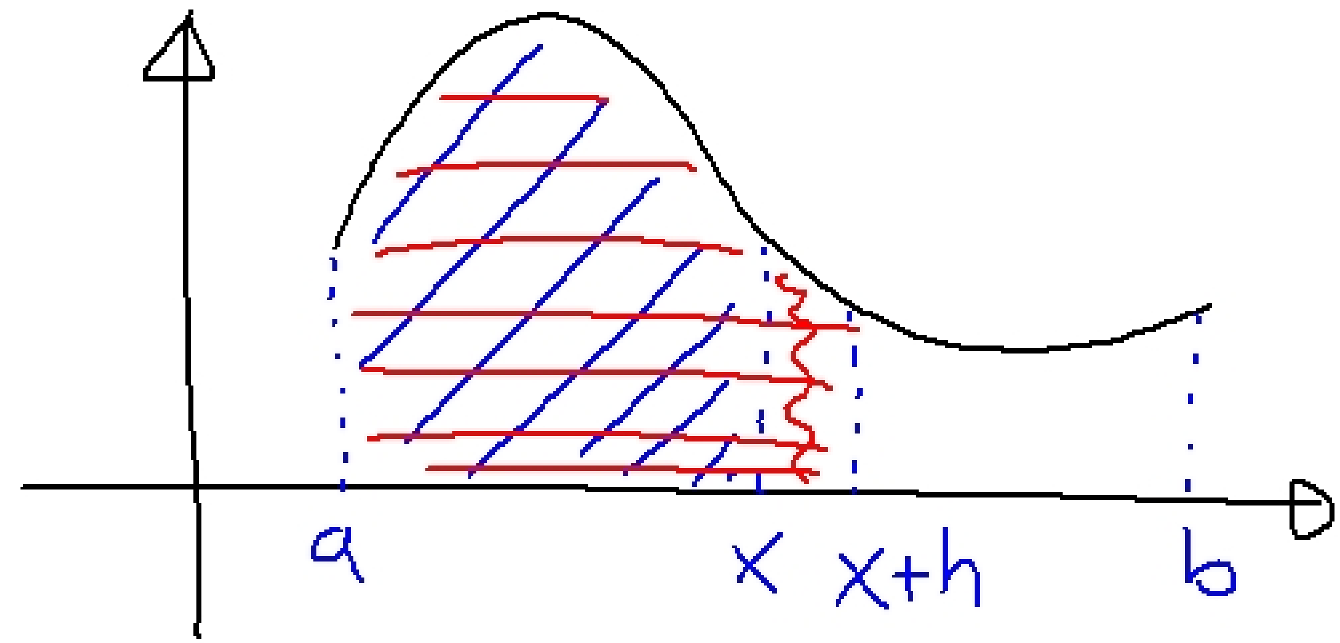
Existens av primitiva funktioner

Sats: Om f är en kontinuerlig funktion på intervallet $[a,b]$ så är $F(x)=\int_a^x f(t)\,dt$ $\underbrace{}_{}$ $x\in[a,b]$ primitiv funktion till f, dvs $F'(x)=f(x)$.

på samma intervall

Bevis:



$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} =$$

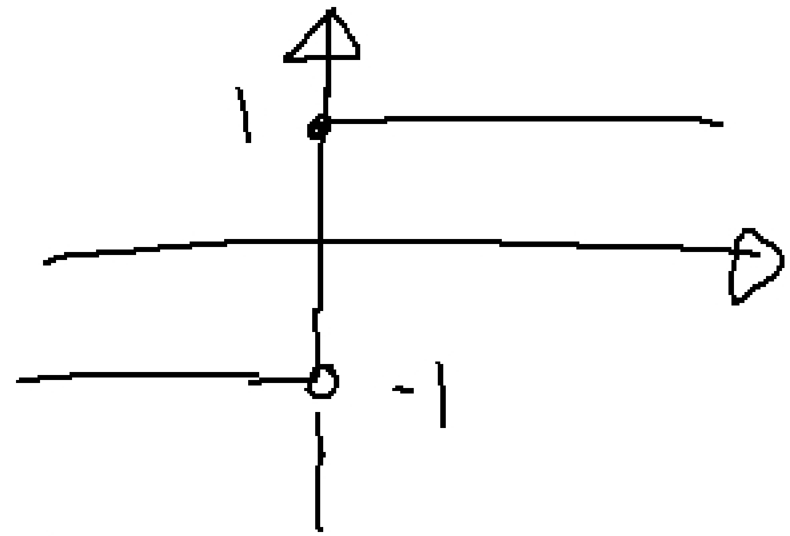
$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h} = \lim_{h\to 0} \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}_{\text{medelvärde}}$$

$$= (\text{enl medelvärdessatsen}) = \lim_{h\to 0} f(\xi) \quad \text{där } \xi \in [x, x+h]$$

$$= f(x)$$

(för $\xi \to x$ när $h \to 0$)

