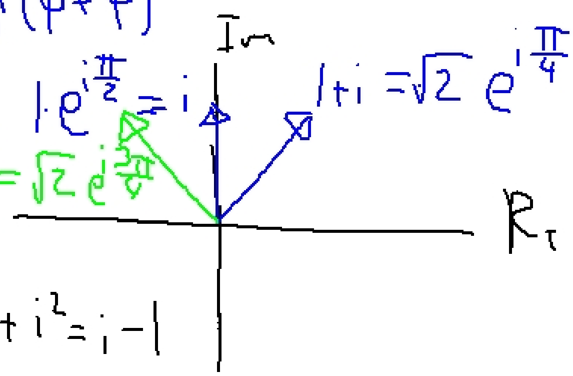
$$= rs\big(\cos(\varphi+\psi) + i\sin(\varphi+\psi)\big)$$

$$= rs\, e^{i(\varphi+\psi)}$$

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$1\cdot e^{i\frac{\pi}{2}} = i$

$-1+i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\frac{3\pi}{4}}$



$i(1+i) = i + i^2 = i - 1$

Ex: Skriv talet $1+\sqrt{3}i$

på polär form.

Lösn: $|1+\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$

$$1+\sqrt{3}i = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

dvs $\cos\varphi = \frac{1}{2}$, $\sin\varphi = \frac{\sqrt{3}}{2}$

så $\varphi = \frac{\pi}{3} + 2\pi n$

$$1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$$

$$z^k = (r e^{i\varphi})^k = r^k e^{i(k\varphi)}$$

$$(\cos\varphi + i \sin\varphi)^k = (\cos(k\varphi) + i \sin(k\varphi))$$