Generaliserade integraler

Ex: $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx$

Substitutionen $t = \tan x$ går ej direct att använda för $\tan\frac{\pi}{2}$ är odefinierad. (OBS! $\tan 0 = \tan \pi = 0$ men integranden är > 0)

Lösn: $$\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx = \lim_{a\to\frac{\pi}{2}^-} \int_0^a \frac{1}{1+\sin^2 x}\,dx +$$

$$+ \lim_{b\to\frac{\pi}{2}^+} \int_b^{\pi} \frac{1}{1+\sin^2 x}\,dx$$

Vi har att

$$\lim_{a\to\frac{\pi}{2}^-} \int_0^a \frac{1}{1+\sin^2 x}\,dx = \left[\begin{array}{l} t=\tan x \\ dx = \frac{dt}{1+t^2} \end{array},\ \sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}\right]$$

$$= \lim_{a\to\frac{\pi}{2}^-} \int_0^{\tan a} \frac{1}{1+\frac{t^2}{1+t^2}}\,\frac{dt}{1+t^2} = \lim_{a\to\frac{\pi}{2}^-} \int_0^{\tan a} \frac{dt}{1+2t^2} =$$

$$= \left[\begin{array}{l} s=\sqrt{2}\,t \\ ds=\sqrt{2}\,dt \end{array}\right] = \lim_{a\to\frac{\pi}{2}^-} \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}\tan a} \frac{ds}{1+s^2} =$$

$$= \lim_{a\to\frac{\pi}{2}^-} \frac{1}{\sqrt{2}} \left(\arctan(\sqrt{2}\tan a) - 0\right)$$

$$= \frac{\pi}{2\sqrt{2}}$$

P55

$$\lim_{b\to\frac{\pi}{2}^+} \int_b^{\pi} \frac{1}{1+\sin^2 x}\,dx = \lim_{b\to\frac{\pi}{2}^+} \left[\frac{1}{\sqrt{2}}\arctan t\right]_{\sqrt{2}\tan b}^{0}$$

$$= \frac{\pi}{2\sqrt{2}}$$

Alltså

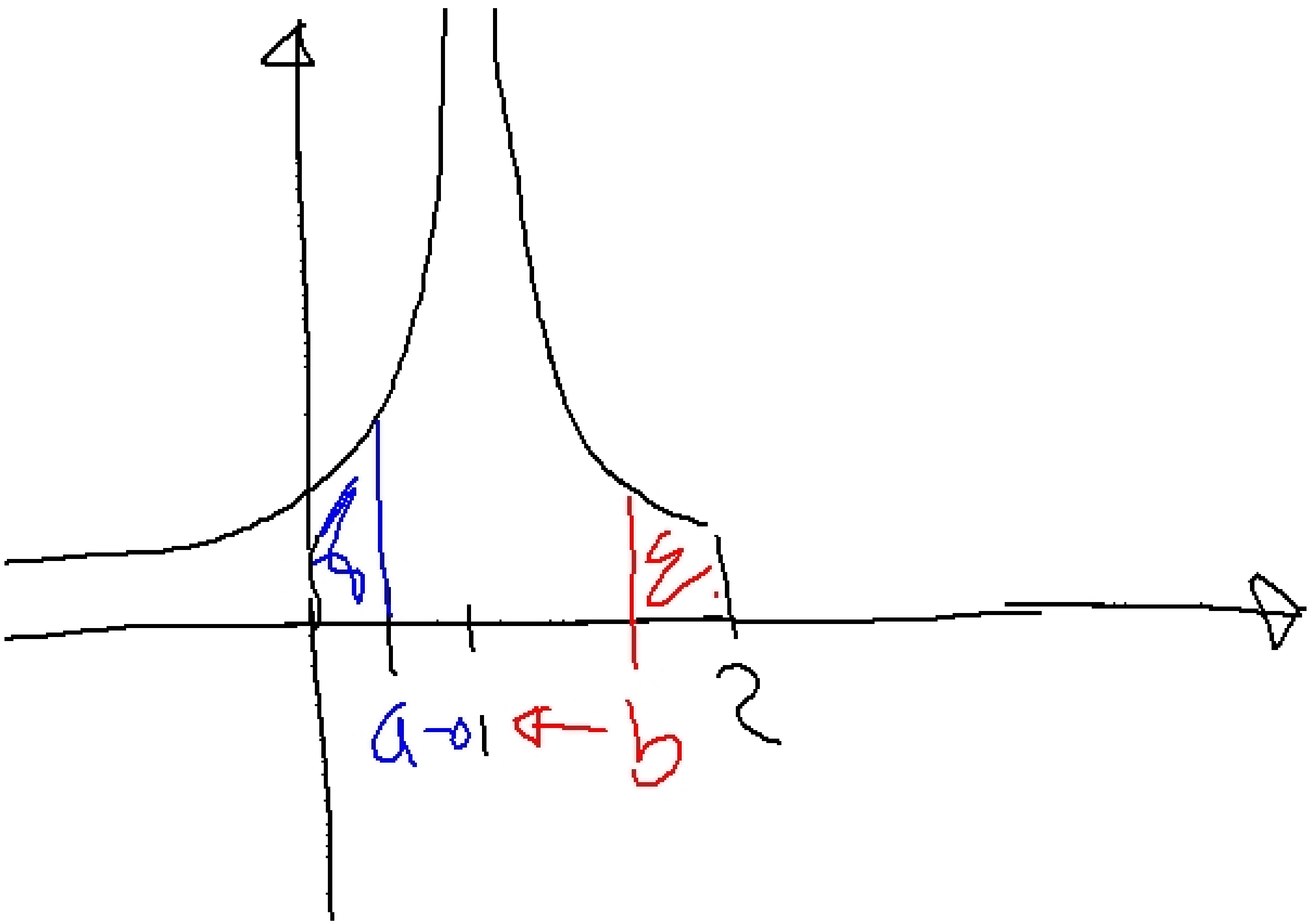
$$\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx = \frac{\pi}{\sqrt{2}}$$

