3.1

Express each of the following in the form $a + ib$ where a and b are real numbers. If the result is multivalued, be sure to state all the values. Use MATLAB as an aid to checking your work.

2. $e^{1/2+2i}$ **3.** $e^{1/2-2i}$ **4.** e^{-i} **5.** $e^{1/2+2i}e^{-1/2-2i}$ **6.** $e^{(-i)^7}$ **7.** $(e^{-i})^7$

8. $e^{1/(1+i)}$ **9.** $e^{e^{-i}}$ **10.** $e^{i\arctan 1}$ **11.** $e^{(-2)^{1/2}}$ **12.** $(e^{-2})^{1/2}$

13. Find *all* solutions of $e^z = e$ by equating corresponding parts (reals and imaginaries) on both sides of the equation.

Recalling that an analytic function of an analytic function is analytic, state the domain of analyticity of each of the following functions. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy–Riemann equations, and find $f'(z)$ in terms of z.

14. $f(z) = e^{iz}$ **15.** $e^{1/z}$ **16.** e^{e^z}

Using L'Hopital's Rule, evaluate the following:

17. $\lim\limits_{z\to i} \dfrac{z-i}{e^z - e^i}$ **18.** $\lim\limits_{\theta\to\pi} \dfrac{1+e^{i\theta}}{1-e^{2i\theta}}$, where θ is real

19. Consider the identity $e^{z_1+z_2} = e^{z_1}e^{z_2}$, which we proved somewhat tediously in this section. Here is an elegant proof which relies on our knowing that $\frac{de^z}{dz} = e^z$ and $e^0 = 1$.

a) Taking a as a constant, show that $\frac{d(e^z e^{a-z})}{dz} = 0$ by using the usual formula for the derivative of a product, as well as the derivative of e^z, and the chain rule. Note that you cannot combine the exponents, as this has not been justified.

b) Since $e^z e^{a-z}$ has just been shown to be a constant, which we will call k, evaluate k in terms of a by using the fact that $e^0 = 1$.

c) Using $e^z e^{a-z} = k$ as well as k found above, and $z = z_1$, $a = z_1 + z_2$, show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

20. Using a method similar to that in Example 1, find the fifth derivative of $e^t \sin t$.

21. In Example 1, we evaluated the seventh derivative of $e^{2t}\cos 2t$. Check this result by using the function *diff* in the Symbolic Mathematics Toolbox of MATLAB.

For the following closed bounded regions, R, where does the given $|f(z)|$ achieve its maximum and minimum values, and what are these values?

22. R is $|z - 1 - i| \le 2$ and $f(z) = e^z$ **23.** R is $|z| \le 1$ and $f(z) = e^{(z^2)}$

24. a) Suppose we want the nth derivative, with respect to t, of $f(t) = \frac{t}{t^2+1}$. Notice that $f(t) = \mathrm{Re}\left(\frac{1}{t-i}\right)$ and that the nth derivative of the function in the brackets is easily taken. Using the method of Example 1, as well as the binomial theorem (which perhaps should be reviewed), show that

$$f^{(n)}(t) = \frac{(-1)n!(n+1)!}{(t^2+1)^{n+1}} \sum_{k=0}^{(n+1)/2} \frac{(-1)^k t^{n+1-2k}}{(2k)!(n+1-2k)!} \quad \text{for } n \text{ odd,}$$

3.2

(continued)
all of them. Use MATLAB to check your result where possible. Note that MATLAB yields only one value.

1. $\sin(2+3i)$ **2.** $\cos(-2+3i)$ **3.** $\tan(2+3i)$ **4.** $(\sin i)^{1/2}$ **5.** $\sin(i^{1/2})$
6. $\sin(e^i)$ **7.** $\cos(2i\arg(2i))$ **8.** $\sin(\cos(1+i))$ **9.** $\tan(i\arg(1+\sqrt{3}i))$
10. $\arg(\tan i)$ **11.** $e^{i\cos i}+e^{-i\cos i}$

12. Prove the identity $\sin^2 z+\cos^2 z=1$ by the following two methods:
a) Use the definitions of sine and cosine contained in Eqs. (3.2–5) and (3.2–6).
b) Use $\cos^2 z+\sin^2 z=(\cos z+i\sin z)(\cos z-i\sin z)$ as well as Euler's identity generalized to complex z.

Using the definitions of the sine and cosine, Eqs. (3.2–5) and (3.2–6), prove the following.