Ex 7.7:

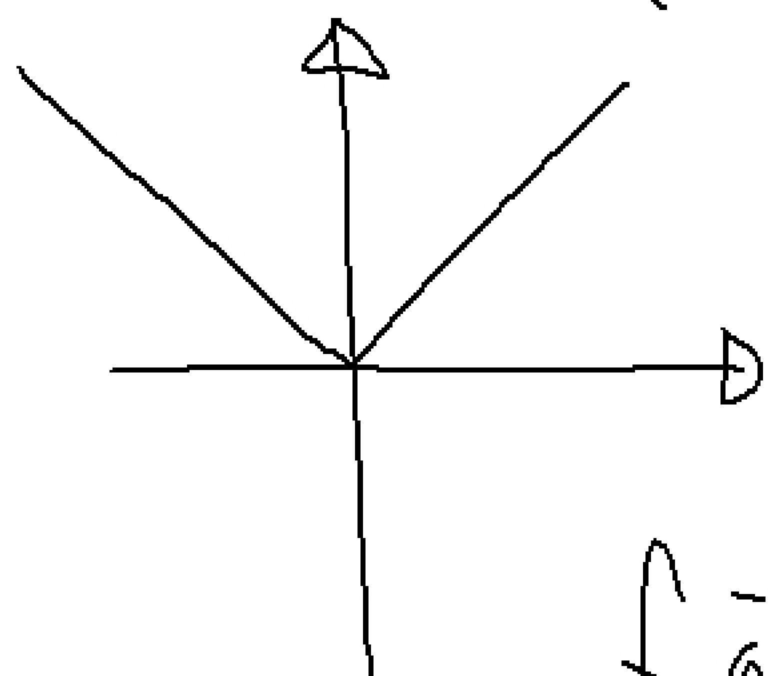


$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

ej kontinuerlig

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} x & \text{om } x \geq 0 \\ -x & \text{om } x < 0 \end{cases}$$

$$F(x) = |x|$$



Går ej att deriveras i $x=0$ så villkoret att f är kontinuerlig är väsentligt i satsen.

Men $f(x)=|x|$ är kontinuerlig
så vi kan använda satsen
för att ta fram en primitiv
funktion. Gör det med
$a=-1$. Alltså räkna

$$\int_{-1}^{x}|t|\,dt=\int_{-1}^{0}|t|\,dt+\int_{0}^{x}|t|\,dt=$$

$$=-\int_{-1}^{0}t\,dt+\begin{cases}\int_{0}^{x}t\,dt\ (x\geq 0)=\frac{x^2}{2}\\ \int_{0}^{x}-t\,dt\ (x<0)=-\frac{x^2}{2}\end{cases}$$

$$= -\left[\frac{t^2}{2}\right]_{-1}^{0} + \begin{cases} \frac{x^2}{2} & x \geq 0 \\ -\frac{x^2}{2} & x < 0 \end{cases}$$

$$= \frac{1}{2} + \begin{cases} \frac{x^2}{2} & x \geq 0 \\ -\frac{x^2}{2} & x < 0 \end{cases} = \begin{cases} \frac{x^2}{2} + \frac{1}{2} & x \geq 0 \\ -\frac{x^2}{2} + \frac{1}{2} & x < 0 \end{cases}$$

blir deriverbar även i origo.

Hemläxa: verifiera detta!

Trigonometriska integrander

Ex: $\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx =$

$= \int \frac{\sin x}{1-\cos^2 x}\,dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x\,dx \end{bmatrix} =$

$= -\int \frac{dt}{1-t^2} = \int \frac{dt}{(t-1)(1+t)} = \int \frac{\frac{1}{2}}{t-1} + \frac{-\frac{1}{2}}{1+t}\,dt$

$= \frac{1}{2}\int \frac{1}{t-1} - \frac{1}{1+t}\,dt = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C =$

$$= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C$$

Substitution $t = \tan \frac{x}{2}$:

$$\boxed{\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}}$$

$$\boxed{= \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

Så $\sin x$ är en rationell funktion i $\tan \frac{x}{2}$.

