Rationella integrander

$$\frac{p(x)}{q(x)} = p_0(x) + \frac{p_1(x)}{q(x)} \quad \text{där grad}\, p_1 < \text{grad}\, q$$

Ex: $\displaystyle\int_0^1 \frac{dx}{x^2-4} = \left[\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \right.$

$$= \frac{A}{x-2} + \frac{B}{x+2}; \quad A(x+2) + B(x-2) = 1$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \end{cases}$$

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$$\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2} =$$

$$\frac{1}{4}\int_0^1 \frac{dx}{x-2} - \frac{1}{4}\int_0^1 \frac{dx}{x+2} =$$

$$= \frac{1}{4}\Big[\ln|x-2|\Big]_0^1 - \frac{1}{4}\Big[\ln|x+2|\Big]_0^1 =$$

$$= \frac{1}{4}\left(\ln\overset{0}{|1-2|} - \ln|-2|\right) - \frac{1}{4}\left(\ln(1+2) - \ln(2)\right) =$$

$$= -\frac{1}{4}\ln 2 - \frac{1}{4}\ln 3 + \frac{1}{4}\ln 2 = -\frac{1}{4}\ln 3.$$

$$\lim_{x \to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x \to 2} \frac{A\cancel{(x-2)}}{\cancel{x-2}} + \frac{B(x-2)}{x+2}$$

$$= \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$$

$$= \lim_{x \to 2} A + \frac{B(x-2)}{(x+2)} = A$$

Ex: $\displaystyle \int \frac{dx}{x^2-2x-15} = \int \frac{dx}{(x-5)(x+3)}$

$$\left[\frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} = \right.$$

$$\left. = \frac{\frac{1}{8}}{x-5} + \frac{-\frac{1}{8}}{x+3} \right] = \frac{1}{8}\int \frac{dx}{x-5} -$$

$$-\frac{1}{8}\int \frac{dx}{x+3} = \frac{1}{8}\ln|x-5| - \frac{1}{8}\ln|x+3| + C$$

Ex: $$\int \frac{x^3+4x^2+x+1}{x^2-2x-3}\,dx$$

$$x^3+4x^2+x+1=(x+6)(x^2-2x-3)+16x+19$$

$$\int \frac{x^3+4x^2+x+1}{x^2-2x-3}\,dx = \int x+6\,dx + \int \frac{16x+19}{x^2-2x-3}\,dx$$

$$\left[\frac{16x+19}{x^2-2x-3} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \frac{\frac{67}{4}}{x-3} + \frac{-\frac{3}{4}}{x+1}\right]$$

$$= \int x + 6 + \frac{\frac{67}{4}}{x-3} + \frac{-\frac{3}{4}}{x+1}\,dx =$$

$$= \frac{x^2}{2} + 6x + \frac{67}{4}\ln|x-3| - \frac{3}{4}\ln|x+1| + C$$

Ex: $\displaystyle \int \frac{x}{(x-1)^2}\,dx = \left[\frac{x}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}\right.$

$$\left. = \frac{1}{(x-1)^2} + \frac{1}{x-1}\right] = \int \frac{1}{(x-1)^2} + \frac{1}{x-1}\,dx =$$

$$= -\frac{1}{x-1} + \ln|x-1| + C$$

Ex: $\displaystyle\int \frac{x+5}{x^2-6x+9}\,dx = \int \frac{x+5}{(x-3)^2}\,dx$

$$= \int \frac{8}{(x-3)^2}\,dx + \int \frac{1}{x-3}\,dx =$$

($x+5 = x-3+8$ så

$$\frac{x+5}{(x-3)^2} = \frac{(x-3)}{(x-3)^2} + \frac{8}{(x-3)^2} = \frac{1}{x-3} + \frac{8}{(x-3)^2}$$)

$$= -8\frac{1}{(x-3)} + \ln|x-3| + C$$

Ex: $\displaystyle\int \frac{dx}{x^2+2x+2}$ (x^2+2x+2 går ej att faktoriseras med reella.)

$$\int \frac{dx}{(x+1)^2+1} = \left[\begin{array}{l} t = x+1 \\ dt = dx \end{array}\right] = \int \frac{dt}{1+t^2} =$$

$$= \arctan t + C = \arctan(x+1) + C$$

Ex: $\int \frac{x-3}{x^2+6x+13}dx = \int \frac{x-3}{(x+3)^2+4}dx$

$$= \int \frac{x-3}{4\left(\left(\frac{x+3}{2}\right)^2+1\right)}dx = \frac{1}{4}\int \frac{x-3}{\left(\frac{x+3}{2}\right)^2+1}dx$$

$$= \left[\begin{array}{l} t = \frac{x+3}{2} \\ dt = \frac{dx}{2} \end{array}\right] = \frac{1}{2}\int \frac{2t-6}{t^2+1}dt =$$

$$= \frac{1}{2}\int \frac{2t}{1+t^2}dt - 3\int \frac{dt}{1+t^2} = \frac{1}{2}\ln|1+t^2| - 3\arctan t + C =$$