Ex: $$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \lim_{a\to 1^-}\int_0^a \frac{dx}{\sqrt{|x-1|}} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{a\to 1^-}\int_0^a \frac{dx}{\sqrt{1-x}} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{\sqrt{x-1}} =$$

$$= \lim_{a \to 1^-} \left[-2\sqrt{1-x} \right]_0^a + \lim_{b \to 1^+} \left[2\sqrt{x-1} \right]_b^2$$

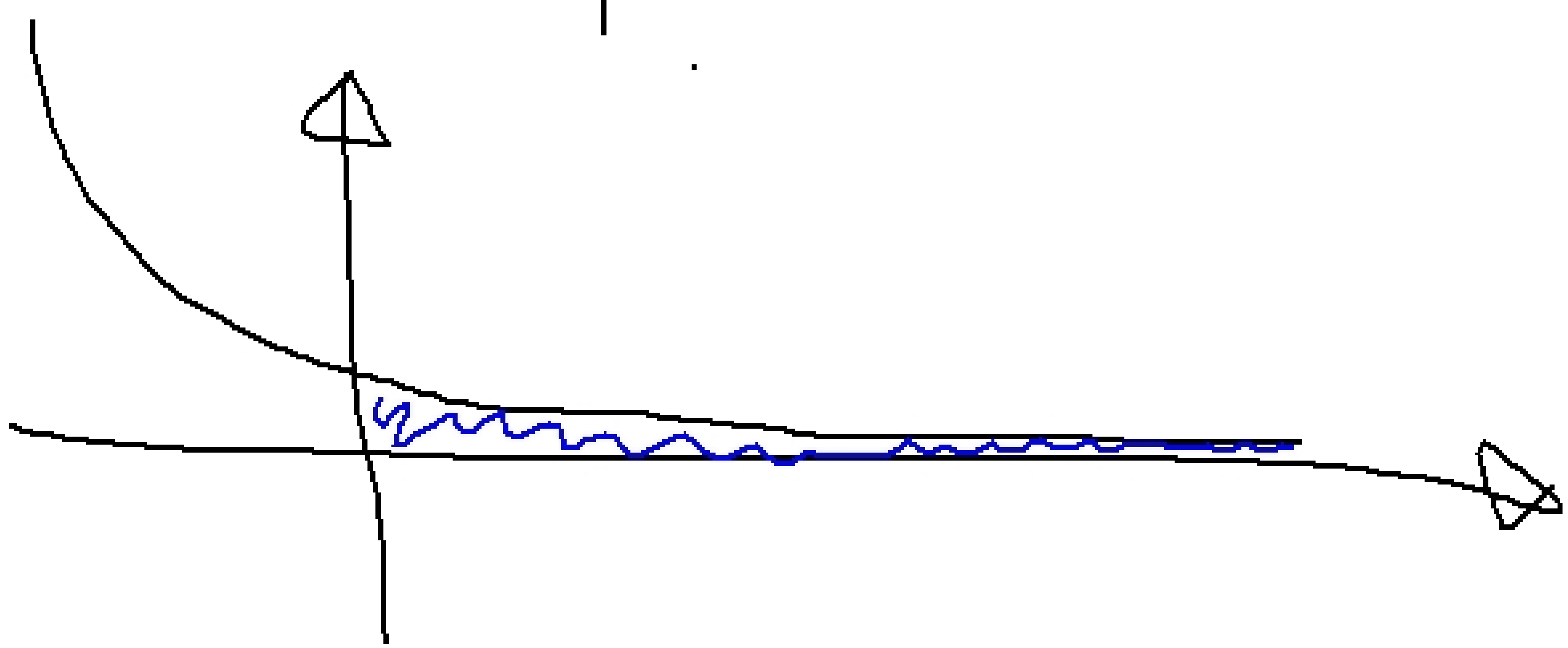
$$= 0 + 2 \quad + 2 - 0 = 4.$$



Ex: $\displaystyle\int_0^\infty e^{-x}\,dx = \lim_{a\to\infty}\int_0^a e^{-x}\,dx$

