EXERCISES 4.3

1. a) Let C be an arbitrary simple closed contour. Use Green's theorem to find a simple interpretation of the line integral $(1/2)\oint_C (-y\,dx + x\,dy)$.

b) Consider $\oint [\cos y\,dx + \sin x\,dy]$ performed around the square with corners at $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Evaluate this integral by doing an equivalent integral over the area enclosed by the square.

c) Suppose you know the area enclosed by a simple closed contour C. Show with the aid of Green's theorem that you can easily evaluate $\oint \bar{z}\,dz$ around C.

To which of the following integrals is the Cauchy–Goursat theorem directly applicable?

2. $\displaystyle\oint_{|z|=1} \frac{\sin z}{z+2i}\,dz$ **3.** $\displaystyle\oint_{|z+3i|=1} \frac{\sin z}{z+2i}\,dz$ **4.** $\displaystyle\oint_{|z-3i|=6} e^{\bar{z}}\,dz$

5. $\displaystyle\oint_{|z+i|=1} \operatorname{Log} z\,dz$ **6.** $\displaystyle\oint_{|z-1-i|=1} \operatorname{Log} z\,dz$

7. $\displaystyle\oint_{|z|=1/2} \frac{1}{(z-1)^4+1}\,dz$ **8.** $\displaystyle\oint_{|z|=3} \frac{dz}{1-e^z}$

9. $\displaystyle\oint_{|z|=b} \frac{dz}{z^2+bz+1},\quad 0 < b < 1$ **10.** $\displaystyle\int_0^{1+i} z^3\,dz$ along $y = x$

11. In the discussion of Green's theorem in the appendix to this chapter, it is shown that if $P(x, y)$ and $Q(x, y)$ are a pair of functions with continuous partial derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ inside some simply connected domain D and if $\oint_C P\,dx + Q\,dy = 0$ for every simple closed contour in D, then $\partial Q/\partial x = \partial P/\partial y$ in D.

Let $f(z) = u(x, y) + iv(x, y)$ be a function such that the first partial derivatives of u and v are continuous in a simply connected domain D. Given that $\oint f(z)dz = 0$ for every simple closed contour in D, use the preceding result to show that $f(z)$ must be analytic in D.

This is a converse of the Cauchy–Goursat theorem. There is another derivation that eliminates the requirement that the partial derivatives be continuous in D. Only $u(x, y)$ and $v(x, y)$ are assumed continuous. The resulting converse of the Cauchy–Goursat theorem is known as *Morera's theorem.*

Prove the following results by means of Cauchy–Goursat theorem. Begin with $\oint e^z\,dz$ performed around $|z| = 1$. Use the parametric representation $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Separate your equation into real and imaginary parts.

12. $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + \theta)]\,d\theta = 0$ **13.** $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + \theta)]\,d\theta = 0$

Prove that the following identities hold for any integer $n \ge 0$.
Hint: Use the contour and technique of the preceding problem but a different integrand.

14. $\displaystyle\int_0^{2\pi} e^{\sin n\theta}\cos(\theta - \cos n\theta)\,d\theta = 0$ **15.** $\displaystyle\int_0^{2\pi} e^{\sin n\theta}\sin(\theta - \cos n\theta)\,d\theta = 0$

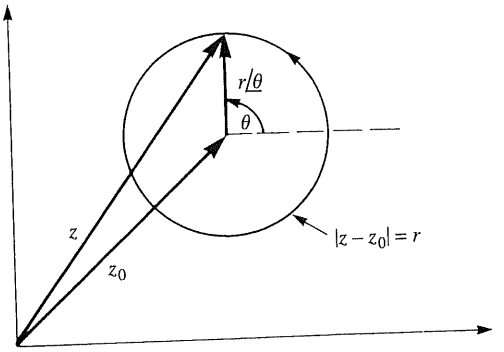

Figure 4.3–8

16. Show that for real a, where $|a| > 1$, we have $\int_0^{2\pi} \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}\,d\theta = 0$.

Hint: Consider $\oint \frac{1}{z-a}\,dz$, where the integral is taken around the unit circle. Represent the circle parametrically as in the previous four problems.

17. Let n be any integer, r a positive real number, and z_0 a complex constant. Show that for $r > 0$,

$$\oint_{|z - z_0| = r} (z - z_0)^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases}$$

Hint: Refer to the derivation of Eq. (4.3–9) and follow a similar procedure. Consider the change of variable $z = z_0 + re^{i\theta}$ indicated in Fig. 4.3–8.