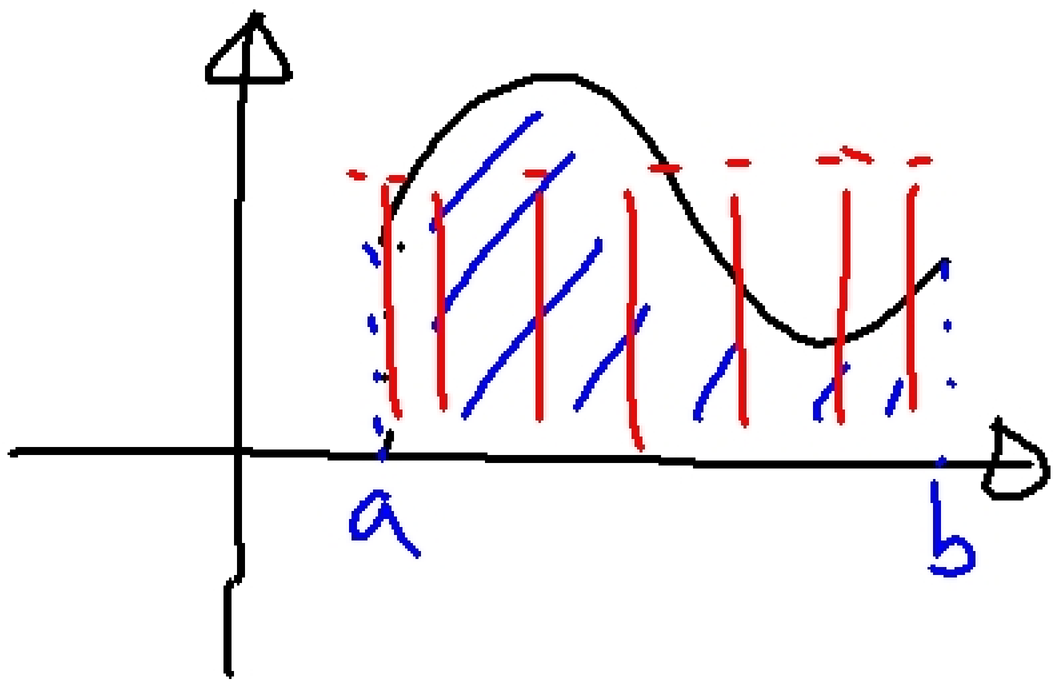
$$= \frac{1}{2}\ln\left(1+\left(\frac{x+3}{2}\right)^2\right) - 3\arctan\left(\frac{x+3}{2}\right) + C$$

Ex: $$\int \frac{x^2+2x+3}{x^3+3x^2+2x}\,dx = \int \frac{x^2+2x+3}{x(x+1)(x+2)}\,dx$$

$$= \int \frac{\frac{3}{2}}{x}\,dx + \int \frac{-2}{x+1}\,dx + \int \frac{\frac{3}{2}}{x+2}\,dx$$

$$= \frac{3}{2}\ln|x| - 2\ln|x+1| + \frac{3}{2}\ln|x+2| + C$$

Medelvärden



$$\frac{1}{b-a}\int_a^b f(t)\,dt$$

Ex: $v = \frac{s}{t}$ om v konstant

Generellt ser det $v = \frac{ds}{dt}$

Medelhastigheten blir så

(när t går från 1 till 3)

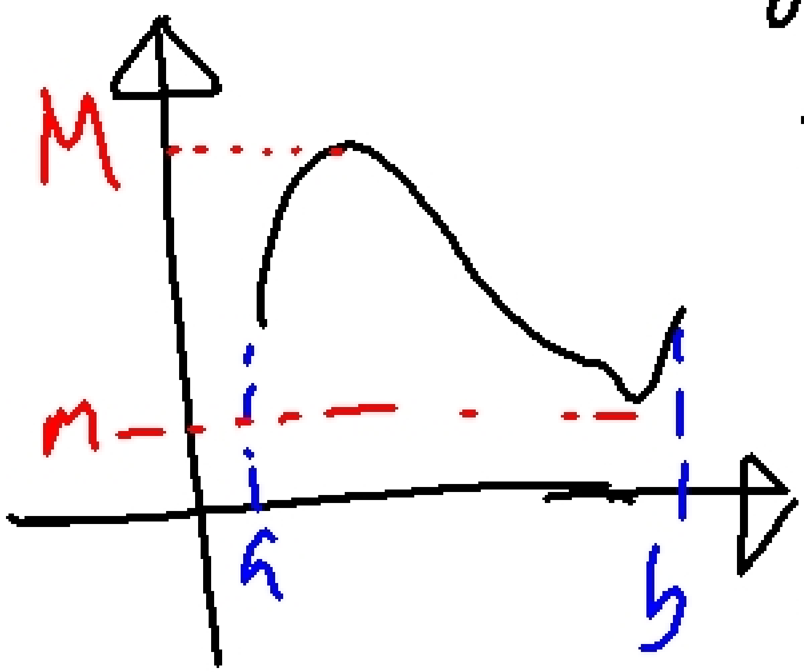
tex $\frac{1}{3-1}\int_1^3 v(t)\,dt = \frac{1}{3-1}\int_1^3 \frac{ds}{dt}\,dt$

$$= \frac{1}{3-1}\int_{s(1)}^{s(3)} ds = \frac{1}{2}\left(s(3)-s(1)\right)$$

<u>Sats</u> (Integralkalkylens medelvärdes-sats) Om f är en kontinuerlig funktion på intervallet $[a,b]$ då finns det en punkt $c \in [a,b]$ sådan att

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

Bevis: En kontinuerlig funktion antar sitt största och minsta värde på ett slutet och begränsat intervall, säg



$$\max_{[a,b]} f(x) = M \quad \text{och} \quad \min_{[a,b]} f(x) = m$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Vi får därmed

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Då en kontinuerlig funktion antar alla mellanliggande värden måste det finnas en punkt c så att

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$