In part (a) we could have integrated on y instead of on x by a similar maneuver. Note that in Exercise 1 of this section the integral $\int xy\, ds$ along the same contour is evaluated. ●

EXERCISES 4.1

1. Using the contour of Example 1, show that

$$\int_{0,1}^{1,0} xy\, ds = \int_0^1 x(1-x^2)\sqrt{1+4x^2}\, dx = \int_0^1 y\sqrt{5/4-y}\, dy.$$

Hint: Recall from elementary calculus that $ds = (\pm)\sqrt{1+(dy/dx)^2}\, dx = (\pm)\sqrt{1+(dx/dy)^2}\, dy$, and that $ds \geq 0$. Evaluate the contour integral by integrating either on x or y. One is slightly easier.

Let C be that portion of the curve $y = x^2$ lying between (0, 0) and (1, 1). Let $F(x, y) = x + y + 1$. Evaluate these integrals along C.

2. $\displaystyle\int_{0,0}^{1,1} F(x, y)dx$ 3. $\displaystyle\int_{0,0}^{1,1} F(x, y)dy$

Let C be that portion of the curve $x^2 + y^2 = 1$ lying in the first quadrant. Let $F(x, y) = x^2y$. Evaluate these integrals along C.

4. $\displaystyle\int_{0,1}^{1,0} F(x, y)dx$ 5. $\displaystyle\int_{0,1}^{1,0} F(x, y)dy$ 6. $\displaystyle\int_{0,1}^{1,0} F(x, y)ds$

7. Show that $\int_{0,-1}^{0,1} y\, dx = -\pi/2$. The integration is along that portion of the circle $x^2 + y^2 = 1$ lying in the half plane $x \geq 0$. Be sure to consider signs in taking square roots.

8. Evaluate $\int_{3,0}^{0,-1} x\, dy$ along the portion of the ellipse $x^2 + 9y^2 = 9$ lying in the first, second, and third quadrants.

4.2 COMPLEX LINE INTEGRATION

We now study the kind of integral encountered most often with complex functions: the complex line integral. We will find that it is closely related to the real line integrals just discussed.

We begin, as before, with a smooth arc that connects the points A and B in the xy-plane. We now regard the xy-plane as being the complex z-plane. The arc is divided into n smaller arcs and, as shown in Fig. 4.2–1, successive endpoints of the subarcs have coordinates $(X_0, Y_0), (X_1, Y_1), \ldots, (X_n, Y_n)$. Alternatively, we could say that the endpoints of these smaller arcs are at $z_0 = X_0 + iY_0$, $z_1 = X_1 + iY_1$, etc. A series of vector chords are then constructed between these points. As in our discussion of real line integrals, the vectors progress from A to B when we are integrating from A to B along the contour. Let Δz_1 be the complex number corresponding to the vector

The length L of the path of integration is simply the circumference of the given quarter circle, namely, $\pi/2$. Thus, applying the ML inequality,

$$\left|\int_{1+i0}^{0+i1} e^{1/z}\,dz\right| \le e\frac{\pi}{2}. \qquad \bullet$$

EXERCISES 4.2

1. In Example 1 we determined the approximate value of $\int_{0+i0}^{1+i}(z+1)dz$ taken along the contour $y = x^2$. Find the exact value of the integral and compare it with the approximate result.
2. Consider $\int_{0+i0}^{1+2i} z\,dz$ performed along the contour $y = 2x(2-x)$. Find the approximate value by means of the two-term series $f(z_1)\Delta z_1 + f(z_2)\Delta z_2$. Take $z_1, z_2, \Delta z_1, \Delta z_2$ as shown in Fig. 4.2–6. Now find the exact value of the integral and compare it with the approximate result.
3. Consider $\int_{0+i0}^{1+2i} dz$ along the contour of Exercise 2. Evaluate this by using a two-term series approximation as is done in that problem. Explain why this result agrees perfectly with the exact value of the integral.

Evaluate $\int_i^1 \bar{z}\,dz$ along the contour C, where C is

4. the straight line segment lying along $x + y = 1$;
5. the parabola $y = (1-x)^2$;
6. the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Compare the answers to Exercises 4, 5, and 6.

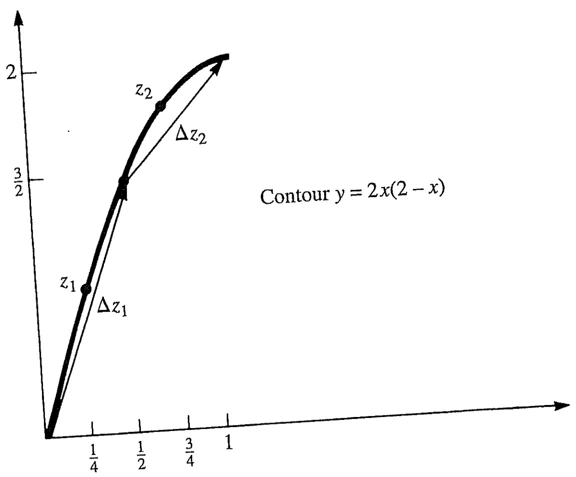


Figure 4.2–6

7. Evaluate $\int e^z\,dz$
a) from $z = 0$ to $z = 1$ along the line $y = 0$;
b) from $z = 1$ to $z = 1 + i$ along the line $x = 1$;
c) from $z = 1 + i$ to $z = 0$ along the line $y = x$. Verify that the sum of your three answers is zero. This is a specific example of a general result given in the next section.

