is, at $x = 1$. Thus $|f(z)| \le e$ in R, and the constant M in part (d) of the theorem here equals e.

EXERCISES 2.2

1. a) Let $f(z) = z$. Show by using an argument like that presented in Example 1 that $\lim_{z \to z_0} f(z) = z_0$, where z_0 is an arbitrary complex number.

b) Using the definition of continuity, explain why $f(z)$ is continuous for any z_0.

2. Let $f(z) = c$ where c is an arbitrary constant. Using the definitions of limit and continuity, prove that $f(z)$ is continuous for all z.

Assuming the continuity of the functions $f(z) = z$ and $f(z) = c$, where c is any constant (proved in Exercises 1 and 2), use various parts of Theorem 2 to prove the continuity of the following functions in the domain indicated. Take $z = x + iy$.

3. $f(z) = iz^3 + i$, all z **4.** $\dfrac{1+i}{z^2+9}$, all $z \ne \pm 3i$

5. $z^4 + \dfrac{1+i}{z^2+3z+2}$, all $z \ne -1, -2$ **6.** $f(z) = |z+i| + (1+i)z$, all z

7. $f(z) = z^2 + x^2 - y^2$, all z **8.** $f(z) = \dfrac{z-i}{\bar{z}-i}$, all $z \ne -i$

9. The function $(\sin x + i \sin y)/(x - iy)$ is obviously undefined at $z = 0$. Show that it fails to have a limit as $z \to 0$ by comparing the values assumed by this function as the origin is approached along the following three line segments: $y = 0, x > 0$; $x = 0, y > 0$; $x = y, x > 0$.

10. Prove that the following function is continuous at $z = i$. Give an explanation like that provided in Example 4.

$$f(z) = \frac{z-i}{z^2 - 3i - 2} \quad z \ne i \quad \text{and} \quad f(i) = i.$$

11. a) Consider the function $f(z) = \frac{z^2-5z+6}{z^2-4}$ defined for $z \ne \pm 2$. How should this function be defined at $z = 2$ so that $f(z)$ is continuous at $z = 2$?

b) Consider the function $f(z) = \frac{z^4+10z^2+9}{z^2-4iz-3}$ defined for $z \ne 3i$ and $z \ne i$. How should this function be defined at $z = 3i$ and $z = i$ so that $f(z)$ is continuous everywhere?

12. In this problem we prove rigorously, using the definition of the limit at infinity, that

$$\lim_{z \to \infty} \frac{z}{1+z} = 1.$$

a) Explain why, given $\varepsilon > 0$, we must find a function $r(\varepsilon)$ such that $\left|\frac{1}{z+1}\right| < \varepsilon$ for all $|z| > r$.

b) Using one of the triangle inequalities, show that the preceding inequality is satisfied if we take $r > 1 + 1/\varepsilon$.

13. The following problem refers to Theorem 2(d). Consider $f(z) = z - i$.

a) In the region R described by $|z| \le 1$, we have $|f(z)| \le M$. Find M assuming $|f(z)| = M$ for some z in R. State where in this closed region $|f(z)| = M$.

z_0 along a line making a 45° angle with the horizontal, we have $\Delta y = \Delta x$, or $m = 1$. The expression becomes $(2x_0 + 2y_0)/(1 + i)$.

EXERCISES 2.3

Sketch the following real functions $f(x)$ over the indicated interval. A plot obtained from MATLAB or a graphing calculator is encouraged. In each case, find the one value of x where the derivative with respect to x fails to exist. State whether the function is continuous at this point. A rigorous justification is not required.

1. $f(x) = \sin|x|, -1 \le x \le 1$ **2.** $f(x) = (x-1)^{2/3}$ (use the *real* root) for $0 \le x \le 2$

3. In Example 1, we showed that $f(z) = \bar{z}$ is not differentiable. Obtain this conclusion by using the definition in Eq. (2.3–3) and show that this results in your having to evaluate $\lim_{\Delta z \to 0} 2\arg(\Delta z)$. Why does this limit not exist?

For what values of the complex variable z do the following functions have derivatives?

4. c (a constant) **5.** $1 + iy$ **6.** z^6 **7.** z^{-5} **8.** $y + ix$ **9.** $xy(1+i)$
10. $x^2 + iy$ **11.** $x + i|y|$ **12.** $e^x + ie^{2y}$ **13.** $y - 2xy + i(-x + x^2 - y^2)$
14. $(x-1)^2 + iy^2 + z^2$ **15.** $f(z) = \cos x - i \sinh y$
16. $f(z) = 1/z$ for $|z| > 1$ and $f(z) = z$ for $|z| \le 1$

17. Find two functions of z, neither of which has a derivative anywhere in the complex plane, but whose nonconstant sum has a derivative everywhere.

