EXERCISES 5.2

1. From home, you walk one mile east, turn 90° and walk 1/2 mile north, then turn 90° and walk 1/4 mile west, then turn 90° and walk 1/8 mile south, then turn 90° and go 1/16 th mile east. You continue on this journey, always turning 90° counterclockwise and making each segment of your trip equal to half the length of the previous one. There are an infinite number of such segments in the journey. If you plot your spiral voyage in the complex plane you can make each segment correspond to the terms in the infinite series contained in Eq. (5.2–8) if $z = i/2$.

a) When you have completed your trip what is the straight line distance, in miles, between your destination and your home?

b) How many miles have your feet actually traversed?

c) If you walk at three miles per hour, how long did your trip take? Assume that the rotations are instantaneous. Since the trip involved an infinite number of segments, why didn't it require an infinite amount of time?

d) Suppose you followed a path like the one described above but each segment is 90 percent of the length of the previous one. You begin with a segment of length one. How long does your journey take?

2. a) Follow an argument like that used in Example 1 to establish that the sequence $z, z^2, z^3, \ldots, z^n, \ldots$ has limit 0 for $|z| < 1$. How should the quantity $N(\varepsilon, z)$ in Eq. (5.2–1) be chosen in this problem?

b) Use the preceding result, as well as that contained in Example 1, to show that the sequence $z(1+e^{-z}), z^2(1+e^{-2z}), \ldots, z^n(1+e^{-nz}), \ldots$ has limit 0 for z in the half-disc domain $|z| < 1$, $\text{Re}(z) > 0$. Employ an argument like that in Example 2.

Use the nth term test to prove that the following series are divergent in the indicated regions.

3. $\displaystyle\sum_{n=1}^{\infty} (2iz)^n \quad \text{for } |z| \geq \frac{1}{2}$ **4.** $\displaystyle\sum_{n=0}^{\infty} (n+1)(i+1)^n(z+1)^n \quad \text{for } |z+1| \geq \frac{1}{\sqrt{2}}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(n)(i-1)^n}{(z-2i)^n} \quad \text{for } |z-2i| \leq \sqrt{2}$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+2}{n}\right)^n (z+1+i)^n \quad \text{for } |z+1+i| \geq \frac{1}{2}$

Use the ratio test to prove the absolute convergence, in the indicated domains, of the following series. Where does the ratio test assert that each series diverges?

7. $\displaystyle\sum_{n=1}^{\infty} n^2\left(z+\frac{1}{2}\right)^n \quad \text{for } |z+1/2| < 1$ **8.** $\displaystyle\sum_{n=0}^{\infty} n!e^{n^2 z} \quad \text{for } \text{Re}(z) < 0$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{(2+i)^n}{(z+i)^n(n+i)^2} \quad \text{for } |z+i| > \sqrt{5}$ **10.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n}{z}\right)^n \quad \text{for } |z| > e$

11. Make the substitutions $z = e^{i\theta}$ in Eq. (5.2–9), $N = n - 1$, assume θ is real, and separate Eq. (5.2–9) into its real and imaginary parts to show that

a) $1 + \cos\theta + \cos 2\theta + \cdots + \cos(N\theta) = \cos[N\theta/2]\dfrac{\sin[(N+1)\theta/2]}{\sin[\theta/2]}$;

b) $\sin\theta + \sin 2\theta + \cdots + \sin(N\theta) = \sin[N\theta/2]\,\dfrac{\sin[(N+1)\theta/2]}{\sin[\theta/2]}$.

c) Explain why we cannot let $N \to \infty$ in the preceding formulas and hope to obtain a meaningful result for the sum of an infinite series.

d) Let $N = 10$. Using MATLAB, make a polar plot of the left side of part (a) for $0 \le \theta \le 2\pi$. Also, as a check, plot the right side and verify that identical curves are obtained. Notice that some care must be exercised at $\theta = 0$ and $\theta = 2\pi$ to avoid dividing by zero.

12. The infinite series in Example 3 must converge if $z = .5 + .5i$. With this value, use MATLAB to sum the first n terms in the series. Let $n = 1, 2, \ldots, 25$. Plot these partial sums as points in the complex plane, labeling them with the corresponding n as far as practicable. Compute the sum of the infinite series and indicate its value on the plot.

In Exercises 13 and 14, find the sum of the series by making a suitable change of variables in Eq. (5.2–8). In each case, state where in the complex plane the series converges to the sum.

