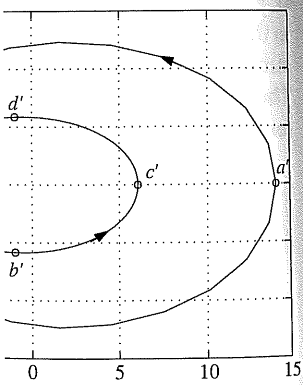
(6.12–10)

)) was encircled once in the

determine how many roots

of the transformation $w =$
lure if we follow the method
d to use MATLAB to carry
$= 3$. The results are shown
e images of the four points
ɔing can also be performed

$= 3$ and encircle $z = 0$ once,
wice in the positive sense.
2–10), two solutions inside



w-plane
ıge of $|z| = 3$ under the
ısformation
$: e^z - 2z$
(b)

The preceding problem illustrates the utility of the principle of the argument in determining whether an equation $f(z) = 0$ has solutions in a given region of the complex plane. This property is further developed in the following chapter, where we study the Nyquist method.

We proved the Fundamental Theorem of Algebra in section 4.6 with the aid of Liouville's theorem; i.e., we showed that the equation $P(z) = 0$, where $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 (n \geq 1, a_n \neq 0)$, has a root in the complex plane. An extension of the theorem, Exercise 18 in section 4.6, shows that there are n roots. If we call them $z_1, z_2, \ldots, z_n$, then $P(z)$ is a constant times $(z - z_1)(z - z_2) \cdots (z - z_n)$. An alternate and simple proof of the fundamental theorem, based on the principle of the argument, is now available to us. Using this principle, we first derive Rouché's theorem (Exercise 8), from which the fundamental theorem follows (Exercise 9). Rouché's theorem, by itself, is also useful in locating the roots of both algebraic and transcendental equations (see Exercises 10–15).

EXERCISES 6.12

Let $f(z)$ be each of the following functions and take C as the circle indicated. Sketch $f(z)$ in the w-plane as z moves counterclockwise around the circle. Without using the argument principle, determine the number of zeros and poles of $f(z)$ inside C. Check your result by using the principle of the argument, Eq. (6.12–9).

1. $f(z) = z$, C is $|z - 2| = 3$
2. $f(z) = 1/z$, C is $|z - 2| = 3$
3. $f(z) = (z + 1)/z$, C is $|z| = 4$
4. $f(z) = \dfrac{1}{(z-1)^2}$, C is $|z| = 2$
5. $f(z) = \text{Log}\, z$, C is $|z - e| = 2$
6. $\dfrac{\sin z}{z}$, C is $|z| = \pi/2$

7. Show that if $f(z)$ has a pole of order p at α, then the residue of $f'(z)/f(z)$ at α is $-p$. *Hint:* $f(z)$ can be expressed as $g(z)/(z - \alpha)^p$, where $g(\alpha) \neq 0$ and $g(z)$ is analytic at α. Why?

8. Let $f(z)$ and $g(z)$ be analytic on and everywhere inside a simple closed contour C. Suppose $|f(z)| > |g(z)|$ on C. We will prove that $f(z)$ and $(f(z) + g(z))$ have the same number of zeros inside C. This is known as *Rouché's theorem* and is a very clever result. It was published by the French mathematician Eugène Rouché (1832–1910) when he was about 30 years old. He is known almost entirely for deriving this formula.

   a) Explain why
$$\frac{\Delta_C \arg f(z)}{2\pi} = N_f$$
and
$$\frac{\Delta_C \arg(f(z) + g(z))}{2\pi} = N_{f+g},$$
where N_f is the number of zeros of $f(z)$ inside C and N_{f+g} is the number of zeros of $f(z) + g(z)$ inside C.

b) Show that

$$N_{f+g} = \frac{1}{2\pi}\Delta_C \arg f(z) + \frac{1}{2\pi}\Delta_C \arg\left(1 + \frac{g(z)}{f(z)}\right).$$

Hint: $f + g = f[1 + g/f]$.

c) If $|g|/|f| < 1$ on C, explain why $\Delta_C \arg[1 + g/f] = 0$.
Hint: Let $w(z) = 1 + g/f$. As z goes along C, suppose that $w(z)$ encircles the origin of the w-plane. This implies that $w(z)$ assumes a negative real value for some z. Why does this contradict our assumption $|g|/|f| < 1$ on C?

d) Combine the results of parts (a), (b), and (c) to show that $N_f = N_{f+g}$.

9. Let $h(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 z^0$ be a polynomial of degree n. We will prove that $h(z)$ has exactly n zeros (counted according to multiplicities) in the z-plane. This is a version of the *Fundamental Theorem of Algebra*, which was discussed in section 4.6.

a) Let

$$f(z) = a_n z^n,$$
$$g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 z^0.$$

Note that $h = f + g$. Consider a circle C of radius $r > 1$ centered at $z = 0$. Show that on C

$$\left|\frac{g(z)}{f(z)}\right| < \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n| r}.$$

How does this inequality indicate that for sufficiently large r we have $|g(z)| < |f(z)|$ on C?

b) Use Rouché's theorem (see Exercise 8) to argue that, for C chosen with a radius as just described, the number of zeros of $h(z) = f(z) + g(z)$ inside C is identical to the number of zeros of $f(z)$ inside C. How many zeros (counting multiplicities) does $f(z)$ have?