13. $\frac{d}{dz}\sin z=\cos z$ and $\frac{d}{dz}\cos z=-\sin z$ **14.** $\cos^2 z=\frac{1}{2}+\frac{1}{2}\cos 2z$
15. $\sin(z+2\pi)=\sin z$ and $\cos(z+2\pi)=\cos z$

16. Show that the equation $\sin z=0$ has solutions in the complex z-plane only where $z=n\pi$ and $n=0,\pm1,\pm2,\ldots$. Thus like $\cos z$, $\sin z$ has zeros only on the real axis.
17. Show that $\sin z-\cos z=0$ has solutions only for real values of z. What are the solutions?

Where in the complex plane do each of the following functions fail to be analytic?

18. $\tan z$ **19.** $\frac{1}{\cos(iz)}$ **20.** $\frac{1}{\sin z\sin[(1+i)z]}$ **21.** $\frac{1}{\sqrt{3}\sin z-\cos z}$

22. Let $f(z)=\sin\left(\frac{1}{z}\right)$.
a) Express this function in the form $u(x,y)+iv(x,y)$. Where in the complex plane is this function analytic?
b) What is the derivative of $f(z)$? Where in the complex plane is $f'(z)$ analytic?
23. Using MATLAB, obtain a three-dimensional plot like Fig. 3.2–2 for $|\sin z|$. Verify that the plot shows that $\sin z=0$ for $z=n\pi$, $n=0,\pm1,\pm2,\ldots$, as is proven in Exercise 16.
24. Using MATLAB, obtain a three-dimensional plot like Fig. 3.2–2 for the real and imaginary parts of $\cos z$. Verify that plots satisfy $\text{Re}(\cos z)=0$ and $\text{Im}(\cos z)=0$ for $z=\pm\pi/2,\pm3\pi/2,\ldots$, as Example 2 demonstrates.
25. Show that $|\cos z|=\sqrt{\sinh^2 y+\cos^2 x}$.
Hint: Recall that $\cosh^2\theta-\sinh^2\theta=1$.
26. Show that $|\sin z|=\sqrt{\sinh^2 y+\sin^2 x}$.
27. Show that $|\sin z|^2+|\cos z|^2=\sinh^2 y+\cosh^2 y$.
28. Show that

$$\tan z=\frac{\sin(2x)+i\sinh(2y)}{\cos(2x)+\cosh(2y)}.$$

(3.2-13)

Now $\operatorname{Log}|-e| = \operatorname{Log} e = 1$ while the principal argument of $-e$ (a negative real number) is π. Thus $\arg(-e) = \pi + 2k\pi$. Therefore,

$$\log e^{1+3\pi i} = 1 + i(\pi + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots.$$

The choice $k = 1$ will yield $\log e^{1+3\pi i} = 1 + 3\pi i$. However, the principal value of $\log e^{1+3\pi i}$ is obtained with $k = 0$ and yields $\operatorname{Log} e^{1+3\pi i} = 1 + \pi i$. ●

EXERCISES 3.4

Find all values of the logarithm of each of the following numbers and state in each case the principal value. Put answers in the form $a + ib$.

1. e **2.** $1 - i$ **3.** $-ie^2$ **4.** $-\sqrt{3} + i$ **5.** e^i **6.** e^{1+4i} **7.** $(-\sqrt{3} + i)^4$
8. $e^{\log(i \sinh 1)}$ **9.** e^{e^i} **10.** $\operatorname{Log}(\operatorname{Log} i)$

11. For what values of z is the equation $\operatorname{Log} z = \overline{\operatorname{Log} \bar{z}}$ true?

Give solutions to the following equations in Cartesian form.

12. $\operatorname{Log} z = 1 + i$ **13.** $(\operatorname{Log} z)^2 + \operatorname{Log} z = -1$

Use logarithms to find all solutions of the following equations.

14. $e^z = e$ **15.** $e^z = e^{-z}$ **16.** $e^z = e^{iz}$ **17.** $(e^z - 1)^2 = e^{2z}$
18. $(e^z - 1)^2 = e^z$ **19.** $(e^z - 1)^3 = 1$ **20.** $e^{4z} + e^{2z} + 1 = 0$ **21.** $e^{e^z} = 1$

22. Is the set of values of $\log i^2$ the same as the set of values of $2 \log i$? Explain.

Prove that if θ is real, then

23. $\operatorname{Re}[\log(1 + e^{i\theta})] = \operatorname{Log}\left|2 \cos \theta \left(\frac{\theta}{2}\right)\right|$ if $e^{i\theta} \neq -1$

24. $\operatorname{Re}[\log(re^{i\theta} - 1)] = \frac{1}{2}\operatorname{Log}(1 - 2r\cos + r^2)$ if $r \geq 0$ and $re^{i\theta} \neq 1$.

