Substitution

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$

$$\Big[f\circ g(x)\Big]_a^b = \int_a^b f'(g(x))\,g'(x)\,dx$$

$$\int_{g(a)}^{g(b)} f'(t)\,dt = \int_a^b f'(g(t))\,g'(t)\,dt$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\int dy = \int \frac{dy}{dt}\,dt = \int \frac{dy}{dx}\frac{dx}{dt}\,dt = \int \frac{dy}{dx}\,dx$$

Ex:

$$\int \underbrace{x\, e^{\overbrace{x^2}^{t}}\, dx}_{\frac{dt}{2}} = \begin{bmatrix} t = x^2 \\ dt = 2x\,dx \end{bmatrix} =$$

$$= \int e^t \frac{dt}{2} = \frac{e^t}{2} + C = \frac{e^{x^2}}{2} + C$$

$$\text{Ex: } \int_{-1}^{1} \sqrt{1-x^2}\,dx = \left[\begin{array}{l} x = \cos t \\ dx = -\sin t\,dt \end{array}\right] =$$

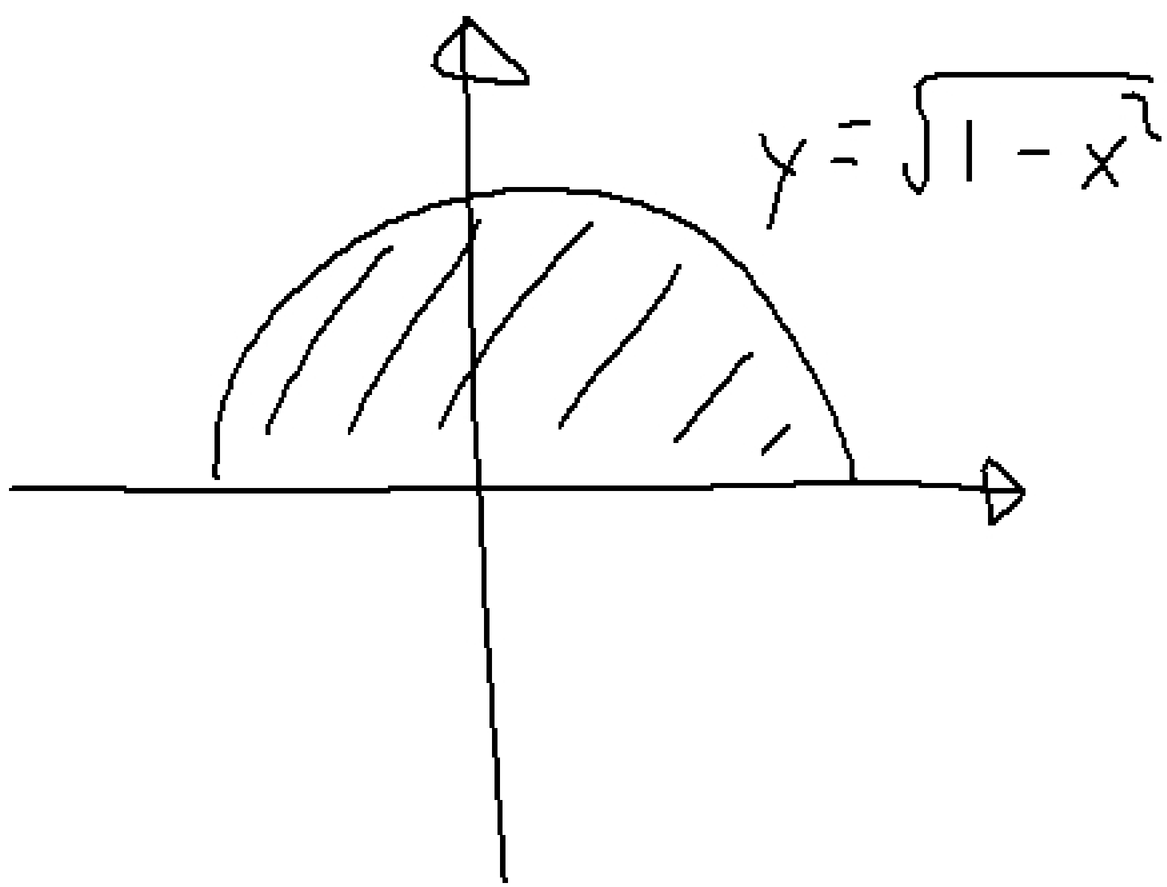
$$= \int_{\arccos(-1)}^{\arccos 1} \sqrt{1-\cos^2 t}\,(-\sin t)\,dt =$$

$$= \int_{\pi}^{0} |\sin t|\,(-\sin t)\,dt =$$

$$= -\int_{0}^{\pi} -\sin^2 t\,dt = \int_{0}^{\pi} \sin^2 t\,dt =$$

$$= \int_0^{\pi} \frac{1-\cos 2t}{2}\,dt = \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} =$$

$$= \frac{\pi}{2}.$$



$$\text{Area} = \frac{\pi}{2}$$

Ex: $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x \, dx = \left[\begin{matrix} t = \sin x \\ dt = \cos x \, dx \end{matrix}\right]$

$$= \int_{0 = \sin 0}^{\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}} t \, dt = \left[\frac{t^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} =$$

$$= \frac{1}{4}.$$

$t = \cos x$

$dt = -\sin x \, dx$

$$-\int_1^{\frac{1}{\sqrt{2}}} t \, dt = \left[-\frac{t^2}{2}\right]_1^{\frac{1}{\sqrt{2}}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Ex: $$\int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx =$$