18. Let $f(z) = u(x,y) + iv(x,y)$. Assume that the second derivative $f''(z)$ exists. Show that

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} \quad \text{and} \quad f''(z) = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}.$$

Hint: See the derivation of Eqs. (2.3–6) and (2.3–8).

19. Show that if $f'(z_0)$ exists, then $f(z)$ must be continuous at z_0.

Hint: Let $z = z_0 + \Delta z$. Consider

$$\lim_{\Delta z \to 0}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \lim_{\Delta z \to 0} \Delta z.$$

Refer to Eq. (2.2–10b) of Theorem 1.

2.4 The Derivative and Analyticity

Finding the Derivative

If we can establish that the derivative of $f(z) = u(x,y) + iv(x,y)$ exists for some z, it is a straightforward matter to find $f'(z)$. In theory, we can work directly with the definition shown if Eq. (2.3–3) but this is generally tedious.

If $u(x,y)$ and $v(x,y)$ are stated explicitly, then we would probably use either Eq. (2.3–6), $f'(z) = \partial u/\partial x + i\partial v/\partial x$, or Eq. (2.3–8), $f'(z) = \partial v/\partial y - i\partial u/\partial y$. The method is illustrated in the example that follows.

2.4

2. a) Find the derivative of $f(z) = 1/z + (x-1)^2 + ixy$ at any points where the derivative exists. Give the numerical value.

   b) Where is this function analytic?

3. a) Where is the function $f(z) = z^3 + z^2 + 1$ analytic?

   b) Find an expression for $f'(z)$ and give the derivative at $1+i$.

4. a) Where does the function $f(z) = z^2 + (x-1)^2 + i(y-1)^2$ have a derivative?

   b) Where is this function analytic? Explain.

   c) Derive a formula that will yield the derivative of this expression at points where the derivative exists, and use this formula to find the numerical value of the derivative at the point $z = 1+i$.

5. a) Show that $f(z) = \frac{1}{e^{2x}\cos 2y + ie^{2x}\sin 2y}$ is an entire function. Pay attention to the possibility of a vanishing denominator.

   b) Obtain an explicit expression for the derivative of this function and find the numerical value of the derivative at $1 + i\pi/4$.

6. a) Show that $z\,[\cos x \cosh y - i \sin x \sinh y]$ is an entire function.

   b) Find an explicit expression for the derivative of this function and give the numerical value of the derivative at $z = i$.

7. Find the derivative at $z = \pi + 2i$ of the function $[\sin x \cosh y + i \cos x \sinh y]^5$.
   Hint: Do not raise the expression in the brackets to the 5th power—show that the function in the brackets is entire and then use part (d) of Theorem 4.

8. a) Where is the function $f(z) = \frac{z}{(1+iz)^4}$ analytic?

   b) Find $f'(-i)$.

Use L'Hopital's Rule to establish these limits:

9. $\dfrac{(z-i)+(z^2+1)}{z^2-3iz-2}$ as $z \to i$ 10. $\dfrac{(z^3+i)}{(z^2+1)z}$ as $z \to i$

11. If $g(z)$ has a derivative at z_0 and $h(z)$ does not have a derivative at z_0, explain why $g(z) + h(z)$ cannot have a derivative at z_0.

12. Find two functions, each of which is nowhere analytic, but whose sum is an entire function. Thus the sum of two nonanalytic functions can be analytic.

13. a) Assume $g(z)$ is analytic and nonzero at z_0 and that $h(z)$ is not analytic at the same point. Show that $f(z) = g(z)h(z)$ cannot be analytic at this point.
    Hint: Assume that $f(z)$ is analytic and show, with the aid of Theorem 5, that a contradiction is obtained.

    b) Using the above result, you can immediately argue that the function $z^2\bar{z}$ cannot be analytic for $z \neq 0$. Using the definition of analyticity, how can you conclude that this function is not analytic at $z = 0$ as well?

14. Consider functions $g(z)$ and $h(z)$ where neither function is analytic anywhere in the complex plane. Find g and h such that their product is an entire function.
    Hint: Express one function as the quotient of a simple entire function and a function that is nowhere analytic. Choose for the second function something nowhere analytic.

15. a) Let $\phi(x, y)$ be a function whose partial derivatives with respect to x and y exist throughout a domain D. Assume $\partial\phi/\partial x = 0$, $\partial\phi/\partial y = 0$ throughout D. Prove that $\phi(x, y)$ is constant in D.