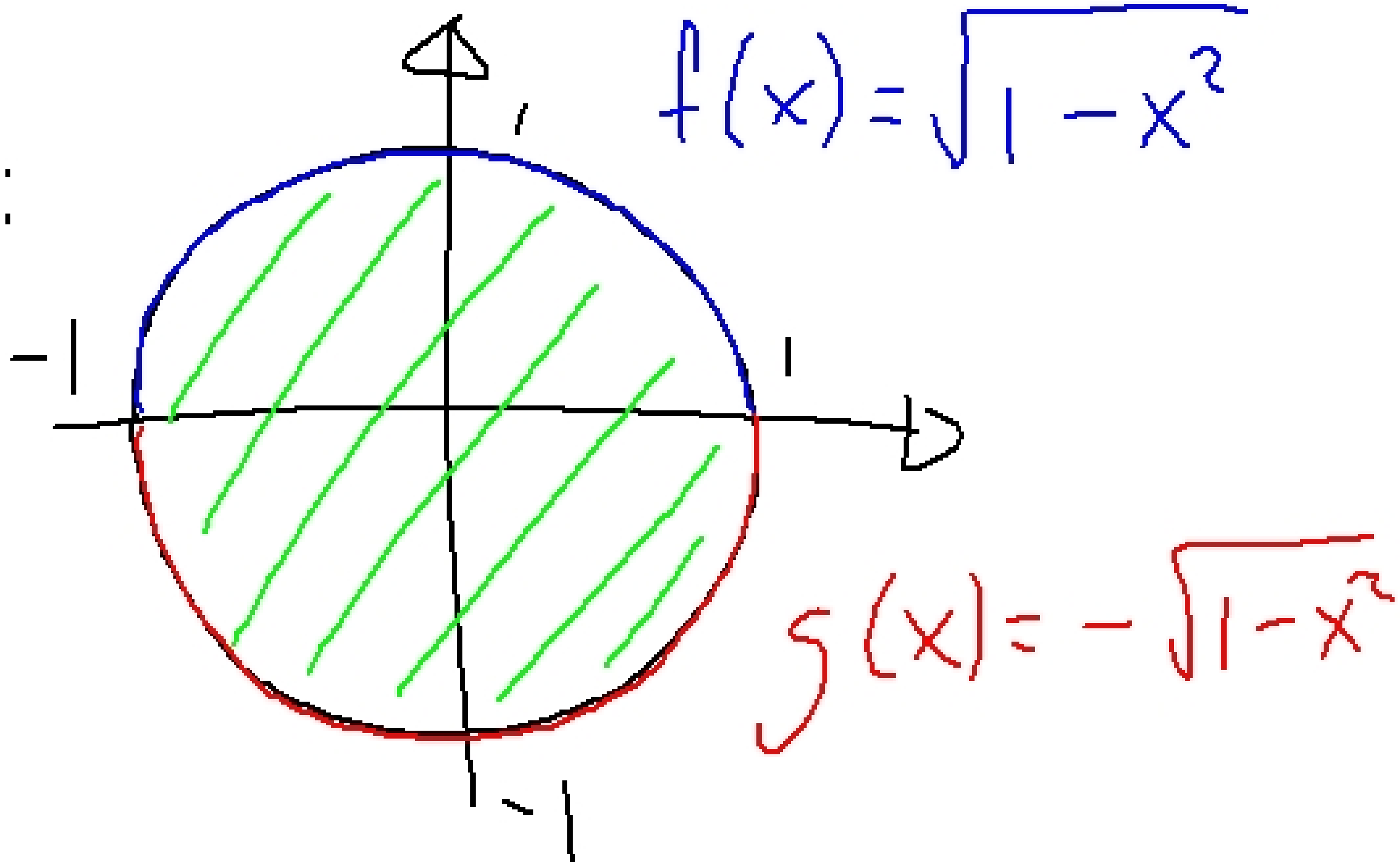
$\displaystyle = \lim_{a\to\infty}\left[-e^{-x}\right]_0^a = \lim_{a\to\infty} -e^{-a}+e^{-0} =$

$= 0 + 1 = 1$



Areor hos plane områden

Ex:



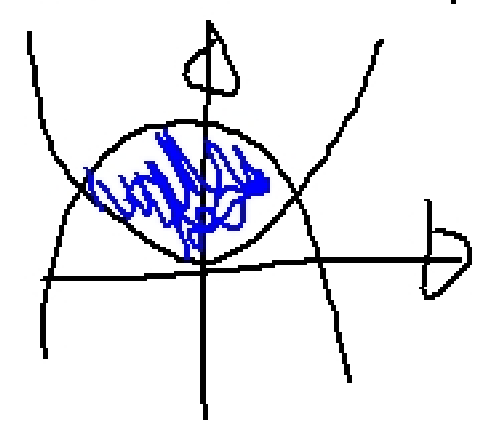
$$\text{Area} = \int_{-1}^{1} f(x) - g(x)\, dx = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$= \begin{bmatrix} x = \sin\varphi \\ dx = \cos\varphi\, d\varphi \end{bmatrix} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\varphi}\, \cos\varphi\, d\varphi =$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\cos\varphi \, d\varphi = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\varphi}{2} \, d\varphi$$

$$= 2\left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \pi$$

Ex: Beräkna arean av det begränsade område som begränsas av kurvorna $f(x) = x^2$ och $g(x) = 2 - x^2$.



