become zero at the point $z = 5$ inside C. We therefore rewrite the integral as

$$\oint_C \frac{\left(\frac{\cos z}{(z-1)^3}\right)}{(z-5)^2}\, dz$$

and apply Eq. (4.5–13) with $n = 1$, $z_0 = 5$, $f(z) = \cos z/(z-1)^3$. Thus

$$\frac{1}{2\pi i}\oint_C \frac{\left(\frac{\cos z}{(z-1)^3}\right)}{(z-5)^2}\, dz = \frac{d}{dz}\frac{\cos z}{(z-1)^3}\bigg|_{z=5} = \frac{-64\sin 5 - 48\cos 5}{4^6}.$$

The value of the given integral is $2\pi i$ times the preceding result.

EXERCISES 4.5

1. To arrive at a formal (nonrigorous) derivation of the Cauchy integral formula, let $f(z)$ be analytic on and inside a simple closed contour C, let z_0 lie inside C, and let C_0 be a circle centered at z_0 and lying completely inside C. From the principle of deformation of contours, we then have

$$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint_{C_0} \frac{f(z)}{z-z_0}\, dz.$$

a) Rewrite the integral on the right by means of the change of variables $z = z_0 + re^{i\theta}$, where r is the radius of C_0 and θ increases from 0 to 2π (see Fig. 4.3–8). Note that $dz/d\theta = ire^{i\theta}$.

b) For the integral obtained in part (a), let $r \to 0$ in the integrand. Now perform the integration and use your result to show that

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0).$$

c) What makes this derivation nonrigorous?

Evaluate the following integrals using the Cauchy integral formula, its extension, or the Cauchy–Goursat theorem where appropriate.

2. $\oint \frac{\sin z}{z-2}\, dz$ around $|z| = 3$

3. $\oint \frac{\sin z}{z-2}\, dz$ around $|z| = 1$

4. $\oint \frac{\cosh z}{(z-3)(z-1)}\, dz$ around $|z = 2|$

5. $\frac{1}{2\pi i}\oint \frac{\cosh(e^z)}{z^2-4z+3}\, dz$ around the square with corners at $z = 2$, $z = 4$, and $z = 3 \pm i$

6. $\oint \frac{e^{iz}}{z^2+z+1}\, dz$ around $\left|z + \frac{1}{2} - 2i\right| = 2$

7. $\frac{1}{2\pi i}\oint \frac{\mathrm{Log}(z)}{z^2+9}\, dz$ around $|z - 4i| = 3$

(continued)

(continued)

8. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{e^{iz}}{(z-i)^2}\,dz$ around $|z-1|=2$

9. $\displaystyle\oint \frac{ze^z}{(z-i)^2}\,dz$ around $|z-1|=2$

10. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{1}{(z+2)(z-i)^2}\,dz$ around $|z-1|=2$

11. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{\cos z}{(z-i)^3}\,dz$ around $|z-1|=2$

12. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{\sin 2z}{z^{15}}\,dz$ around $|z|=2$

13. $\displaystyle\oint \frac{\sin 2z}{z^{16}}\,dz$ around $|z|=2$

14. A student is attempting to perform the integration $\int_{0+i0}^{1+i} \bar{z}\,dz$ along the line $y=\sqrt{\sin(\frac{\pi}{2}x)}$. He studies Theorem 6 in section 4.4 and reasons that if he can find a function $F(z)$ satisfying $dF/dz=\bar{z}$ in a domain containing the path of integration, then he can evaluate the integral as $F(1+i)-F(0+i0)$ without having to use the path of integration. Explain why this will not work.

15. a) Use the extension of the Cauchy integral formula to show that $\oint e^{az}/(z^{n+1})dz = a^n 2\pi i/n!$, where the integration is performed around $|z|=1$.

b) Rewrite the integral of part (a) using the substitution $z=e^{i\theta}\,(0\le\theta\le 2\pi)$ when z lies on the unit circle. Integrating on θ show that, when a is real, $\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta-n\theta)d\theta = 2\pi a^n/n!$, and $\int_0^{2\pi} e^{a\cos\theta}\sin(a\sin\theta-n\theta)d\theta=0$.

16. a) Consider the integral $\oint \frac{dz}{z-a}$ around $|z|=1$, where a is any constant such that $|a|\ne 1$. Using either the Cauchy–Goursat theorem or the Cauchy integral formula, whichever is appropriate, evaluate this integral for the cases $|a|>1$ and $|a|<1$.

b) Explain why the techniques just used cannot be applied to $\oint \frac{dz}{\bar{z}-a}$ around $|z|=1$. However, evaluate this integral for the two cases given above by noticing that on the unit circle we have $\bar{z}=1/z$. Are your answers different from (a)?

17. a) If a is a real number and $|a|<1$ show that

$$\int_0^{2\pi} \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}\,d\theta = 2\pi.$$

Hint: Consider $\oint dz/(z-a)$ around $|z|=1$. What is the value of this integral? Now rewrite this integral using $z=e^{i\theta}, 0\le\theta\le 2\pi$, for z on the unit circle.

Recall that in Exercise 16 of section 4.3, we evaluated the above integral for the case $|a|>1$ and found it to be zero.

18. The rigorous proof of the extended Cauchy integral formula for the first derivative requires for its completion our showing that

$$\lim_{\Delta z_0\to 0} \frac{1}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-(z_0+\Delta z_0))(z-z_0)} - \oint_C \frac{f(z)dz}{(z-z_0)^2}\right| = 0.$$

Complete the proof.

is not prime is uniquely expressible as the product of positive prime integers. The result was known to the ancient Greeks circa 300 BC.