25. **a)** Consider the identity $\log z_1 + \log z_2 = \log(z_1 z_2)$. If $z_1 = -ie$ and $z_2 = -2$, find specific values for $\log z_1$, $\log z_2$, and $\log(z_1 z_2)$ that satisfy the identity.

b) For z_1 and z_2 given in part (a) find specific values of $\log z_1$, $\log z_2$, and $\log(z_1/z_2)$ so that the identity $\log(z_1/z_2) = \log z_1 - \log z_2$ is satisfied.

26. Consider the identity $\log z^n = n \log z$, where n is an integer, which is valid for appropriate choices of the logarithms on each side of the equation. Let $z = 1 + i$ and $n = 5$.

a) Find values of $\log z^n$ and $\log z$ that satisfy $n \log z = \log z^n$.

b) For the given z and n is $n \operatorname{Log} z = \operatorname{Log} z^n$ satisfied?

c) Suppose $n = 2$ and z is unchanged. Is $n \operatorname{Log} z = \operatorname{Log} z^n$ then satisfied?

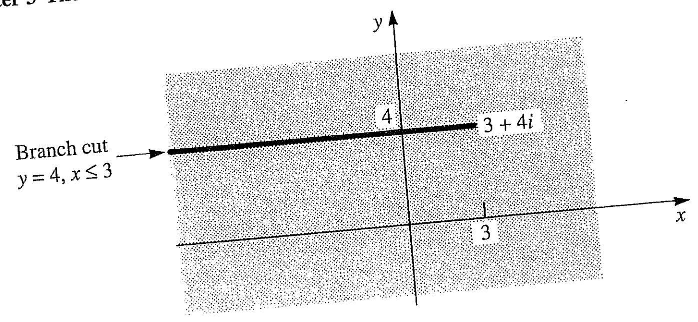

Figure 3.5–7

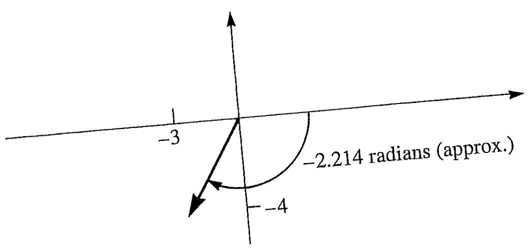

Figure 3.5–8

Part (b): $f(0) = \text{Log}(-3-4i) = \text{Log}\,5 + i\arg(-3-4i)$. Since we are dealing with the principal branch, we require in the domain of analyticity that $-\pi < \arg(-3-4i) < \pi$. From Fig. 3.5–8 we find this value of $\arg(-3-4i)$ to be approximately -2.214. Thus $f(0) \doteq \text{Log}\,5 - i2.214$.

EXERCISES

1. Use $\text{Log}\,z = (1/2)\,\text{Log}(x^2+y^2) + i\arg z$, where $\arg z = \tan^{-1}(y/x)$ or, where appropriate $(x=0)$, $\arg z = \pi/2 - \tan^{-1}(x/y)$, and Eq. (2.3–6) or (2.3–8) to show that $d(\text{Log}\,z)/dz = 1/z$ in the domain of Fig. 3.5–3. The inverse functions are here evaluated so that $\arg z$ is the principal value.

2. Suppose that

$$f(z) = \log z = \text{Log}\,r + i\theta, \quad 0 \le \theta < 2\pi.$$

a) Find the largest domain of analyticity of this function.

b) Find the numerical value of $f(-e^2)$.

c) Explain why we cannot determine $f(e^2)$ within the domain of analyticity.

Consider a branch of $\log z$ analytic in the domain created with the branch cut $x = 0,\ y \ge 0$. If for this branch, $\log(-1) = -i\pi$, find the following

3. $\log 1$ **4.** $\log(-ie)$ **5.** $\log(-e+ie)$ **6.** $\log(-\sqrt{3}+i)$ **7.** $\log(\text{cis}(3\pi/4))$

Consider a branch of $\log z$ analytic in the domain created with the branch cut $x = -y$, $x \geq 0$. If, for this branch, $\log 1 = -2\pi i$, find the following.