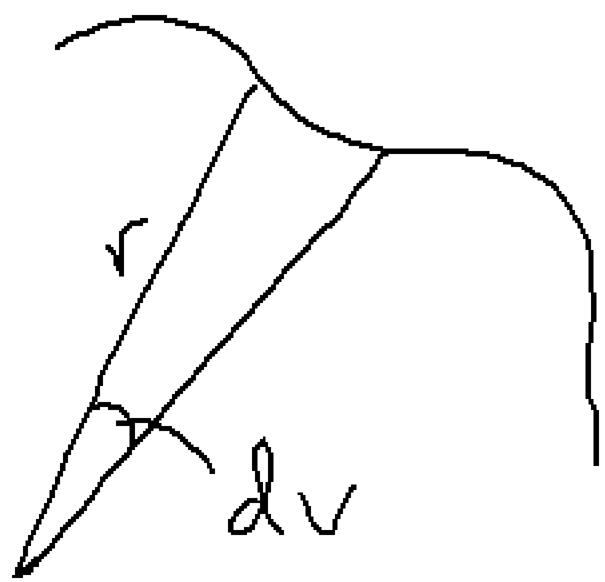
Kurvorna skär varandra i $x = \pm 1$

$$\text{Arean} = \left| \int_{-1}^{1} f(x) - g(x)\, dx \right|$$

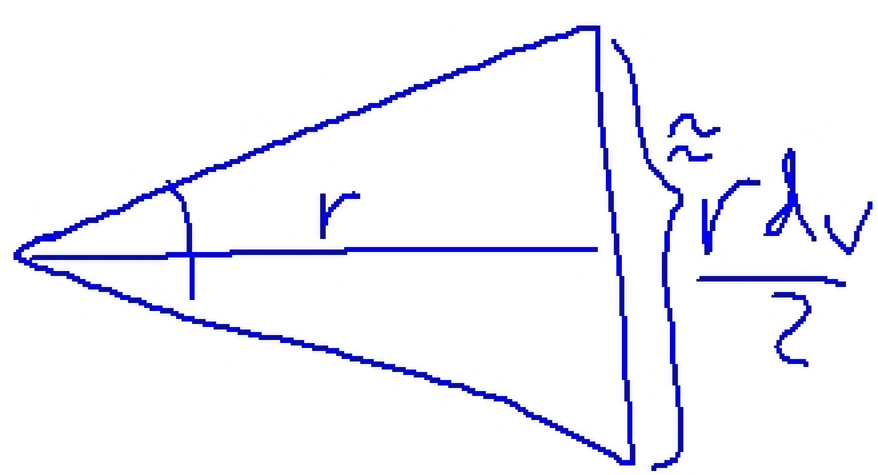
$$= \left| \int_{-1}^{1} 2x^2 - 2\, dx \right| = 2 \left| \int_{-1}^{1} x^2 - 1\, dx \right|$$

$$= 2 \left| \left[\frac{x^3}{3} - x \right]_{-1}^{1} \right| = 2 \left| \frac{1}{3} - 1 + \frac{1}{3} - 1 \right| = \frac{8}{3}.$$