Hint: Let x_0, y_0 be a point in D. Suppose you move from this point to another point x_1, y_0 which is to the right or left of the original point. Assume that these points are sufficiently close together that the horizontal straight-line path connecting them lies within D. Argue that the values of $\phi(x, y)$ assumed at these two points must be the same. Observe that the same kind of argument works if you move from x_0, y_0 to a point above or below, while you remain in D. Now notice that *any* two points in D can be connected by a curve consisting of small steps parallel to the x and y axes (a staircase). Argue that $\phi(x, y)$ would not change along such a path—therefore the value of $\phi(x, y)$ is the same for any two points in D and is thus constant.

b) Using the preceding and the C–R equations, prove that *if an analytic function is purely real in a domain D, then the function must be constant in D*. Explain why the preceding statement is true if we substitute the word "imaginary" for "real."

16. Suppose $f(z) = u + iv$ is analytic. Under what circumstances will $g(z) = u - iv$ be analytic?

Hint: Consider the functions $f(z) + g(z)$ and $f(z) - g(z)$. Then refer to Exercise 15. You may also use Theorem 5.

17. Consider an analytic function $f(z) = u + iv$ whose modulus $|f(z)|$ is equal to a constant k throughout some domain. Show that this can occur only if $f(z)$ is constant throughout the domain.

Hint: The case $k = 0$ is trivial. Assuming $k \neq 0$, we have $u^2 + v^2 = k^2$ or $k^2/(u + iv) = u - iv$. Now refer to Exercise 16.

18. a) Assume that both $f(z)$ and $\overline{f(\bar{z})}$ are defined in a domain D and that $f(z)$ is analytic in D. Assume that $f(\bar{z}) = \overline{f(z)}$ in D. Show that $f(\bar{z})$ cannot be analytic in D unless $f(z)$ is a constant.

Hint: $f(z) + f(\bar{z})$ is real. Why? Now use the result of Exercise 15.

b) Use the preceding result to argue in a few lines that $(\bar{z})^3 + \bar{z}$ is nowhere analytic.

19. Using an argument like that presented in Example 7, show that the following functions are nowhere analytic:

a) $(\bar{z} + 1)^2$ **b)** $\bar{z}^3$

20. This problem introduces us to complex functions of a real variable. They will usually appear in the form $f(t) = u(t) + iv(t)$, where u and v are real functions of the real variable t (a letter chosen to suggest time). The rule for differentiating $f(t)$ is identical to that used in real calculus. We use Eq. (2.3–1), rewritten here as $f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$. As $f(t)$ is defined only for a real variable, the concept of analyticity does not pertain. All of the rules learned in elementary calculus for differentiating a real function of a real variable apply to $f(t)$, with the additional obvious statement that $f'(t) = u'(t) + iv'(t)$. We can plot in the complex plane (the u, v-plane) the locus assumed by $f(t)$ as the parameter t varies through some interval.

a) Consider $f(t) = \cos t + i \sin t$. Draw the locus in the complex plane describing $f(t)$ as t advances from 0 to 2π.

b) For the function of part (a), find $f'(t)$ and show that for any t the vector representation of $f'(t)$ is perpendicular to that of $f(t)$.

2.5

2. Where in the complex plane will the function $\phi(x, y) = \sin(xy)$ satisfy Laplace's equation? Is this a harmonic function?

3. Consider the function $\phi(x, y) = e^{ky} \sin(mx)$. Assuming this function is harmonic throughout the complex plane, what must be the relationship between the real constants k and m? Assume that $m \neq 0$.

4. Find the value of the integer n if $x^n - y^n$ is harmonic.

> Putting $z = x + iy$, show the following by direct calculation.
>
> 5. $\text{Im}(1/z)$ is harmonic throughout any domain not containing $z = 0$.
>
> 6. $\text{Re}(z^3)$ is harmonic in any domain.

7. Find two values of k such that $\cos x[e^y + e^{ky}]$ is harmonic.

8. If $g(x)[e^{2y} - e^{-2y}]$ is harmonic, $g(0) = 0$, $g'(0) = 1$, find $g(x)$.

9. a) Consider $\phi(x, y) = x^3y - y^3x + y^2 - x^2 + x$. Show that this can be the real part or the imaginary part of an analytic function.

   b) Assuming the preceding is the real part of an analytic function, find the imaginary part.

   c) Assuming that $\phi(x, y)$ is the imaginary part of an analytic function find the real part. Compare your answer to that in part (b).

   d) If $\phi(x, y) + iv(x, y)$ is an analytic function and if $u(x, y) + i\phi(x, y)$ is also analytic, where $\phi(x, y)$ is an arbitrary harmonic function, prove that, neglecting constants, $u(x, y)$ and $v(x, y)$ must be negatives of each other. Is this confirmed by your answers to parts (b) and (c)?