$$= \begin{bmatrix} t = \cos x \\ dt = -\sin x \, dx \end{bmatrix} = -\int_1^{\frac{1}{\sqrt{2}}} \frac{dt}{t} =$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \frac{dt}{t} = \Big[\ln|t| \Big]_{\frac{1}{\sqrt{2}}}^{1} = \ln 1 - \ln\left(\frac{1}{\sqrt{2}}\right) =$$

$$= 0 - \left(-\frac{1}{2}\right) \ln 2 = \frac{1}{2} \ln 2 .$$

Ex: $$\int_1^2 \frac{dx}{x+3} = \left[\begin{matrix} t = x+3 \\ dt = dx \end{matrix}\right] =$$

$$= \int_4^5 \frac{dt}{t} = \left[\ln t\right]_4^5 =$$

$$= \ln 5 - \ln 4$$

Ex: $$\int \frac{5x}{4+4x^2}\,dx = \left[\begin{matrix} t = 1+x^2 \\ dt = 2x\,dx \end{matrix}\right] =$$

$$= \int \frac{\frac{5}{2}}{4t}\,dt = \frac{5}{8}\ln|t| + C = \frac{5}{8}\ln(1+x^2) + C$$

Ex: $\displaystyle\int_0^{\pi} \sin 5x \, dx = \left[\begin{array}{l} t = 5x \\ dt = 5\,dx \end{array}\right] =$

$$= \int_0^{5\pi} \sin t \, \frac{dt}{5} = \left[-\frac{\cos t}{5} \right]_0^{5\pi} =$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Ex: Lös ekvationen $\frac{dx}{dt} = x$

Lösn: $\int \frac{1}{x} \frac{dx}{dt} dt = \int 1 \, dt$

$$\ln|x| = \int \frac{1}{x} dx = t + C$$

$$x = De^{t}$$

Partiell integration

$$(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

så

$$f(x)\cdot g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

dvs

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$f(x)\big(g(x)+C\big) - \int f'(x)\big(g(x)+C\big)\,dx$$

Ex:

Ex: $\int \ln|x|\,dx = \int \ln|x| \cdot 1\,dx$

$= \ln|x| \cdot x - \int \frac{1}{x} x\,dx =$

$= x\ln|x| - x + C$

Ex: $\int_0^{3\pi} x\cos x\,dx = \left[x\sin x\right]_0^{3\pi} -$

$- \int_0^{3\pi} 1 \cdot \sin x\,dx = \left[x\sin x\right]_0^{3\pi} - \left[-\cos x\right]_0^{3\pi} = -2$

Ex: $\int \arctan x \, dx =$

$$= \int \arctan x \cdot 1 \, dx =$$

$$= x \arctan x - \int \frac{x}{1+x^2} \, dx =$$

$$\left(\int \frac{x}{1+x^2} dx = \begin{bmatrix} t = 1+x^2 \\ dt = 2x\,dx \end{bmatrix} = \int \frac{dt}{2t} = \frac{\ln|t|}{2} + C \right.$$

$$= \frac{1}{2} \ln (1+x^2) + C$$

$$= x \arctan x - \frac{1}{2} \ln (1+x^2) + C$$

Ex: Bestäm primitiva funktionerna till

$$\sin 2x\, e^{3x}$$

$$\int \sin 2x\, e^{3x}\,dx = \sin 2x \frac{e^{3x}}{3} - \int 2\cos 2x \frac{e^{3x}}{3}\,dx$$

$$= \sin 2x \frac{e^{3x}}{3} - \frac{2}{3}\cos 2x \frac{e^{3x}}{3} + \frac{2}{3}\int -2\sin 2x \frac{e^{3x}}{3}\,dx$$

$$\Rightarrow \left(1+\frac{4}{9}\right)\int \sin 2x\, e^{3x}\,dx = \sin 2x \frac{e^{3x}}{3} - \frac{2}{9}\cos 2x\, e^{3x} + C$$

$$\int \sin 2x\, e^{3x}\,dx = \frac{9}{13}\left(\sin 2x\,\frac{e^{3x}}{3} - \frac{2}{9}\cos 2x\, e^{3x}\right) + C$$

Ex:

$$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x\,dx =$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,dx = 1 - \int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,dx =$$

$$= \begin{bmatrix} t = \cos x \\ dt = -\sin x \, d\cancel{x} \end{bmatrix} = 1 + \int_1^0 t^2 \, dt$$

$$= 1 - \int_0^1 t^2 \, dt = 1 - \left[\frac{t^3}{3} \right]_0^1 =$$

$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

Ex: $\displaystyle\int_0^1 x e^x \, dx = \left[x e^x \right]_0^1 - \int_0^1 1 \cdot e^x \, dx$

$$= e - \left[e^x \right]_0^1 = e - e + 1 = 1$$

Ex: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \sin 3x \, dx =$

$$\left(\begin{aligned} \cos(2x+3x) &= \cos 2x \cos 3x - \sin 2x \sin 3x \\ \cos(3x-2x) &= \cos 3x \cos 2x + \sin 3x \sin 2x \end{aligned}\right)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x - \cos 5x}{2} \, dx = \left[\frac{\sin x}{2} - \frac{\sin 5x}{10}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{\sin\frac{\pi}{2}}{2} - \frac{\sin\left(-\frac{\pi}{2}\right)}{2} - \frac{\sin\frac{5\pi}{2}}{10} + \frac{\sin\left(-\frac{5\pi}{2}\right)}{10} = 1 - \frac{1}{5} = \frac{4}{5}$$