13. $1 + (z-1)^2 + (z-1)^4 + (z-1)^6 + \cdots$ **14.** $1 + 1/z + 1/z^2 + 1/z^3 + \cdots$

15. a) Prove that $\sum_{n=1}^{\infty} nz^{n-1} = 1/(1-z)^2$ for $|z| < 1$ by using series multiplication, i.e., Theorem 5, and the result $\sum_{j=1}^{\infty} z^{j-1} = 1/(1-z)$ for $|z| < 1$.

b) Using the result derived in part (a) and an additional series multiplication, show that

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2} z^{n-1} = \frac{1}{(1-z)^3}.$$

The identity $\sum_{j=1}^{n} j = n(n+1)/2$ can be helpful here.

16. Consider the series $\sum_{k=1}^{\infty} kz^{k-1} = 1 + 2z + 3z^2 + \cdots$. Show that its nth partial sum, $1 + 2z + 3z^2 + \cdots + nz^{n-1}$, is given by

$$\frac{z^n[n(z-1)-1]+1}{(1-z)^2} \quad \text{for } z \neq 1.$$

Hint: Refer to Eq. (5.2–9), which gives the nth partial sum of the series studied in Example 3. Notice that if we differentiate the series in that equation, we will obtain the series for the $(n-1)$st partial sum in the present problem.

17. Using the kind of argument presented in Example 3, prove that $\sum_{k=1}^{\infty} kz^{k-1} = 1/(1-z)^2$ for $|z| < 1$. Use the nth partial sum derived in Exercise 16.

5.3 UNIFORM CONVERGENCE OF SERIES

In Example 3 of the previous section, we showed that the series $\sum_{j=1}^{\infty} z^{j-1}$ converges to $1/(1-z)$ for $|z| < 1$. To accomplish this, we found a number N such that $|S(z) - S_n(z)| < \varepsilon$ for $n > N$. Here S_n was the nth partial sum, and $S(z)$ the sum of the given series. We should recall (see Eq. 5.2–12) that our N depended on both ε and z.

As shown in the exercises, Theorem 10 can be used to establish the following theorem.

THEOREM 11 (Analyticity of the Sum of a Series) If $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly to $S(z)$ for all z in R and if $u_1(z), u_2(z), \ldots$ are all analytic in R, then $S(z)$ is analytic in R. •

The preceding theorem guarantees the existence of the derivative of the sum of a uniformly convergent series of analytic functions. We have a way to arrive at this derivative.

THEOREM 12 (Term-by-Term Differentiation) Let $\sum_{j=1}^{\infty} u_j(z)$ converge uniformly to $S(z)$ in a region R. If $u_1(z), u_2(z), \ldots$ are all analytic in R, then at any interior point of this region

$$\frac{dS}{dz} = \sum_{j=1}^{\infty} \frac{du_j(z)}{dz}. \tag{5.3–9}$$ •

The theorem states that when a uniformly convergent series of analytic functions is differentiated term by term, we obtain the derivative of the sum of the original series.

We illustrate the preceding with our geometric series. Since $1/(1-z) = \sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + \cdots$, where convergence is uniform for $|z| \le r$ (with $r < 1$), we have upon differentiation

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = \frac{d}{dz}(1 + z + z^2 + \cdots) = 1 + 2z + 3z^2 + \cdots,$$

or

$$\frac{1}{(1-z)^2} = \sum_{j=1}^{\infty} jz^{j-1}, \quad |z| < r, \quad r < 1. \tag{5.3–10}$$

This result was obtained relatively painlessly with the use of Theorem 12. Without this theorem the same result could be had from the more difficult manipulation required in Exercises 15–17 of section 5.2.

EXERCISES 5.3

Use the Weierstrass M test to establish the uniform convergence of the following series in the indicated regions. State the convergent series of constants that is employed.

1. $\sum_{j=1}^{\infty} (-z)^{j-1}$ for $|z| \le .999$

Hint: Consider a convergent real geometric series of constants.

2. $\sum_{j=0}^{\infty} \frac{j}{j+1}(-z)^j$ for $|z| \le r$, where $r < 1$ (See previous hint.)