- Polärt



$$\text{Area} = \frac{r^2 dv}{2}$$

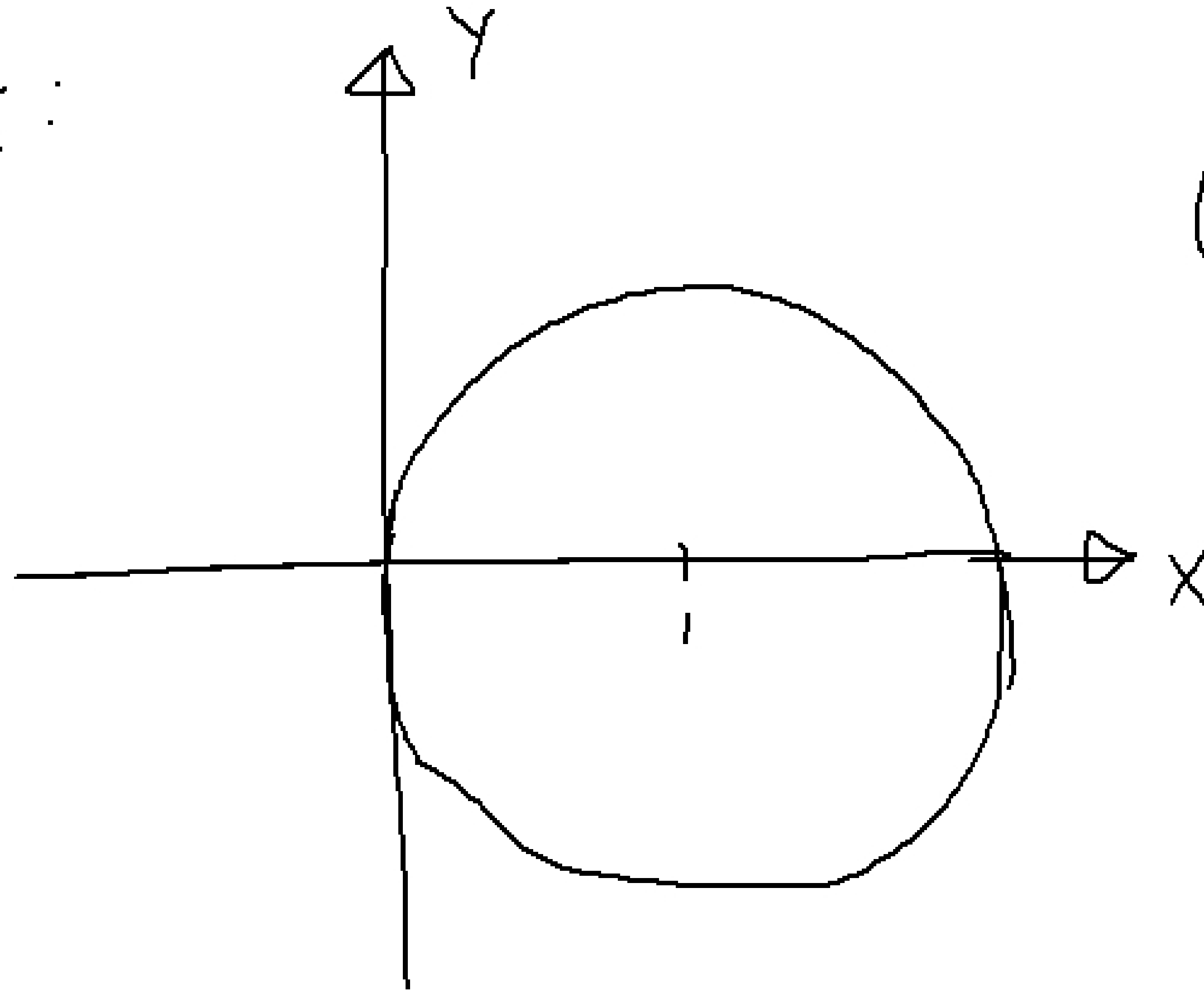


$$A = \frac{1}{2}\int_{v_1}^{v_2} (r(v))^2 \, dv$$

Ex: enhetscirkel-skivans area.

$$A = \frac{1}{2}\int_0^{2\pi} 1^2 \, dv = \frac{2\pi}{2} = \pi$$

Ex:



$$(x-1)^2+y^2=1$$

Sätt

$$x = r\cos v$$
$$y = r\sin v$$

$$(r\cos v - 1)^2 + (r\sin v)^2 = 1$$

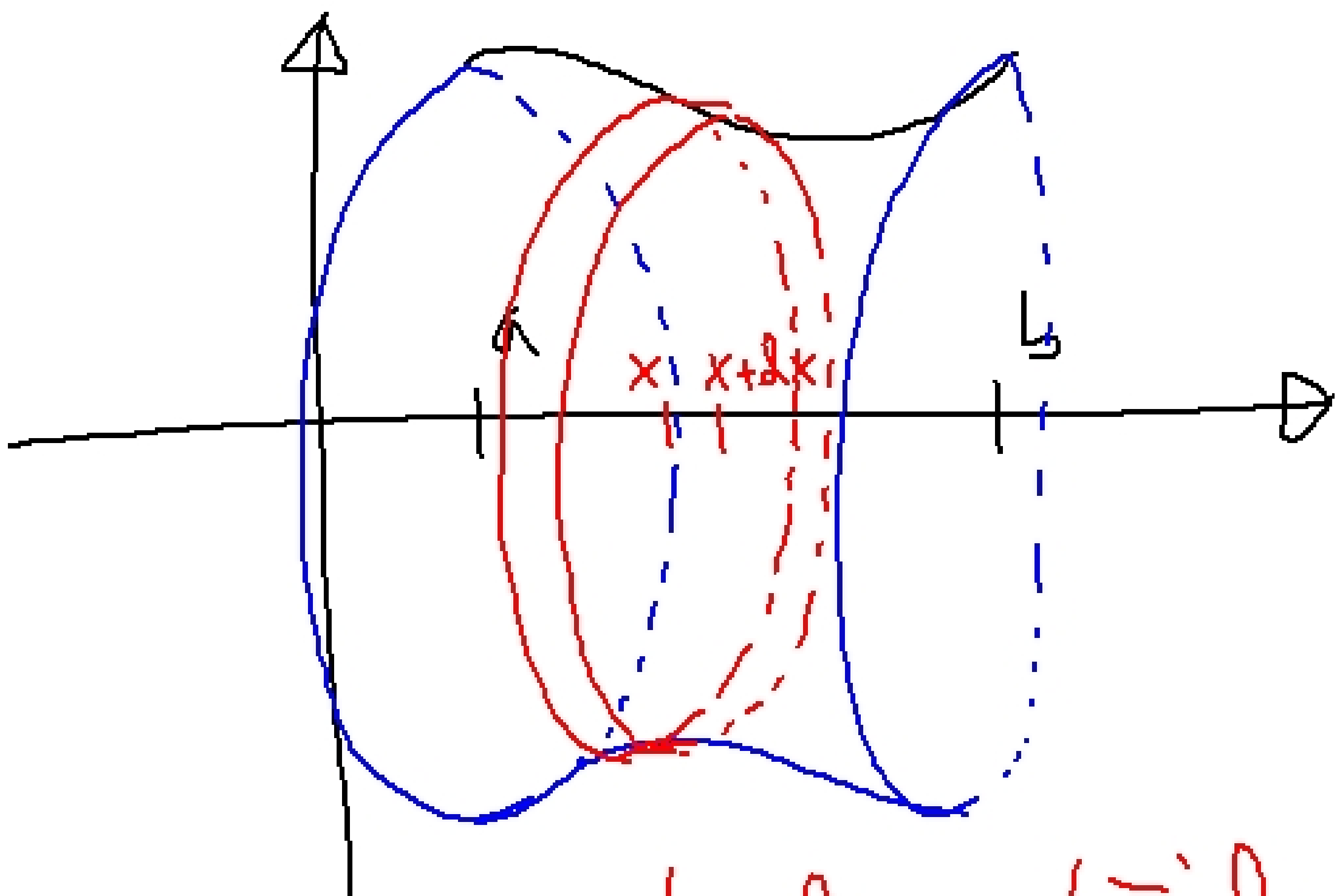
$$r = 2\cos v \qquad \text{Area} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos v)^2\,dv$$

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2 v \, dv = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2v}{2}\, dv =$$

$$= 2\left[\frac{v}{2} + \frac{\sin 2v}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \pi$$

