# On Existence and Uniqueness of Solutions of Ordinary Differential Equations

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**Abstract**

This paper concentrates on questions regarding existence and uniqueness to the generic initial value problem in the theory of ordinary differential equations, that is $\dot{x} = f(x,t)$, $x(0) = x_0$, where $f$ is some Lipschitz function in a neighbourhood of $(x_0, 0) \in \mathbb{R}^n \times \mathbb{R}$. This is a strictly weaker condition than is often formulated in elementary books on differential equations.

## Introduction and Notation

There are several problems in physics which lead to the study of certain differential equations. These equations are often quite complicated, and analytic solutions may not always be accesible. In these cases numerical aides must be used. However, this raises some delicate questions regarding the reliablity and rigidity of numerical computations. For example,

- Does a solution of the problem always exist ?
- If a solution exists, is it then unique ?
- When can there be several solutions ?

In any case, a numerical algorithm, applied on some specific equation, will probably yield some kind of a plot of a possible solution. We must then ask us whether this plot really is the whole truth. Even if we are garanted the existence of a solution, we can *a priori* not be certain that this is *the* solution to the problem. There may very well be other solutions. Of course, from a purely physical point of view, a mathematical model which generates such a differential equation, must have some serious flaw. Nevertheless, these equations do arise sometimes, and it is important to recognize them.

In this paper, we will consider ordinary differential equations ( ODE ) in $\mathbb{R}^n$. More precisely, let $V \subseteq \mathbb{R}^n$ and $I \subseteq \mathbb{R}$ and let $f : I \times V \to \mathbb{R}^n$ be a function with some regularity ( to be made precise later ). We will analyse some questions regarding existence and uniqueness of solutions to the standard initial value problem ( IVP )

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0,$$

where $\dot{x}$ denotes the derivative of $x$ with respect to $t$. We let $I$ be some small closed interval around $t_0$ ( which after translation can be assumed to be 0 ) . The exact geometry of $V$ will be clear in the discussion below.

The presentations in this paper will be somewhat theoretical, and sometimes very concise, so I strongly urge the reader not to pass the sections too quickely, but take some moments to contemplate the material. Some theorems and lemmata are proven in more general settings than is actually needed, which, of course, only make them more beutiful.

# Basic Concepts

## Metric spaces and Cauchy sequences

There will be a need for determining distances between elements in certain spaces. We therefore introduce the concept of a metric. Let us state the first definition. The reader is asked to convince herself that the properties below are reasonable in a definition of a distance map, i.e. the distance between two elements ( let us henceforth call elements points ) $x$ and $y$ ought to be the distance between $y$ and $x$, and furthermore, if the distance between $x$ and $y$ is zero, then $x$ and $y$ should be the same point, and *vice versa*.

**Definition.** *Let $X$ be a set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric if for all $x, y, z \in X$*

$$\begin{array}{rl} (i) & d(x,y) = 0 \iff x = y \\ (ii) & d(x,y) = d(y,x) \\ (iii) & d(x,y) \leq d(x,z) + d(z,y) \end{array}$$

*If $X$ has a defined metric $d$ , then $X$, sometimes written $(X, d)$, is said to be a metric space.*

Note from the definition that a metric is always positive, since for all $x$ and $y$ in $X$

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y).$$

The notation of a distance makes it possible to talk about a neighbourhood of a point, i.e. the set of points on a certain distance from a given point. We define the open ball of radius $r > 0$ around $p \in X$ as

$$B_p(r) = \{\, x \in X \ ; \ d(p,x) < r \,\}\,.$$

We can also generalize the concept of convergence to a metric space $(X, d)$. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in $X$ converges to $x \in X$ if

$$\forall\, \varepsilon > 0 \quad \exists\, N \quad \forall\, n > N \qquad d(x, x_n) < \varepsilon.$$

There are numerous educational examples of metrics. Of course, $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ is a metric, which can easily be checked. A more well-known example is the norm-induced metric on $\mathbb{R}^n$, defined as

$$d(x,y) = |x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2},$$

where $x_i, y_i$ for $i = 1, \ldots, n$ denote the coordinates of $x, y \in \mathbb{R}^n$.

Let us now turn to the study of Caucy sequences. We introduce them to simplify tests of convergence in certain cases. Let $\{x\}_{n\in\mathbb{N}}$ be a sequence of points in a metric space $(X, d)$. A sequence is called Cauchy if

$$\forall\, \varepsilon > 0 \quad \exists\, N \quad \forall\, m, n > N \qquad d(x_m, x_n) < \varepsilon.$$

There are some apparent similarities between the definition of a convergent sequence and a Cauchy sequence. However, the definition of a Cauchy sequence is strictly weaker, which is important to realize. Of course, if $\{x\}_{n\in\mathbb{N}}$ is convergent to $x \in X$, then for all $\varepsilon$ there exists $N$ such that for all $n > N$, $d(x, x_n) < \varepsilon/2$. Choose $m > N$

$$d(x_m, x_n) \le d(x_m, x) + d(x, x_n) < \varepsilon.$$

Hence every convergent seqence is Cauchy, but the converse may not always be true. However, if every Cauchy sequence in $X$ do converge, $X$ is said to be complete. A common example of a complete metric space is $(\mathbb{R}, |\cdot|)$, that is, the real line with the absolute value as metric. This fact is however not entirely trivial.