(continued)

(continued)

3. $\displaystyle\sum_{j=0}^{\infty} \frac{z^j}{j!}$ for $|z| \le r$, where $r < \infty$

Hint: Recall the series for e^r, where r is real.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{|n-i|}{n^3} z^n$ for $|z| \le r$, where $r < 1$

Hint: Recall (see Exercises 16–19, section 5.1) that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-nz}}{\operatorname{Log}(n+i)}$ for $\operatorname{Re}(z) \ge a$, where $a > 0$

6. In this problem, we prove the Weierstrass M test. We are given a series $\sum_{j=1}^{\infty} u_j(z)$ whose sum is $S(z)$ when z lies in a region R. We have also at our disposal a convergent series of constants $\sum_{j=1}^{\infty} M_j$ such that throughout R we have $|u_j(z)| \le M_j$ and wish to show this guarantees uniform convergence of the original series as well as absolute convergence.

a) Using the comparison test from real calculus (or see Exercises 16–19 of section 5.1), explain why the series $\sum_{j=1}^{\infty} u_j(z)$ must be absolutely convergent. Recall that absolute convergence guarantees ordinary convergence.

b) Using the definition of convergence explain why

$$|S(z) - S_n(z)| = \left|\lim_{k\to\infty} \sum_{j=1}^{k} u_j(z) - \sum_{j=1}^{n} u_j(z)\right| = \left|\lim_{k\to\infty} \sum_{j=n+1}^{k} u_j(z)\right| \quad \text{if we take } k > n.$$

c) Explain why $|\lim_{k\to\infty} \sum_{j=n+1}^{k} u_j(z)| \le \lim_{k\to\infty} \sum_{j=n+1}^{k} M_j$.

Hint: Recall the triangle inequality and its generalization.

d) Prove that given $\varepsilon > 0$ there must exist a number N such that $\lim_{k\to\infty} \sum_{j=n+1}^{k} M_j < \varepsilon$ for $n > N$.

Hint: Study the difference between the sum of the series $\sum_{j=1}^{\infty} M_j$ and the nth partial sum of the same series.

e) How do the results contained in steps (b), (c), and (d) establish the uniform convergence of the series $\sum_{j=1}^{\infty} u_j(z)$?

7. a) Let $z' = iy$ in Eq. (5.3–8). Now taking the corresponding parts of both sides of the resulting equation, obtain the expansions

$$\tan^{-1} y = y - y^3/3 + y^5/5 - \cdots \quad \text{and} \quad \frac{1}{2}\operatorname{Log}(1+y^2) = y^2/2 - y^4/4 + y^6/6 - \cdots.$$

for y real, $|y| < 1$.

b) What value for y in the first of the above series yields the following result?

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left[1 - \frac{1}{3}\frac{1}{3} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\frac{1}{3^3} + \cdots\right]$$

c) Assuming that you know $\sqrt{3}$, you can use the preceding series to compute π. Using MATLAB, compute the nth partial sum for the above series for π and list these values is not simple—it initially ran to over 350 pages—but has now been pared down. At one point, a computer was used by de Branges to confirm the validity of the work, but the proof itself does not rely on a machine.

History has dealt disparagingly with Bieberbach, the author of the conjecture, and rightly so. Aside from his fame in authoring the problem, he is remembered as a notorious uniform-wearing Nazi and vicious anti-Semite, who sought to eliminate Jews from the profession of German mathematics.†

EXERCISES 5.4

1. Derive the Maclaurin series expansions in Eq. (5.4–21).
2. Theorem 15 on Taylor series was derived for expansions about the origin $z_0 = 0$. Follow the suggestions given in that derivation and give a proof valid for any z_0.

State the first three nonzero terms in the following Taylor series expansions. The function to be expanded and the center of expansion are indicated. Give also the nth (general term) in the series and state the circle within which the series representation is valid. Use the principal branch of any multivalued functions.

3. $\dfrac{1}{z}$, $z = i$ **4.** e^z, $z = 2 + i$ **5.** $\operatorname{Log} z$, $z = e$ **6.** $\dfrac{1}{z^2}$, $z = 1 + i$