10. Show that all the roots of $z^4 + z^3 + 1 = 0$ are inside $|z| = 3/2$.
Hint: Use Rouché's theorem (Exercise 8), taking $f(z) = z^4$, $g(z) = z^3 + 1$. Note that $|g(z)| \le 1 + |z|^3$.

11. Show that all roots of the equation in Exercise 10 are outside $|z| = 3/4$.
Hint: Same as above, but take $f(z) = z^3 + 1$, $g(z) = z^4$. Note that $|f(z)| \ge 1 - |z|^3$.

12. In Exercises 10 and 11, we investigated the solutions of $z^4 + z^3 + 1 = 0$ and found them to lie between the circles $|z| = 3/4$ and $|z| = 3/2$. Most computational software packages have a program for computing the roots of a polynomial. In MATLAB, the program is called *roots*. Using *roots* or something comparable find all the roots of this quartic equation and verify that they do lie as predicted.

13. Use Rouché's theorem to show that $3z^2 - e^z = 0$ has two solutions inside $|z| = 1$.
Hint: Take $f(z) = 3z^2$.

14. Use Rouché's theorem to show that $5\sin z - e^z = 0$ has one solution inside the square $|x| \le \pi/2$, $|y| \le \pi/2$. Explain why this root must be real.

Hint: Recall that $|\sin z| = \sqrt{\sinh^2 y + \sin^2 x}$.

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8.2

b) Each semiinfinite line is mapped into the w- (or u, v-) plane by the transformation $w = z^2$. Find the equation and sketch the image in the w-plane of each line under this transformation. Give the equations in the form $v = f(u)$.

c) Using the equation of each image in the w-plane, find their point of intersection and prove, using these equations, that the angle of intersection of these image curves is the same as that found for the lines in part (a).

What are the critical points of the following transformations?

4. $w = z - z^{-1}$ **5.** $w = \cos z$ **6.** $w = ze^z$ **7.** $w = \dfrac{z-i}{z+i}$

8. $w = iz + \operatorname{Log} z$

9. a) What is the image of the semicircular arc, $|z| = 1, 0 \le \arg z \le \pi$, under the transformation $w = z + 1/z$?
Hint: Put $z = e^{i\theta}$.

b) What is the image of the line $y = 0, x \ge 1$ under this same transformation?

c) Do the image curves found in parts (a) and (b) have the same angle of intersection in the w-plane as do the original curves in the xy-plane? Explain.

10. Show that under the mapping $w = 1/z$ the image in the w-plane of the infinite line $\operatorname{Im} z = 1$ is a circle. What is its center and radius?

11. Find the equation in the w-plane of the image of $x + y = 1$ under the mapping $w = 1/z$. What kind of curve is obtained?

12. Consider the straight line segment directed from $(2, 2)$ to $(2.1, 2.1)$ in the z-plane. The segment is mapped into the w-plane by $w = \operatorname{Log} z$.

a) Obtain the approximate length of the image of this segment in the w-plane by using the derivative of the transformation at $(2, 2)$.

b) Obtain the exact value of the length of the image. Use a calculator to convert this to a decimal, and compare your result with part (a).

c) Use the derivative of the transformation to find the angle through which the given segment is rotated when mapped into the w-plane.

13. The square boundary of the region $1 \le x \le 1.1, 1 \le y \le 1.1$ is transformed by means of $w = e^z$.

a) Use the derivative of the transformation at $(1, 1)$ to obtain a numerical approximation to the area of the image of the square in the w-plane.

b) Obtain the exact value of the area of the image, and compare your result with part (a).

8.3 One-to-one Mappings and Mappings of Regions

It is now necessary to study with some care the correspondence that the analytic transformation $w = f(z)$ creates between points in the z-plane and points in the w-plane. Let all the points in a region R be mapped into the w-plane so as to form an image region R'. Let z_1 be any point in R. Since $f(z)$ is single valued in R, z_1 is mapped into a unique point $w_1 = f(z_1)$. Given the point w_1, can we assert that it is the image of a unique point, that is, if $w_1 = f(z_1)$ and $w_1 = f(z_2)$, where z_1 and z_2 are points in R, does it follow that $z_1 = z_2$? The following definition is useful in dealing with this question.