Rotationsvolymer

rotation kring x-axeln



Volym $= \int_a^b \pi (f(x))^2 \, dx$

Cylinderns höjd: dx

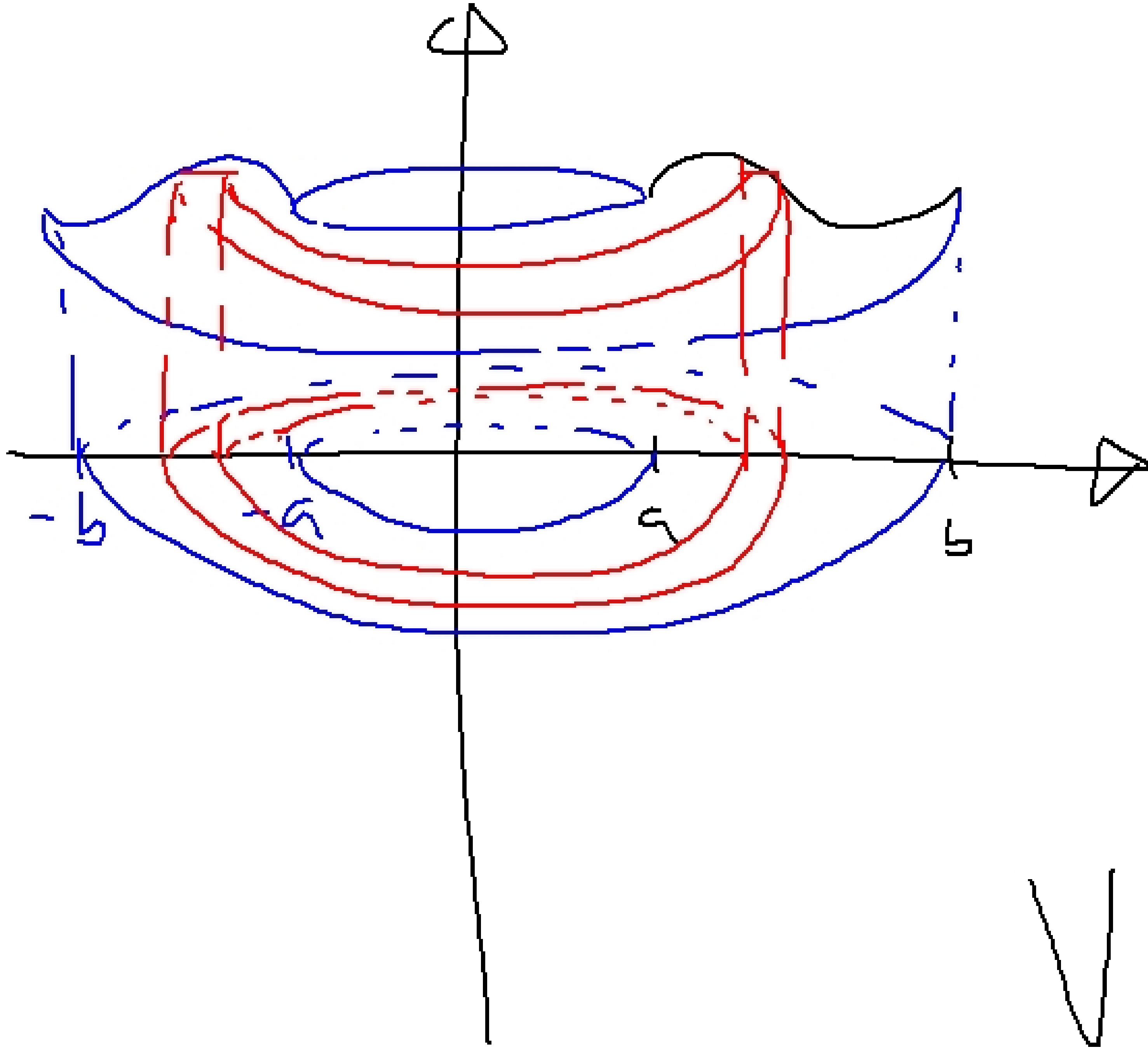
basarea $= \pi (f(x))^2$

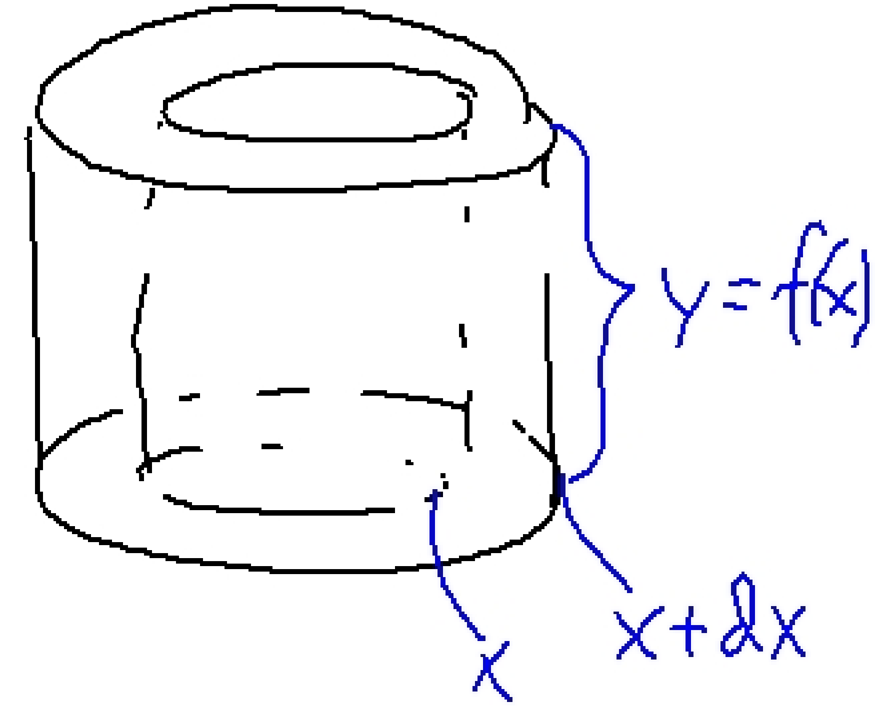
$\Rightarrow$ Cylinderns volym $= \pi (f(x))^2 dx$

Ex: volymen hos ett klot med radie ett.

$$\int_{-1}^{1} \pi \left(\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} 1 - x^2 \, dx =$$

$$= \pi \left[x - \frac{x^3}{3} \right]_{-1}^{1} = \pi \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{4\pi}{3}$$