7. $\cosh z - \cos z$, $z = 0$ **8.** z^i, $z = 1$ **9.** i^z, $z = 0$

10. a) Find all the coefficients in the expansion of z^5 about z_0 (a constant) and write out the entire series in terms of z and z_0.
    b) For what values of z is the preceding series a valid expansion of the function?
    c) Explain how you could have obtained the result in (a) by using the binomial theorem. Note that $z = (z - z_0) + z_0$.
11. a) Explain why $z^{1/2}$ (principal branch) cannot be expanded in a Maclaurin series. Also, explain why the series expansion sought in Eq. (5.1–5) does not exist.
    b) Explain whether this same branch of $z^{1/2}$ can be expanded in a Taylor series about 1. If so, find the first three terms and state the circle within which the expansion is valid.
12. Consider the two infinite series, $1 + z + z^2 + z^3 + \cdots$ and $1 + (z/\operatorname{Log} 2) + (z^2/\operatorname{Log} 3) + (z^3/\operatorname{Log} 4) + \cdots$. Both of these series will converge to $1/(1 - z)$ at $z = 0$. Yet the second series is not the Taylor expansion of $1/(1 - z)$. Does this not contradict Theorem 16, which asserts that the Taylor series expansion of a function is the power series expansion of the function? Explain.

Without actually obtaining the coefficients in the following Taylor series, determine the center and radius of the circle within which each converges to the function on the left. Use the principal branch of any multivalued functions.

13. $\dfrac{1}{z - i} = \displaystyle\sum_{n=0}^{\infty} c_n (z + 1)^n$ **14.** $\dfrac{1}{z^3 + 1} = \displaystyle\sum_{n=0}^{\infty} c_n (z - i)^n$ *(continued)*

† For more on this unsavory piece of history, see Sanford Segal, *Mathematicians under the Nazis* (Princeton, NJ: Princeton University Press, 2003).

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15. $\frac{1}{\cos z} = \sum_{n=0}^{\infty} c_n (z - 1 - i)^n$ **16.** $\frac{1}{\text{Log}\, z} = \sum_{n=0}^{\infty} c_n (z - 1 - 2i)^n$

17. $\frac{1}{z^{1/2} - 1} = \sum_{n=0}^{\infty} c_n (z - 2)^n$ **18.** $\frac{1}{z^{1/2} - 1} = \sum_{n=0}^{\infty} c_n (z - 2)^n$

Without obtaining the series, determine the interval along the x-axis for which the indicated real Taylor series converges to the given real function. Convergence at the endpoints of the interval need not be considered.

19. $\frac{1}{1 - x}$ expanded about $x = -1$
(How do your findings confirm Eqs. (5.1–2), (5.1–6), and (5.1–7)?)

20. $\frac{1}{x^2 + 9}$ expanded about $x = 2$ **21.** $\frac{1}{\sin x}$ expanded about $x = 1/4$

22. $\frac{1}{\sin x}$ expanded about $x = 2$ **23.** $\tan x$ expanded about $x = 2$

24. $\sqrt{x}$ expanded about $x = e$ **25.** $\frac{1}{x^3 + 1}$ expanded about $x = 1$

26. *An apparent paradox.* Let us approximate e^z by its Nth partial sum, as obtained from Eq. (5.4–20). We have $e^z \approx 1 + z + \cdots + \frac{z^{(N-1)}}{(N-1)!}$. We would expect that the approximation improves with increasing N. Notice, however, that this approximation is a polynomial of degree $N - 1$ and therefore (see section 4.6) has $N - 1$ roots in the complex plane. Thus there are exactly $N - 1$ locations in the complex plane where the polynomial approximation vanishes (some of these may be multiple roots). Recall, however, that $e^z \neq 0$ for all z. Thus it would appear that as more and more terms are included in the partial sum, the number of points in the complex plane at which the partial sum cannot adequately represent the exponential function increases and that there is no point in our trying to improve the approximation with more terms. Try to resolve this paradox by means of a MATLAB program that finds the roots of the polynomial approximation. Use the MATLAB command called *roots*. Make a plot that shows, for each N, the distance of the nearest root to $z = 0$. Do not take $N = 1$ as the partial sum is a constant and so there will be no roots. Take $N = 2 \ldots 50$.

27. Use Eq. (5.4–20) and a triangle inequality to prove that in the disc $|z| \leq 1$ we have $|e^z - 1| \leq (e - 1)|z|$.

28. a) Let $z^N - z_0^N = \sum_{n=1}^{N} c_n (z - z_0)^n$ valid for all z. N is a positive integer. Show that $c_n = N! z_0^{N-n} / [n!(N - n)!]$.

b) Replace z in the above with $z + z_0$ and show that

$$(z + z_0)^N = \sum_{n=0}^{N} \frac{N! z_0^{N-n}}{n!(N - n)!} z^n.$$

This is the familiar binomial expansion.