Ex: $$\int \frac{1}{\sin x}\,dx = \left[\begin{array}{l} t = \tan\frac{x}{2} \\ dt = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \\ dx = \frac{2}{1+t^2}\,dt \end{array}\right] =$$

$$= \int \frac{1}{\frac{2t}{\cancel{1+t^2}}}\,\frac{2dt}{\cancel{1+t^2}} = \int \frac{2}{2t}\,dt = \int \frac{dt}{t}$$

$$= \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

$$\boxed{\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \boxed{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

vilket ger $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$

vilket ger $\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| = \frac{1}{2}\ln\left(\tan^2\frac{x}{2}\right) =$

$= \ln\left|\tan\frac{x}{2}\right|$

Substitutionen $t = \tan\frac{x}{2}$ fungerar för alla rationella uttryck i cos och sin men kan ge lite krångliga räkningar.

$$\int f(\tan x)\,dx = \left[\begin{array}{l} t = \tan x \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] = \int \frac{f(t)}{1+t^2}\,dt.$$

Ex:
$$\int \frac{dx}{1+\tan x} = \left[\begin{array}{l} t = \tan x \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] = \int \frac{dt}{(1+t)(1+t^2)} =$$

$$= \int \frac{A}{1+t} + \frac{Bt+C}{1+t^2}\,dt = \int \frac{\frac{1}{2}}{1+t} + \frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}\,dt$$

$$= \frac{1}{2}\ln|t+1| - \frac{1}{2}\int \frac{t-1}{1+t^2}\,dt = \frac{1}{2}\ln|1+t| -$$

$$- \frac{1}{4}\int \frac{2t}{1+t^2}\,dt + \frac{1}{2}\arctan t + C = \frac{1}{2}\ln|1+t| -$$

$$-\frac{1}{4}\ln(1+t^2)+\frac{1}{2}\arctan t + C =$$

$$= \frac{1}{2}\ln|1+\tan x| - \frac{1}{4}\ln|1+\tan^2 x| +$$

$$+\frac{1}{2}x + C$$

Övn: $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx = \int \frac{\tan x}{\tan x+1}\,dx = \left[\begin{array}{l} t=\tan x \\ dx=\frac{dt}{1+t^2}\end{array}\right.$

$$= \int \frac{t}{(1+t)}\frac{dt}{(1+t^2)} = \int \frac{A}{1+t}+\frac{Bt+C}{1+t^2}\,dt$$

$$= \int \frac{-\frac{1}{2}}{1+t}+\frac{\frac{1}{2}t+\frac{1}{2}}{1+t^2}\,dt =$$

$$= -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln|1+t^2| + \frac{1}{2}\arctan t + C$$

$$= -\frac{1}{2}\ln|1+\tan x| + \frac{1}{4}\ln|1+\tan^2 x| + \frac{1}{2}x + C.$$

Ex: $\int \sin^{2n+1} x \, dx = \int \sin^{2n} x \sin x \, dx =$

$$= \int (1-\cos^2 x)^n \sin x \, dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x \, dx \end{bmatrix} =$$

$$= -\int (1-t^2)^n \, dt = -\int \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{2k} \, dt =$$

$$= - \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int t^{2k} \, dt =$$

$$= - \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{t^{2k+1}}{2k+1} + C =$$

$$= - \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\cos^{2k+1} x}{2k+1} + C$$

Ex: $\int \sin^4 x \, dx = \int \sin x \sin^3 x \, dx =$

$$= -\cos x \sin^3 x + \int \cos x \, 3 \sin^2 x \cos x \, dx$$

$$= -\cos x \sin^3 x + 3 \int \cos^2 x \sin^2 x \, dx =$$

$$= -\cos x \sin^3 x + 3\int (1-\sin^2 x)\sin^2 x\, dx =$$

$$= -\cos x \sin^3 x + 3\int \sin^2 x\, dx - 3\int \sin^4 x\, dx$$

$$\Rightarrow \int \sin^4 x\, dx = \frac{1}{4}\left(-\cos x \sin^3 x + 3\int \sin^2 x\, dx\right)$$

$$= \frac{1}{4}\left(-\cos x \sin^3 x + 3\int \frac{1-\cos 2x}{2}\, dx\right) =$$

$$= \frac{1}{4}\left(-\cos x \sin^3 x + \frac{3}{2}x - \frac{3}{4}\sin 2x + C\right)$$

Algebraiska integrander

Ex: $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

Ex: $\int \frac{dx}{\sqrt{x^2+1}} = \left[\begin{array}{l} x = \operatorname{sh} t \\ dx = \operatorname{ch} t\, dt \end{array}\right] =$

$= \int \frac{\operatorname{ch} t\, dt}{\sqrt{\operatorname{sh}^2 t + 1}} = \int \frac{\operatorname{ch} t}{\operatorname{ch} t}\, dt = \int dt = t + C$

$= \ln\left(x + \sqrt{1+x^2}\right) + C$