8. $\log i$ **9.** $\log(\sqrt{3} + i)$ **10.** $\log(-ie)$

11. Consider the function $f(z) = \text{Log}(z - i)$.

a) Describe the branch cut that must be used to create the largest domain of analyticity for this function.

b) Find the numerical value of $f(-i)$.

c) Explain why $g(z) = [\text{Log}(z - i)]/(z - 2i)$ has a singularity in the domain found in part (a), but $h(z) = [\text{Log}(z - i)]/(z + 2 - i)$ is analytic throughout the domain.

12. a) Show that $-\text{Log}\, z = \text{Log}(1/z)$ is valid throughout the domain of analyticity of $\text{Log}\, z$.

b) Find a nonprincipal branch of $\log z$ such that $-\log z = \log(1/z)$ is not satisfied somewhere in your domain of analyticity of $\log z$. Prove your result.

13. Show that $\text{Log}[(z - 1)/z]$ is analytic throughout the domain consisting of the z-plane with the line $y = 0$, $0 \leq x \leq 1$ removed. Thus a branch cut is not always infinitely long.

14. Show that $f(z) = \text{Log}(z^2 + 1)$ is analytic in the domain shown in Fig. 3.5–9.

Hint: Points satisfying $\text{Re}(z^2 + 1) \leq 0$ and $\text{Im}(z^2 + 1) = 0$ must not appear in the domain of analyticity. This requires a branch cut (or cuts) described by $\text{Re}((x + iy)^2 + 1) \leq 0$, $\text{Im}((x + iy)^2 + 1) = 0$. Find the locus that satisfies both these equations.

15. a) Show that $\text{Log}(\text{Log}\, z)$ is analytic in the domain consisting of the z-plane with a branch cut along the line $y = 0$, $x \leq 1$ (see Fig. 3.5–10).

Hint: Where will the inner function, $\text{Log}\, z$, be analytic? What restrictions must be placed on $\text{Log}\, z$ to render the outer logarithm an analytic function?

b) Find $d(\text{Log}(\text{Log}\, z))/dz$ within the domain of analyticity found in part (a).

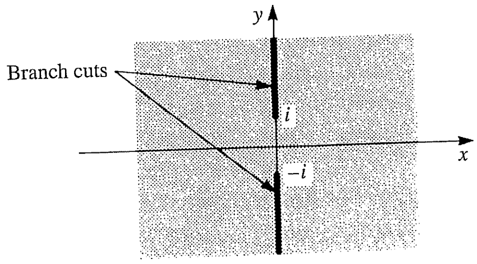

Figure 3.5–9

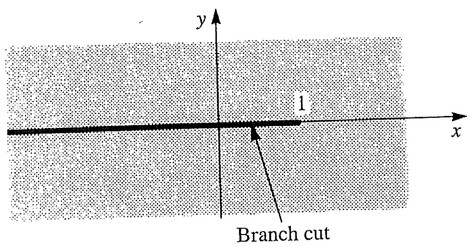

Figure 3.5–10

EXERCISES 3.6

Find all values of the following in the form $a+ib$ and state the principal value. It should be possible to check the principal value by means of a computer equipped with a numerical software package like MATLAB.

1. 1^{2i} **2.** i^{-1} **3.** $(\sqrt{3}+i)^{1-2i}$ **4.** $(e^i)^i$ **5.** $e^{(e^i)}$ **6.** $(1.1)^{1.1}$

7. $\pi^{i/2}$ **8.** $(\operatorname{Log} i)^{\pi/2}$ **9.** $(1+i\tan 1)^{\sqrt{2}}$ **10.** $(\sqrt{2})^{1+i\tan 1}$

11. Show that all possible values of z^i are real if $|z| = e^{n\pi}$, where n is any integer.

Using Eq. (3.1–5) or Eq. (3.1–7) and the definition in Eq. (3.6–1), prove that for any complex values α, β, and z, we have the following.

12. The values of $1/z^{\beta}$ are identical to the values of $z^{-\beta}$.

13. The values of $z^{\alpha}z^{\beta}$ are identical to the values of $z^{\alpha+\beta}$.

14. Use Eq. (3.6–1) to show that

a) if n is an integer, then z^n has only one value and it is the same as the one given by Eq. (1.4–2);

b) if n and m are integers and n/m is an irreducible fraction, then $z^{n/m}$ has just m values and they are identical to those given by Eq. (1.4–13);

c) if c is an irrational number, then z^c has an infinity of different values;

d) if c is complex with $\operatorname{Im} c \neq 0$, then z^c has an infinity of different values.

15. The following puzzle appeared without attribution in the Spring 1989 *Newsletter of the Northeastern Section of the Mathematical Association of America*. What is the flaw in this argument?