10. Suppose that $f(z) = u + iv$ is analytic and that $g(z) = v + iu$ is also. Show that v and u must both be constants.
Hint: $-if(z) = v - iu$ is analytic (the product of analytic functions). Thus $g(z) \pm if(z)$ is analytic and must satisfy the C–R equations. Now refer to Exercise 15 of section 2.4.

11. Find the harmonic conjugate of $e^x \cos y + e^y \cos x + xy$.

12. Find the harmonic conjugate of $\tan^{-1}(x/y)$ where $-\pi < \tan^{-1}(x/y) \leq \pi$.

13. Show, if $u(x, y)$ and $v(x, y)$ are harmonic functions, that $u + v$ must be a harmonic function but that uv need not be a harmonic function. Is $e^u e^v$ a harmonic function?

> If $v(x, y)$ is the harmonic conjugate of $u(x,y)$ show that the following are harmonic functions.
>
> 14. uv 15. $e^u \cos v$ 16. $\sin u \cosh v$

17. Consider $f(z) = z^2 = u + iv$.

   a) Find the equation describing the curve along which $u = 1$ in the xy-plane. Repeat for $v = 2$.

   b) Find the point of intersection, in the first quadrant, of the two curves found in part (a).

   c) Find the numerical value of the slope of each curve at the point of intersection, which was found in part (b), and verify that the slopes are negative reciprocals.

18. a) Show that $f(z) = e^x \cos y + ie^x \sin y = u + iv$ is entire.

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b) Consider the curve in the xy plane along which $u = 1$. Using MATLAB, generate the portion of this curve lying in the first quadrant. Restrict y to satisfy $0 \le y \le \pi/2$ and place the same restriction on x. Repeat the preceding but consider $u = 1/2$. Make both plots on a single set of coordinates.

c) Repeat part (b) but generate the curves for which $v = 1$ and $1/2$. Make the plots on the coordinate systems of part (a) so that the orthogonality of the intersections is apparent.

d) Find mathematically the point of intersection of the curves $u = 1$ and $v = 1/2$. Verify this from your plot.

e) Taking derivatives, find the slopes of the curves $u = 1$ and $v = 1/2$ at their point of intersection and verify that they are negative reciprocals. Confirm your result by inspecting the plot. Note that the curves in the plot will not appear to intersect orthogonally unless you have used the same scale for the horizontal and vertical axes.

19. Consider $f(z) = z^3 = u + iv$.

a) Find the equation describing the curve along which $u = 1$ in the xy-plane. Repeat for $v = 1$. In each case, sketch the curves in the first quadrant.

b) Find mathematically the point of intersection (x_0, y_0) in the first quadrant of the two curves. This is most easily done if you let $z = r \operatorname{cis} \theta$. First find the intersection in polar coordinates.

c) Find the slope of each curve at the point of intersection. Verify that these are negative reciprocals of each other.

20. a) Let $x = r\cos\theta$ and $y = r\sin\theta$, where r and θ are the usual polar coordinate variables. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be a function that is analytic in some domain that does not include $z = 0$. Use Eqs. (2.4–5a,b) and an assumed continuity of second partial derivatives to show that in this domain u and v satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 0. \qquad (2.5\text{–}14)$$

This is Laplace's equation in the polar variables r and θ.

b) Show that $u(r, \theta) = r^2 \cos 2\theta$ is a harmonic function.

c) Find $v(r, \theta)$, the harmonic conjugate of $u(r, \theta)$, and show that it too satisfies Laplace's equation everywhere.

2.6 Some Physical Applications of Harmonic Functions

A number of interesting cases of natural phenomena that are described to a high degree of accuracy by harmonic functions will be discussed in this section.

Steady-State Heat Conduction†

Heat is said to move through a material by *conduction* when energy is transferred by collisions involving adjacent molecules and electrons. For conduction, the time rate of flow of heat energy at each point within the material can be specified by means of a vector. Typically this vector will vary in both magnitude and direction throughout the material. In general, a variation with time must also be considered. However, we

†For a detailed discussion of this subject, see F. Kreith and M. Bohn, *Principles of Heat Transfer*, 6th ed., (Pacific Grove, CA: Brooks Cole, 2001).