EXERCISES 4.6

Use Gauss' mean value theorem in its various versions (see Eqs. (4.6–1) and (4.6–3)) and integrations around appropriate circles to prove the following:

1. $\frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}} = 1$ **2.** $\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$ (Do Exercise 1 first.)

3. $\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2\left(\frac{\pi}{6} + ae^{i\theta}\right)d\theta = \frac{3}{4}$, where $a > 0$

4. $\int_{-\pi}^{\pi} \frac{a + \cos n\theta}{a^2 + 1 + 2a\cos n\theta}d\theta = \frac{2\pi}{a}$, where $a > 1$, n integer $\left(\textit{Hint: } f(z) = \frac{1}{z^n + a}.\right)$

5. $\int_0^{2\pi} \mathrm{Log}\,[a^2 + 1 + 2a\cos(n\theta)]d\theta = 4\pi\,\mathrm{Log}\,a$, where $a > 1$, n integer

Hint: $a^2 + 1 + 2a\cos(n\theta) = |a + e^{in\theta}|^2$.

6. Show by direct calculation (do the integration) that the average value of the function $g(x, y) = x^2 - y^2$ on the circle $|z| = r$ is equal to the value of $g(x, y)$ at the center of the circle, for all $r > 0$. Also show that the average value of the function $h(x, y) = x^2 + y^2$ on the same circle is never equal to the value of $h(x, y)$ at the center for any $r > 0$. Perform the integrations with the usual polar substitutions $x = r\cos\theta$, $y = r\sin\theta$. Explain why these results should be so different by referring to Gauss' mean value theorem.

7. a) Let $u(x, y)$ be a harmonic function. Let u_0 be the value of u at the center of the circle, of radius r, shown in Fig. 4.6–4. The values of u at four equally spaced points on the circumference are u_1, u_2, u_3, u_4.

Note that u_1 and u_3 lie on the diameter parallel to the x axis while u_2 and u_4 lie on the diameter parallel to the y axis. Refer to Eq. (4.6–3a) and use an approximation to the integral to show that

$$u_0 \approx \frac{u_1 + u_2 + u_3 + u_4}{4}.$$

b) Use a calculator or computer to evaluate the harmonic function $e^x \cos y$ at the four points (1.1, 1), (0.9, 1), (1, 1.1), and (1, 0.9). Compare the average of these results with $e^x \cos y$ evaluated at (1, 1).

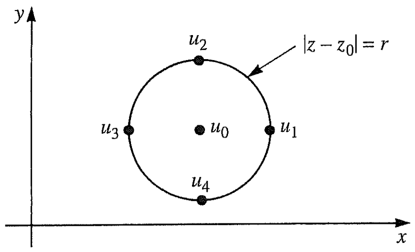

Figure 4.6–4

c) The approximation in the equation of part (a) generally improves as the radius of the circle shrinks to zero; it is perfect if $r = 0$. Using MATLAB, make a plot of the right side of this equation for the function of part (b). The radius r should vary from 0.1 to 1.0. Compare the values obtained with the value of the function at the center of the circle.

8. Let $f(z)$ be a nonconstant function that is continuous and nonzero throughout a closed bounded region R. Let $f(z)$ be analytic at every interior point of R. Show that the minimum value of $|f(z)|$ in R must occur on the boundary of R.
Hint: Consider $g(z) = 1/f(z)$ and recall the maximum modulus theorem.

For the following closed regions R and functions $f(z)$, find the values of z in R where $|f(z)|$ achieves its maximum and minimum values. If your answers do not lie on the boundary of R, give an explanation. Give the values of $|f(z)|$ at its maximum and minimum in R.

9. $f(z) = z$, R is $|z - 1 - i| \le 1$
10. $f(z) = z^2$, R is $|z - 1 - i| \le 2$
11. $f(z) = e^z$, R same as in Exercise 9
12. $f(z) = \sin z$ and R is the rectangle $1 \le y \le 2$, $0 \le x \le \pi$.
Hint: See the result for $|\sin z|$ in Exercise 26, section 3.2.

13. Let $u(x, y)$ be real, nonconstant, and continuous in a closed bounded region R. Let $u(x, y)$ be harmonic in the interior of R. Prove that the maximum value of $u(x, y)$ in this region occurs on the boundary. This is known as the *maximum principle*.
Hint: Consider $F(z) = u(x, y) + iv(x, y)$, where v is the harmonic conjugate of u. Let $f(z) = e^{F(z)}$. Explain why $|f(z)|$ has its maximum value on the boundary. How does it follow that $u(x, y)$ has its maximum value on the boundary?

14. For $u(x, y)$ described in Exercise 13 show that the minimum value of this function occurs on the boundary. This is known as the *minimum principle*.
Hint: Follow the suggestions given in Exercise 13 but show that $|f(z)|$ has its minimum value on the boundary.

15. Consider the closed bounded region R given by $0 \le x \le 1, 0 \le y \le 1$. Now $u = (x^2 - y^2)$ is harmonic in R. Find the maximum and minimum values of u in R and state where they are achieved.

16. A long cylinder of unit radius, shown in Fig. 4.6–5, is filled with a heat-conducting material. The temperature inside the cylinder is described by the harmonic function

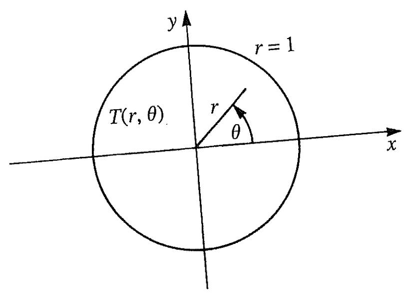

Figure 4.6–5