Euler's identity?

$$e^{i\theta} = (e^{i\theta})^{2\pi/2\pi} = (e^{2\pi i})^{\theta/2\pi} = (1)^{\theta/2\pi} = 1.$$

Using the principal branch of the function, evaluate the following.

16. $f'(i)$ if $f(z) = z^{2+i}$ **17.** $f'(-128i)$ if $f(z) = z^{8/7}$

18. $f'(-8i)$ if $f(z) = z^{1/3+i}$

Let $f(z) = z^z$, where the principal branch is used. Evaluate the following.

19. $f'(z)$ **20.** $f'(i)$

21. Let $f(z) = z^{\sin z}$, where the principal branch is used. Find $f'(i)$.

22. Find $(d/dz)2^{\cosh z}$ using principal values. Where in the complex z-plane is $2^{\cosh z}$ analytic?

23. Find $f'(i)$ if $f(z) = i^{(e^z)}$ and principal values are used.

24. Let $f(z) = 10^{(z^3)}$. This function is evaluated such that $f'(z)$ is real when $z = 1$. Find $f'(1+i)$. Where in the complex plane is $f(z)$ analytic?

25. Let $f(z)$
$f'(i\pi/2$
26. a) Let
1, 2, .
gener
each
b) The p
Greg
431–4
roxim
plotted
just de
on this
Investi
journal
c) Explain
Does th
27. a) Consid
use the
functio
on the n
Hint: P
crossed
used in
cut. Sinc
on and o
is also a
b) Assume
log. Expl
and diffe
is undefin
function,

3.7 INVERS

If we know the
means of the ide
is the inverse of
Suppose we
find w and whet
Let $z = \sin$
complex numbe

Now with $p = e$

Now proceeding to $z = -1$ along the path indicated in Fig. 3.8–7, we find that $\theta = -\pi$, $r = |z| = 1$, θ_1 is, after some variation, again π, $r_1 = |z - 1| = |-2| = 2$. Using these values in Eq. (3.8–13) together with $k + m = 0$, we obtain

$$f(-1) = \sqrt{1}\,\underline{/-\frac{\pi}{2} + k\pi}\;\sqrt{2}\,\underline{/\frac{\pi}{2} + m\pi} = \sqrt{2}e^{i(k+m)\pi} = \sqrt{2}.$$

Note that the path taken from 1/2 to −1 in Fig. 3.8–7 remains within the domain of analyticity. ●

EXERCISES 3.8

A certain branch of $z^{1/2}$ is defined by means of the branch cut $x = 0$, $y \geq 0$. If this branch has the value -3 when $z = 9$, what values does $f(z)$ assume at the following points? Also, state the numerical value of $f'(z)$ at each point.

1. 1 **2.** $-9i$ **3.** $-1 + i$ **4.** $-9 + 9i\sqrt{3}$

A branch of $(z - 1)^{2/3}$ is defined by means of the branch cut $x = 1$, $y \leq 0$. If this branch $f(z)$ equals 1 when $z = 0$, what is the value of $f(z)$ and $f'(z)$ at the following points?

5. $1 + 8i$ **6.** -1 **7.** $-i$ **8.** $1/2 - i/2$

9. A branch of $(z^2 - 1)^{1/2}$ is defined by means of a branch cut consisting of the line segment $-1 \leq x \leq 1$, $y = 0$.
   a) Prove that this function has branch points at $z = \pm 1$.
   b) Show that if we encircle these branch points by moving once around the ellipse $x^2/2 + y^2 = 1$, we do not pass to a new branch of the function. Present an argument like that in Example 3, part (b).

10. Consider the multivalued function $z^{1/3}(z - 1)^{1/3}$.
   a) Show, by encircling each of them, that this function has branch points at $z = 0$ and $z = 1$.
   b) Show that, unlike Example 3, the line segment $y = 0$, $0 \leq x \leq 1$, which connects the two branch points, cannot serve as a branch cut for defining a branch of this function.
   c) State suitable branch cuts for defining a branch.

Suppose a branch of $(z^2 - 1)^{1/3}$ equals -1 when $z = 0$. There are branch cuts defined by $y = 0$, $|x| \geq 1$. What value does this branch assume at the following points?

11. i **12.** $-i$ **13.** $1 + i$

14. If two functions each have a branch point at z_0, does their product necessarily have a branch point at z_0? Illustrate with an example.

15. If $f(z)$ has a branch point at z_0, does $1/f(z)$ necessarily have a branch point at z_0? Explain.