29. Let a function $f(z)$ be expanded in a Taylor series about z_0. The circle, centered at z_0, within which this series converges is in certain cases larger than, but concentric with, the

Inside $|z| = 1$, we can add the three series together and obtain

$$\frac{z}{(z+1)^2(z-2)} = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1, \tag{5.5–26}$$

where

$$c_n = (-1)^{n+1}\frac{2}{9} + \frac{(-1)^n}{3}(n+1) - \frac{1}{9}\left(\frac{1}{2}\right)^n.$$

EXERCISES 5,5

The following exercises involve our generating a new Taylor series through a change of variables in the geometric series Eq. (5.2–8) or some other familiar expansion. Here a is any constant. Explain how the following are derived:

1. $\displaystyle \frac{1}{1+az} = 1 - az + a^2z^2 - a^3z^3 + \cdots, \quad |z| < \left|\frac{1}{a}\right|$

2. $\displaystyle \frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots, \quad |z| < 1$

3. $\displaystyle \frac{1}{1+a+z} = 1 - (z+a) + (z+a)^2 - \cdots, \quad |z+a| < 1$

4. a) $\displaystyle e^{-z^2} = 1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6} + \cdots, \quad$ all z

b) Use the preceding result to find the 10th derivative of e^{-z^2} at $z = 0$.

5. Differentiate the series of Eq. (5.5–2) to show that

$$\frac{1}{z^3} = 1 - \frac{3\cdot 2}{2}(z-1) + \frac{4\cdot 3}{2}(z-1)^2 - \frac{5\cdot 4}{2}(z-1)^3 + \cdots, \quad |z-1| < 1.$$

6. By differentiating the series of Eq. (5.2–8) several times, find c_n in the expansion $1/(1-z)^4 = \sum_{n=0}^{\infty} c_n z^n$, $|z| < 1$.

7. Use the series in Eq. (5.2–8) and successive differentiation to show that, for $N \geq 0$,

$$\frac{1}{(1-z)^N} = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \frac{(N-1+n)!}{n!(N-1)!}, \quad |z| < 1.$$

8. a) Integrate the series of Exercise 2 along a contour connecting the origin to an arbitrary point z, where $|z| < 1$, to show that

$$\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z| < 1. \tag{5.5–27}$$

b) We might put $z = 1$ in the preceding expansion to obtain $\tan^{-1} 1 = \pi/4 = 1 - 1/3 + 1/5 - \cdots$. This expansion, which could be used to obtain $\pi/4$, is valid, although not

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Why is this so?
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b) The *Cornu Spir*
problems.[‡] It is d
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[†]See L.V. Ahlfors, *Complex*
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and Co., 1997).
[‡]See, e.g., J.D. Kraus, *Antenn*

F for $0 \le P \le 1.5$, and for this interval the data on the curve can be approximated by our using a series approximation to $F(P)$. Using the first five terms in the series of part (a), generate the portion of the Cornu Spiral $0 \le P \le 1.5$, taking P in increments of .1. Do this with a MATLAB program. Repeat this procedure with a 10-term series and compare your two results with a picture of the Cornu Spiral that you can download from the World Wide Web (do a search with the keywords "Cornu Spiral"). To generate the curve for relatively large values of P, you would evaluate numerically the integral defining $F(P)$.

11. a) By considering the first and second derivatives of the geometric series in Eq. (5.2–8), show that $\sum_{n=1}^{\infty} n^2 z^n = (z+z^2)/(1-z)^3$ for $|z| < 1$.

b) Use your result to evaluate $\sum_{n=1}^{\infty} n^2/2^n$.

Use series multiplication to find a formula for c_n in these Maclaurin expansions. In what circle is each series valid?

12. $\dfrac{\cosh z}{1-z} = \sum_{n=0}^{\infty} c_n z^n$ **13.** $\dfrac{\text{Log}(1-z)}{1+z} = \sum_{n=0}^{\infty} c_n z^n$

14. Assume $P(z)/Q(z)$ satisfies the requirements of Rule I for partial fractions. Thus $Q(z)$ has no repeated factor and

$$\frac{P(z)}{Q(z)} = \frac{P}{C(z-a_1)(z-a_2)\cdots(z-a_n)} = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \cdots + \frac{A_n}{z-a_n}.$$

a) Multiply both sides of the preceding equation by $(z-a_1)(z-a_2)\cdots(z-a_n)$ and cancel common factors to show that

$$\frac{P(z)}{C} = A_1(z-a_2)(z-a_3)\cdots(z-a_n) + A_2(z-a_1)(z-a_3)\cdots(z-a_n) + \cdots + A_n(z-a_1)(z-a_2)\cdots(z-a_{n-1}).$$

b) Show how to obtain any coefficient $A_j (j = 1, 2, \ldots, n)$ by setting $z = a_j$ in the previous equation.