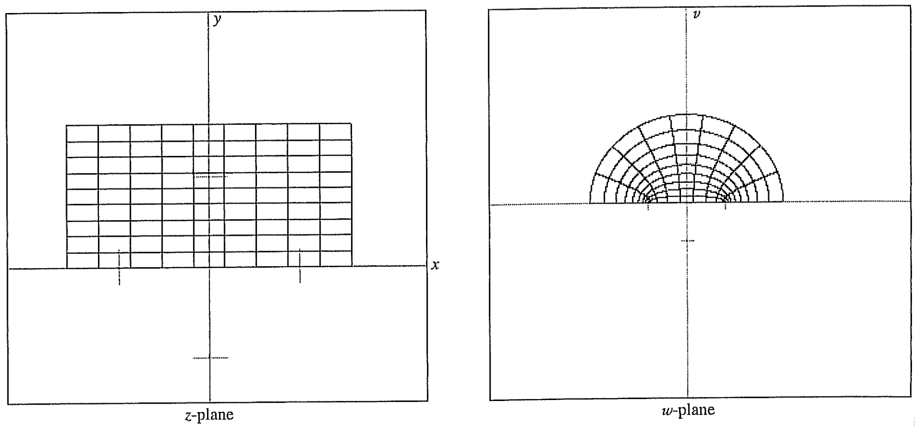

Figure 8.3–6

2. a) Consider the infinite strip $|\text{Re}\, z| \le a$, where a is a constant satisfying $0 < a < \pi/2$. Find the image of this strip, under the transformation $w = \sin z$, by mapping its boundaries.

b) Is the mapping in part (a) one to one?

c) Suppose $a = \pi/2$. Is the mapping now one to one?

How does the transformation $w = \cos z$ map the following regions? Is the mapping one to one in each case?

3. The infinite strip $a \le \text{Re}\, z \le b$, where $0 < a < b < \pi$

4. The infinite strip $0 \le \text{Re}\, z \le \pi$

5. The semiinfinite strip $0 \le \text{Re}\, z \le \pi$, $\text{Im}\, z \ge 0$

6. Consider the region consisting of an annulus with a sector removed shown in Fig. 8.3–7. The region is described by $\varepsilon \le |z| \le R$, $-\pi + \alpha \le \arg z \le \pi - \alpha$. The region is mapped with $w = \text{Log}\, z$.

a) Make a sketch of the image region in the w-plane showing A', B', C', ... (the images of A, B, C, ...). Assume $0 < \varepsilon < R$, $0 < \alpha < \pi$.

b) Is the mapping one to one? Explain.

c) What does the image region of part (a) look like in the limit as $\varepsilon \to 0+$?

7. Consider the wedge-shaped region $0 \le \arg z \le \alpha$, $|z| < 1$. This region is to be mapped by $w = z^4$. What restriction must be placed on α to make the mapping one to one?

8. a) Refer to Example 3 of this section. Show that at their point of intersection the images of $x = x_1$ and $y = y_1$ are orthogonal. Work directly with the equation of each image.

b) In this same example, what inverse transformation $z = g(w)$ will map the upper half of the w-plane onto the semiinfinite strip of Fig. 8.3–5(a)? State the branches of any logarithms and square roots in your function, and verify that point D' is mapped into D, that C' is mapped into C, and that $w = i$ has an image lying inside the strip.

9. The semiinfinite strip $0 < \text{Im}\, z < \pi$, $\text{Re}\, z > 0$ is mapped by means of $w = \cosh z$. Find the image of this domain.

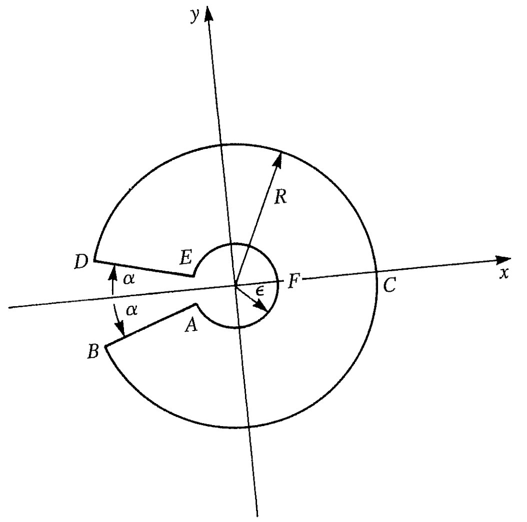

Figure 8.3–7

10. a) Consider the half-disc-shaped domain $|z| < 1$, $\operatorname{Im} z > 0$. Find the image of this domain under the transformation

$$w = \left(\frac{z-1}{z+1}\right)^2.$$

Hint: Map the semicircular arc bounding the top of the disc by putting $z = e^{i\theta}$ in the above formula. The resulting expression reduces to a simple trigonometric function.

b) What inverse transformation $z = g(w)$ will map the the domain found in part (a) back onto the half-disc? State the appropriate branches of any square roots.

11. Following Theorem 2 there is a remark asserting that if $f'(z) = 0$ at any point in a domain, then $w = f(z)$ cannot map that domain one to one. However, in Example 1 we found that a wedge containing $z = 0$ can be mapped one to one by $f(z) = z^2$ even though $f'(0) = 0$. Is there a contradiction here? Explain.

8.4 THE BILINEAR TRANSFORMATION

The bilinear transformation defined by

$$w = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \text{ are complex constants,} \tag{8.4–1}$$

which is also known as the *linear fractional transformation* or the *Möbius transformation*, is especially useful in the solution of a number of physical problems, some of which are discussed in this chapter. The utility of this transformation arises from the way in which it maps straight lines and circles.

Equation (8.4–1) defines a finite value of w for all $z \neq -d/c$. One generally assumes that

$$ad \neq bc. \tag{8.4–2}$$