The function $z(t) = e^{it} = \cos t + i \sin t$ can provide a useful parametric representation of circular arcs (see Fig.3.1–1). If t ranges from 0 to 2π we have a representation of the whole unit circle, while if t goes from α to β we generate an arc extending from $e^{i\alpha}$ to $e^{i\beta}$ on the unit circle. Use this parametric technique to perform the following integrations.

8. $\displaystyle\int_1^{-1} \frac{1}{z}\,dz$ along $|z| = 1$, upper half plane

9. $\displaystyle\int_1^{-1} \frac{1}{z}\,dz$ along $|z| = 1$, lower half plane

10. $\displaystyle\int_1^{i} \bar{z}^4\,dz$ along $|z| = 1$, first quadrant

11. Show that $x = 2\cos t$, $y = \sin t$, where t ranges from 0 to 2π, yields a parametric representation of the ellipse $x^2/4 + y^2 = 1$. Use this representation to evaluate $\int_2^i \bar{z}\,dz$ along the portion of the ellipse in the first quadrant.

12. In Example 3 we evaluated $\int_{1+i}^{2+4i} z^2\,dz$ along the parabola $y = x^2$ by means of the parametric representation $x = \sqrt{t}$, $y = t$. Show that the representation $x = t$, $y = t^2$ can also be used, and perform the integration using this parametrization.

13. a) Find a parametric representation of the shorter of the two arcs lying along $(x-1)^2 + (y-1)^2 = 1$ that connects $z = 1$ with $z = i$.
Hint: See discussion preceding Exercises 8–10 above, where parametrization of a circle is discussed.
b) Find $\int_1^i \bar{z}\,dz$ along the arc of (a), using the parametrization you have found.

14. Consider $I = \int_{0+i0}^{2+i} e^{z^2}\,dz$ taken along the line $x = 2y$. Without actually doing the integration, show that $|I| \le \sqrt{5}e^3$.

15. Consider $I = \int_1^i (1/\bar{z}^4)dz$ taken along the line $x + y = 1$. Without actually doing the integration, show that $|I| \le 4\sqrt{2}$.

16. Consider $I = \int_i^1 e^{i\,\mathrm{Log}\,\bar{z}}\,dz$ taken along the parabola $y = 1 - x^2$. Without doing the integration, show that $|I| \le 1.479e^{\pi/2}$.

17. a) Let $g(t)$ be a complex function of the real variable t.
Express $\int_a^b g(t)dt$ as the limit of a sum. Using an argument similar to the one used in deriving Eq. (4.2–14), show that for $b > a$ we have

$$\left|\int_a^b g(t)dt\right| \le \int_a^b |g(t)|dt. \tag{4.2–18}$$

b) Use Eq. (4.2–18) to prove that

$$\left|\int_0^1 \sqrt{t}\, e^{it}\, dt\right| \le \frac{2}{3}.$$

18. Write a MATLAB program that will enable you to verify the entries in the table in Example 1; i.e., write a program that will yield approximations to $\int_{0+i0}^{1+i} (z+1)dz$ along the contour $y = x^2$ as shown in Fig. 4.2–2. Show also that if you used a 50-term approximation to the integral the result would be $1.00010 + i1.99990$.

4.3 Contour Integration and Green's Theorem

In the preceding section, we discussed piecewise smooth curves, called contours, that connect two points A and B. If these two points happen to coincide, the resulting curve is called a *closed contour*.

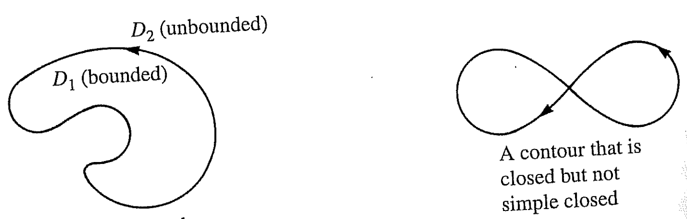
DEFINITION (Simple Closed Contour) A *simple closed contour* is a contour that creates two domains, a bounded one and an unbounded one; each domain has the contour for its boundary. The bounded domain is said to be the *interior* of the contour. •

Examples of two different closed contours, one of which is simple, are shown in Fig. 4.3–1.

A simple closed contour is also known as a *Jordan contour*, named after the French mathematician Camille Jordan (1838–1922). That a piecewise smooth curve forming a simple loop, as in Fig. 4.3–1(a), always creates a bounded domain (inside the loop) and an unbounded domain (outside the loop) seems self-evident but it is not obvious to a pure mathematician. The proof is difficult and was first presented in 1905 by an American, Oswald Veblen. The resulting theorem is named after Jordan, who proposed the hypothesis.

We will often be concerned with line integrals taken around a simple closed contour.

The integration is said to be performed in the *positive sense* around the contour if the interior of the contour is on our left as we move along the contour in the direction of integration.



A simple closed contour
(a)

A contour that is closed but not simple closed
(b)

Figure 4.3–1