c) Show that the result obtained in part (b) is identical to

$$A_j = \lim_{z \to a_j} \left[(z-a_j) \frac{P(z)}{Q(z)} \right] = \frac{P(a_j)}{Q'(a_j)}.$$

Hint: Consider L' Hôpital's Rule.

d) Expand $z/[(z^2+1)(z-2)]$ in partial fractions by using the results of part (b) or (c).

Obtain the following Taylor expansions. Give a general formula for the nth coefficient, and state the circle within which your expansion is valid.

15. $\dfrac{z}{(z-1)(z+2)}$ expanded about $z = 0$ **16.** $\dfrac{z}{(z+1)(z+2)}$ expanded about $z = 1$

(continued)

(*continued*)

17. $\dfrac{1}{z^2}$ expanded about $z = 1 + i$ **18.** $\dfrac{1}{z^3}$ expanded about $z = i$

19. $\dfrac{z+1}{(z-1)^2(z+2)}$ expanded about $z = 2$

20. $\dfrac{1}{(z-1)^2(z+1)^2}$ expanded about $z = 2$

21. $\dfrac{e^z}{(z-2)(z+1)}$ expanded about $z = 0$

22. Use the answer to Exercise 20 to find the value of the 10th derivative of $1/[(z-1)^2(z+1)^2]$ at $z = 2$.

23. $\dfrac{z^3 + 2z^2 + z - 1}{z^2 - 4}$ expanded about $z = 1$.
Hint: The denominator is not of higher degree than the numerator; thus we cannot immediately make a partial fraction decomposition. Show by a long division that the given function can be written as

$$(z+2) + \frac{5z+7}{z^2-4} = (z-1) + 3 + \frac{5z+7}{z^2-4}.$$

Now apply the method of partial fractions.

24. Let $h(z) = f(z)/g(z)$, where $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$, and $g(z_0) = b_0 \neq 0$. We seek a Taylor expansion of $h(z)$ of the form $h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$. Find c_0, c_1, c_2 by long division of the series for $f(z)$ by the series for $g(z)$. Show that you obtain results identical to Eq. (5.5–11).

25. Find the coefficients c_0, c_1, c_2, c_3 in the Maclaurin expansion $\text{Log}(1+z)/\cos z = \sum_{n=0}^{\infty} c_n z^n$, $|z| < 1$, by the series division of Exercise 24 or by the technique used in deriving Eq. (5.5–11).

26. Obtain all the coefficients in the following Maclaurin expansion by doing a long division:

$$\frac{1+z}{1+z+z^2+z^3+\cdots} = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1.$$

Explain why there are only two nonzero coefficients in your result.

27. a) The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are defined by†

$$B_n = n!c_n,$$

where

$$f(z) = \begin{Bmatrix} \dfrac{z}{e^z - 1}, & z \neq 0 \\ 1, & z = 0 \end{Bmatrix} = \sum_{n=0}^{\infty} c_n z^n.$$

†The Bernoulli numbers also appear in other expansions. Tables of the numbers can be found in various handbooks, e.g., M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (New York: Dover Publications, 1965), 810.

$\sin z$, as was just done in this last example. We will postpone the problem of finding the Laurent series in each ring until we consider the subject of residues in the next chapter. In principle, we could find each coefficient by an application of Eq. (5.6–7), but for now we see no easy way to evaluate these integrals. ●

EXERCISES 5.6

Obtain the following Laurent expansions. State the first four nonzero terms. State explicitly the nth term in the series, and state the largest possible annular domain in which your series is a valid representation of the function.

1. $\dfrac{\sinh z}{z^3}$ expanded in powers of z **2.** $\dfrac{\cos(1/z)}{z^3}$ expanded in powers of z

3. $\sin\left(1+\dfrac{1}{z-1}\right)$ expanded in powers of $z-1$

4. $\text{Log}\left[1+\dfrac{1}{z-1}\right]$ expanded in powers of $z-1$

Hint: Make a change of variable in Eq. (5.3–8).

5. $\left(z+\dfrac{1}{z}\right)^7$ expanded in powers of z (Give all the terms.)