## Banach's fixed point theorem

This may be the most important part of the paper. I urge the reader to really read through and understand the wording of the theorem stated below. However, before we start to prove this beutiful theorem, we must first introduce some basic concepts.

Let $(X, d)$ be a metric space, and let $T$ be a map on $X$. A point $p \in X$ such that $T(p) = p$ is called a fixed point of $T$, and if there exists $c < 1$, such that for all $x, y \in X$

$$d(T(x), T(y)) \le c\, d(x, y),$$

$T$ is said to be a strict contraction on $X$. It is clear that a strict contraction can have at most one fixed point. To see this, assume that $p, q \in X$ are two distinct fixed points of $T$. Then

$$d(T(p), T(q)) = d(p, q) \le c\, d(p, q),$$

which is impossible if $d(p, q) > 0$. Hence $p = q$. We are now ready to state the famous theorem of Banach about fixed points of strict contrations on a complete metric space.

**Theorem.** *Let $X$ be a complete metric space, and $T$ a strict contracition on $X$. Then $T$ has a unique fixed point in $X$.*

*Proof.* The part of uniqueness is clear from above. To prove existence, let us define the following recursion formula for $n \ge 0$ and $x_0 \in X$

$$x_{n+1} = T(x_n), \quad x(0) = x_0.$$

We want to show that $\{x_n\}_{n\in\mathbb{N}}$ converges for some initial point $x_0 \in X$. Since $X$ is complete, it is sufficient to show that this sequence is Cauchy. It is clear that

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \le c\, d(x_1, x_0),$$

from which follows inductively

$$d(x_n, x_{n-1}) \le c^{n-1}\, d(x_1, x_0).$$

Using the triangle inequality, it is easy to show that

$$d(x_{n+m}, x_n) \le d(x_{n+m}, x_{n+m-1}) + \ldots + d(x_{n+1}, x_n),$$

and from above, it follows

$$d(x_{n+m}, x_n) \leq (c^{n+m-1} + c^{n+m-2} + \ldots + c^n)\, d(x_1, x_0),$$

which is a geometrical serie in $c$, and hence, we have

$$d(x_{n+m}, x_n) \leq \frac{c^n}{1-c}\, d(x_1, x_0).$$

Now, since $c < 1$, we have that for every $\varepsilon > 0$ there exists $n$, such that

$$\frac{c^n}{1-c}\, d(x_1, x_0) < \varepsilon,$$

and hence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and converges to some point $p \in X$. We have $p = T(p)$, and $p$ is a fixed point of $T$. $\square$

Note that the convergence of the recursion is independent on the initial point $x_0$, and that this is in fact a very constructive proof.

## The Lipschitz condition and continuity

In this subsection we will discuss how to define continuity of functions on metric spaces. Some of the considerations we make will not be necessary for the future developments, but I include this short passage anyway in order to state some definitions in connection with continuity, which will be crucial for us later on.

**Definition.** *Let $(X, d)$ be a metric space. A function $f : X \to X$ is said to be continous, if*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad s.t. \quad d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Of course, any strict contraction will be continous. To see this, let $\varepsilon > 0$ and choose $\delta$ such that $c\delta < \varepsilon$. We then have

$$d(f(x), f(y)) \leq cd(x, y) < \varepsilon.$$

Notice that we never used the fact that $c < 1$.

For most of our purposes, only to assume continuity is often a too weak. A stronger requirement would be differentiablity. However, satsifactory results can be drawn already from the Lipschitz class of functions. So let us define the elements in this set.

**Definition.** *Let $(X, d_X)$ and $(Z, d_Z)$ be metric spaces. A function $f : X \to Z$ is said to be Lipschitz continous ( or Lipschitz ), if*

$$\exists\, A > 0 \quad \forall\, x, y \in X \quad \Rightarrow \quad d_Z(f(x), (y)) < Ad_X(x, y).$$

Any strict contraction is obviously Lipschitz, and copying the proof above, we see that every Lipschitz function is continous. However, there are continous functions which are not Lipschitz. To see this, let us consider the following example.