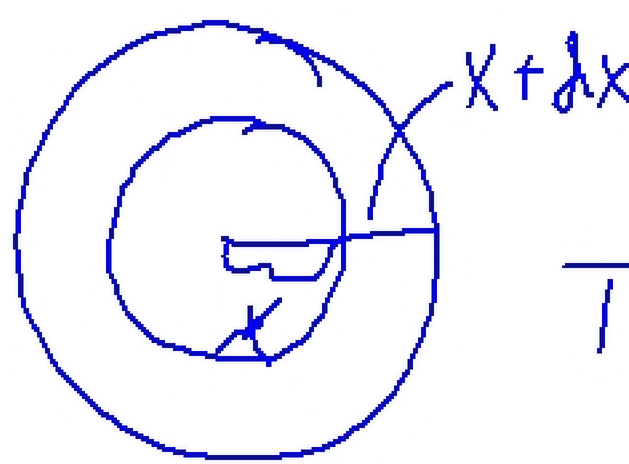




$$dV = 2\pi x y \, dx$$

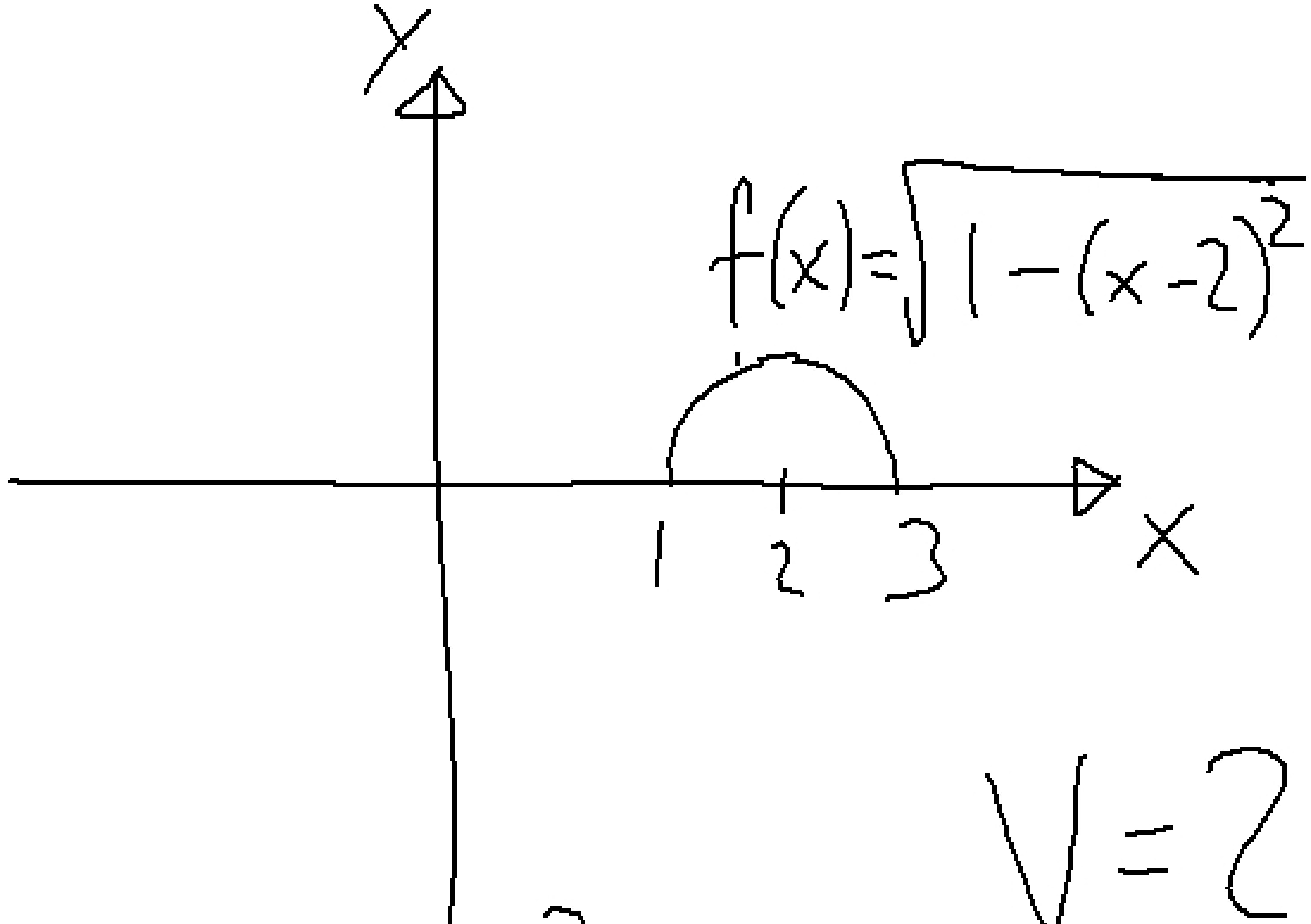
$$V = \int_a^b 2\pi x y \, dx$$

Basarea:



$$\pi(x+dx)^2 - \pi x^2 \approx 2\pi x \, dx$$

Ex: Volymen hos en torus



$$V = 2 \cdot \int_1^3 2\pi x f(x)\, dx =$$

$$= 4\pi \int_1^3 x\sqrt{1-(x-2)^2}\, dx = \left[\begin{array}{l} t = x-2 \\ dt = dx \end{array} \right] =$$

$$= 4\pi \int_{-1}^{1} \sqrt{1-t^2}\,(t+2)\,dt =$$

$$= 4\pi \int_{-1}^{1} t\sqrt{1-t^2}\,dt + 8\pi \int_{-1}^{1} \sqrt{1-t^2}\,dt$$

$$= 4\pi \left[-\frac{1}{2}\frac{2}{3}(1-t^2)^{\frac{3}{2}} \right]_{-1}^{1} + 8\pi \frac{\pi}{2} = 4\pi^2$$