Evaluate the following integrals. The contour is the square centered at the origin with corners at $\pm(2 \pm 2i)$. The result contained in the previous problem as well as the principle of deformation of contours will be useful.

18. $\oint \frac{dz}{z-i}$ **19.** $\oint \frac{dz}{(z-i)^4}$ **20.** $\oint \frac{z\,dz}{z-i}$

21. $\oint \frac{(z+1)^m\,dz}{z^m}$, $m \geq 0$ is an integer (*Hint:* Use the binomial theorem.)

22. $\oint \frac{z^m\,dz}{(z-1)^m}$, $m \geq 0$ is an integer (*Hint:* See previous theorem.)

23. Show that

$$\oint_{|z-3|=2} \frac{\operatorname{Log} z}{(z+1)(z-3)}\,dz = \oint_{|z-3|=2} \frac{\operatorname{Log} z}{4(z-3)}\,dz.$$

Hint: Write $1/[(z+1)(z-3)]$ as a sum of partial fractions.

24. Consider the n-tuply connected domain D whose nonoverlapping boundaries are the simple closed contours $C_0, C_1, \ldots, C_{n-1}$ as shown in Fig. 4.3–9.† Let $f(z)$ be a function

†An n-tuply connected domain has $n - 1$ holes. See section 1.5.

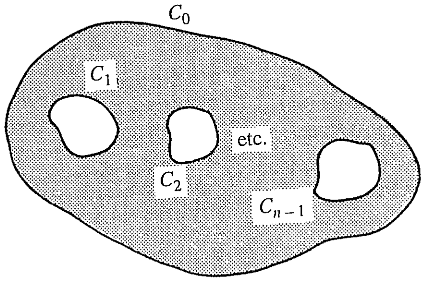

Figure 4.3–9

that is analytic in D and on its boundaries. Show that

$$\oint_{C_0} f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_{n-1}} f(z)\,dz.$$

Hint: Consider the derivation of the principle of deformation of contours. Make a set of cuts similar to those made in Fig. 4.3–4 in order to link up the boundaries.

25. Use the result derived in Exercise 24 to show

$$\oint_{|z|=2} \frac{\sin z}{(z^2-1)}\,dz = \oint_{|z-1|=1/2} \frac{\sin z}{(z^2-1)}\,dz + \oint_{|z+1|=1/2} \frac{\sin z}{(z^2-1)}\,dz.$$

4.4 Path Independence, Indefinite Integrals, Fundamental Theorem of Calculus in the Complex Plane

The Cauchy–Goursat theorem is a useful tool when we must integrate an analytic function around a closed contour. When the contour is not closed, there exist techniques, derivable from this theorem, that can assist us in evaluating the integral. For example, we can prove the following.

THEOREM 5 (Principle of Path Independence) Let $f(z)$ be a function that is analytic throughout a simply connected domain D, and let z_1 and z_2 lie in D. Then if we use contours lying in D, the value of $\int_{z_1}^{z_2} f(z)\,dz$ will not depend on the particular contour used to connect z_1 and z_2.

The preceding theorem is sometimes known as the *principle of path independence*. It is really just a restatement of the Cauchy–Goursat theorem. To establish this principle, we will consider two nonintersecting contours C_1 and C_2, each of which connects z_1 and z_2. Each contour is assumed to lie within the simply connected domain D in which $f(z)$ is analytic. We will show that

$$\int_{\substack{z_1 \\ \text{along } C_1}}^{z_2} f(z)\,dz = \int_{\substack{z_1 \\ \text{along } C_2}}^{z_2} f(z)\,dz. \qquad (4.4\text{–}1)$$

4.4

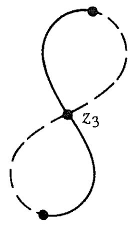

Figure 4.4–5

in a simply connected domain containing C_1 and C_2. Show that

$$\int_{\substack{z_1 \\ \text{along } C_1}}^{z_2} f(z)dz = \int_{\substack{z_1 \\ \text{along } C_2}}^{z_2} f(z)dz.$$

Use Theorem 6 to evaluate the following integrals along the curve $y = \sqrt{x}$.