**Example.** *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 3x^{\frac{2}{3}}$. Let us choose $d(x, y) = |x - y|$. The function $f$ is not Lipschitz, which can be easily seen. Let $y = 0$, and consider*

$$\frac{|f(x) - f(0)|}{|x - 0|} = 3\frac{|x|^{\frac{2}{3}}}{|x|} = 3|x|^{-\frac{1}{3}},$$

*which does not stay bounded by a constant $A$. Notice that $f$ is nevertheless continous.*

## Metrics on function spaces

We will soon have to measure distances between certain functions. For this reason we define a metric on the space of continous functions on a closed and bounded interval $I \subseteq \mathbb{R}$, taking values in $\mathbb{R}^n$. We call this space $C(I)$. There are several alternative metrics on $C(I)$, but we will choose one metric which also yields completeness to the space, namely

$$d(f,g) = \parallel f - g \parallel = \max_{t \in I} |f(t) - g(t)|, \qquad f,g \in C(I),$$

where $|\cdot|$ denotes the norm on $\mathbb{R}^n$. The reader is urged to show that this is in fact a metric. First note that the norm really is defined for all $f,g \in C(I)$, since the maximum of a continous function always exists on closed and bounded subsets of $\mathbb{R}$. The proof of completeness is omitted, but can be found in most elementary text books in analysis.

# Existence and uniqueness of solutions of ODE

## The generic IVP

Let $I = [-1,1]$, and define the open ball of radius $r > 0$ around $p \in \mathbb{R}^n$, with the norm ( absolute value ) $|\cdot|$ as the defined metric, as

$$B_r(p) = \{ x \in \mathbb{R}^n \, ; \, |x - p| < r \} .$$

We are now ready to state the fundamental theorem about the existence and uniqueness of solutions to the generic IVP on the standard form.

**Theorem.** *Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}_n$, and let $f : B_\varepsilon(x_0) \times I \longrightarrow \mathbb{R}^n$ be Lipschitz. Then there exists $a \in (0,1]$ and a continously differentiable function $x : [-a,a] \longrightarrow \mathbb{R}^n$ such that*

$$\dot{x} = f(x,t), \qquad x(0) = x_0.$$

Let us introduce the notation $I_a$ for the closed interval $[-a,a]$.

Before we prove this theorem, it may be educational to examine the necessity of the restrictions above. In most elementary books on differential equations, $f$ is assumed to be continously differentiable as well. However, and this is a very straightforward check, every continously differentiable map $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Lipschitz, so our results are strictly stronger.

**Example.** *Let $f(x,t) = 3x^{\frac{2}{3}}$, and study the IVP*

$$\dot{x} = f(x,t), \qquad x(0) = 0.$$

*As we have seen in the previous example, $f$ is not Lipschitz at $(x,t) = (0,t)$. Indeed, we can easily verify that both $x(t) = 0$ and $x(t) = t^3$ satisfy the equation above. These solutions are different in any neighbourhood of the point $0$. Hence, a solution exists, but it is not unique.*

## A proof of the main theorem

A naive approach to solving the standard problem described above would be to integrate both sides over some interval $[0, t]$. Respecting the initial value we get

$$\int_0^t \dot{x}(s)\, ds = \int_0^t f(x, s)\, ds,$$

or equivalently,

$$x(t) = x_0 + \int_0^t f(x, s)\, ds.$$

Of course, this is just a reformulation of the original problem, and we have not gain any new information about the actual solution, if such exists. However, this new formulation is not completely worthless. We can define the operator $T$ on $C(I)$ as

$$(Tx)(t) = x_0 + \int_0^t f(x, s)\, ds, \qquad x \in C(I).$$

In terms of this operator, solving the standard problem is just equivalent to finding a fixed point to the operator $T$. Indeed, if $x$ is a fixed point to $T$, then $x$ must be continously differentiable, and

$$x(t) = x_0 + \int_0^t f(x, s)\, ds \Rightarrow \dot{x}(t) = f(x, t).$$

We will show that this operator is a strict contraction on $\mathbb{C}(I_a)$ for some small $a > 0$.

*Proof.* Let $f$ be Lipschitz, and $x, y \in C(I)$.

$$\parallel Tx - Ty \parallel = \parallel \int_0^t f(x, s)\, ds - \int_0^t f(y, s)\, ds \parallel \leq \int_0^t \parallel f(x, s) - f(y, s) \parallel\, ds.$$

Now using the Lipschitz condition, we know that there exists $A$ such that for all $x, y \in I$ , $||f(x, s) - f(x, s)|| \leq A(s)||x - y||$, where $A$ is some bounded function of $s$ on an interval $I_a = [-a, a]$, for some small $a > 0$. Let us set $A = \sup_{s \in I_a} A(s)$. This $A$ will of course be finite. Hence, if $t = a$, we have

$$\parallel Tx - Ty \parallel \leq Aa||x - y||.$$

We can choose $a$ such that $aA < 1$. This value of $a$ makes $T$ into a strict contraction on $C(I_a)$. According to the the well-known fixed point theorem of Banach, $T$ will have an unique fixed point in $C(I_a)$ which, from our reasoning above, will be the unique solution to the standard problem. $\square$

*Remark.* Notice that the value of $a$ may be very small if $A$ is large, which is the case if $f$ has a lot of bumbs or some other kind of irregularity.