Obtain the indicated Laurent expansions of $\frac{1}{z+i}$. State the nth term of the series.

6. An expansion valid for $|z|>1$ **7.** An expansion valid for $|z-i|>2$

Expand the following functions in a Laurent series valid in a domain whose outer radius is infinite. State the center and inner radius of the domain. Give the nth term of the series.

8. $1/(z-1)$ expanded in powers of $z+3$

9. $1/(z+2)$ expanded in powers of $z-i$

10. $z/(z-i)$ expanded in powers of $z-1$

11. a) Consider $f(z)=1/[z(z-1)(z+3)]$. This function is expanded in three different Laurent series involving powers of z. State the three domains in which Laurent series are available.

b) Find each series and give an explicit formula for the nth term.

For the following functions, find the Laurent series valid in an annular domain that contains the point $z=2+2i$. The center of the annulus is at $z=1$. State the domain in which each series is valid, and give an explicit formula for the nth term of your series.

12. $f(z)=\dfrac{1}{z(z-2)}$ **13.** $f(z)=\dfrac{1}{z(z-4)}$ **14.** $f(z)=\dfrac{1}{(z-1)(z-3)}$

15. $f(z)=\dfrac{z-i}{z-1}$ **16.** $f(z)=\dfrac{1}{(z-1)^3}+\dfrac{1}{z}$ **17.** $f(z)=\dfrac{1}{(z-1)^3}+z^3$

to the real axis. If z (and hence $\bar{z}$) is in D, then $f(z)$ and $f(\bar{z})$ are conjugates of one another. •

The proof of the preceding is simple if we use Taylor series and some of our new knowledge of analytic continuation. The steps are outlined in Exercise 17. Of course a function can satisfy $\bar{f}(z) = f(\bar{z})$ for all z and not be analytic.

EXERCISES

5.7

1. Consider $f(z) = z^3 - x^3 + 3xy^2 + i(y^3 - 3x^2y)$, where $z = x + iy$.
   a) Show that the zeros of this function on the axis $y = 0$, $-\infty < x < \infty$, are not isolated.
   b) Does the result of (a) contradict the statement that the zeros of an analytic function are isolated? Explain.
2. a) Consider $f(z) = e^z - e^{iy}$. Show that this function is zero everywhere on the y (imaginary) axis of the complex z-plane.
   b) Can you conclude from part (a) and Theorem 19 that every point on the y-axis has a neighborhood throughout which $f(z) = 0$? Explain.
3. a) Are the zeros of $\sin(\pi/(z^2 + 1))$, in the domain $|z| < 1$, isolated?
   b) Find all the zeros of this function in the domain.
   c) Identify any limit (accumulation) points of the set, and state whether they belong to the given domain.

Find the order of the zeros of the following functions at the points indicated.

4. $\cos z$ at $z = n\pi + \pi/2$, n any integer
5. $\operatorname{Log} z$ at $z = 1$
6. $(z^4 - 1)^2/z$ at $z = i$
7. $z^3 \sin z$ at $z = 0$ and also $z = \pi$

8. Show that if $f(z)$ has a zero of order n at z_0, then $[f(z)]^m$ (m is a positive integer) has a zero of order nm at z_0.

Use the result to find the order of the zeros in the following.

9. $(\operatorname{Log} z - 1)^2$ at $z = e$
10. $(\sin z)^4$ at $z = \pi$
11. $(z^3 \sin z)^2$ at $z = 0$

12. a) Let $f(z) = 1 + z + z^2 + \cdots$, $|z| < 1$. Expand this function in a Taylor series about $z = -3/4$; i.e., state c_n in its expansion $\sum_{n=0}^{\infty} c_n(z + 3/4)^n$. You may use your knowledge of the sum of the given series.
    b) Does the Taylor series found in (a) produce an analytic continuation of $f(z)$ into a region extending beyond $|z| = 1$? Explain.
13. a) Find in closed form the function defined by $\int_0^\infty e^{2t} e^{-zt}\,dt$ for $\operatorname{Re} z > 2$.
    b) What is the analytic continuation of this function? What is the largest region in which this continuation is valid?
14. a) Show that the function $f(z) = \int_0^z (2 + 3 \cdot 2w + 4 \cdot 3w^2 + 5 \cdot 4w^3 + \cdots)\,dw$ is analytic in the disc $|z| < 1$ and is undefined for $|z| > 1$.