2. $\displaystyle\int_0^{4+2i} e^{iz}\,dz$ **3.** $\displaystyle\int_0^{4+2i} 1 + z^2\,dz$ **4.** $\displaystyle\int_{1+i}^{4+2i} z + z^{-2}\,dz$

5. $\displaystyle\int_0^{4+2i} e^z \sinh z\,dz$ **6.** $\displaystyle\int_0^{4+2i} e^z \cosh e^z\,dz$ **7.** $\displaystyle\int_{1+i}^{4+2i} \frac{z}{z^2-1}\,dz$

8. a) What, if anything, is incorrect about the following two integrations? The integrals are both along the line $y = x$.

$$\int_{0+i0}^{1+i} z\,dz = \frac{z^2}{2}\bigg|_{0+i0}^{1+i} = \frac{(1+i)^2}{2} = i,$$

$$\int_{0+i0}^{1+i} \bar{z}\,dz = \frac{\bar{z}^2}{2}\bigg|_{0+i0}^{1+i} = \frac{(1-i)^2}{2} = -i.$$

b) What is the correct numerical value of each of the above integrals?

9. Find the value of $\int_e^i \operatorname{Log} z\,dz$ taken along the line connecting $z = e$ with $z = i$. Why is it necessary to specify the contour?

10. Find $\int_{1+i}^{-1-i} \frac{\operatorname{Log} z}{z}\,dz$, where the integral is along a contour not intersecting the branch cut for Log z.

11. Find $\int_1^i z^{1/2}\,dz$. The principal branch of $z^{1/2}$ is used. The contour does not pass through any point satisfying $y = 0$, $x \le 0$.

12. Find $\int_1^i z^{1/2}\,dz$. The branch of $z^{1/2}$ used equals -1 when $z = 1$. The branch cut lies along $y = 0$, $x \le 0$, and the contour does not pass through the branch cut.

13. Find $\int_1^i i^z\,dz$. Use principal values. Why is it not necessary to specify the contour?

14. Perform the integration $\int_0^i \cos z \cosh z\,dz$ by the two methods described below and check that they produce identical results. Why can the contour be left unspecified?

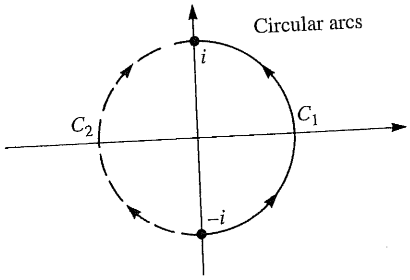

Figure 4.4–6

a) Represent $\cos z$ and $\cosh z$ by exponential functions of z. Perform the multiplication of the resulting expressions and integrate the exponentials.

b) The MATLAB Symbolic Math Toolbox can do many real symbolic integrals. Using this feature, find the function whose derivative is the real function $\cos x \cosh x$ and exploit this result to evaluate the given integral.

15. Consider contours C_1 and C_2 shown in Fig. 4.4–6. We can use the result derived in Example 2 of this section to show that along C_1 we have $\int_{-i}^{i} 1/z\, dz = \pi i$.

a) Explain why we cannot employ the principle of path independence to show that along C_2 we must have $\int_{-i}^{i} 1/z\, dz = \pi i$.

b) Find the correct value of the integral along C_2 by employing a branch of $\log z$ that is analytic in a simply connected domain containing the path of integration.

c) Check the answer to part (b) by switching to the parametric representation of the contour of integration with $z = e^{i\theta}$. Integrate on the variable θ (see Example 1).

16. Do the following problem by employing Theorem 6.

a) Find $\int_0^{2i} dz/(z-i)$ taken along the arc satisfying $|z-i| = 1$, $\operatorname{Re} z \geq 0$.

b) Repeat part (a) with the same limits, but use the arc $|z-i| = 1$, $\operatorname{Re} z \leq 0$.

17. Let z_1 and z_2 be a pair of arbitrary points in the complex plane. Contours C_1 and C_2 each connect points z_1 and z_2. The contours do not otherwise intersect, and neither passes through $z = 0$. Explain why

$$\underset{\text{along } C_1}{\int_{z_1}^{z_2}} \frac{1}{z^2}\, dz = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} \frac{1}{z^2}\, dz.$$

Consider two cases:

a) $z = 0$ does not belong to the domain whose boundaries are C_1 and C_2 (see Fig. 4.4–7).

b) $z = 0$ does belong to the domain whose boundaries are C_1 and C_2 (see Fig. 4.4–8).

18. In elementary calculus the reader learned the *Mean Value Theorem*: If $f(x)$ is continuous for $a \leq x \leq b$, then there exists a number x_1, where $a < x_1 < b$, such that

$$\int_a^b f(x)dx = f(x_1)(b-a).$$

Show that this theorem does not have a counterpart for complex line integrals by doing the following:

a) Show that $\int_1^i (1/z^2)dz = 1 + i$, where the integral is along $x + y = 1$.