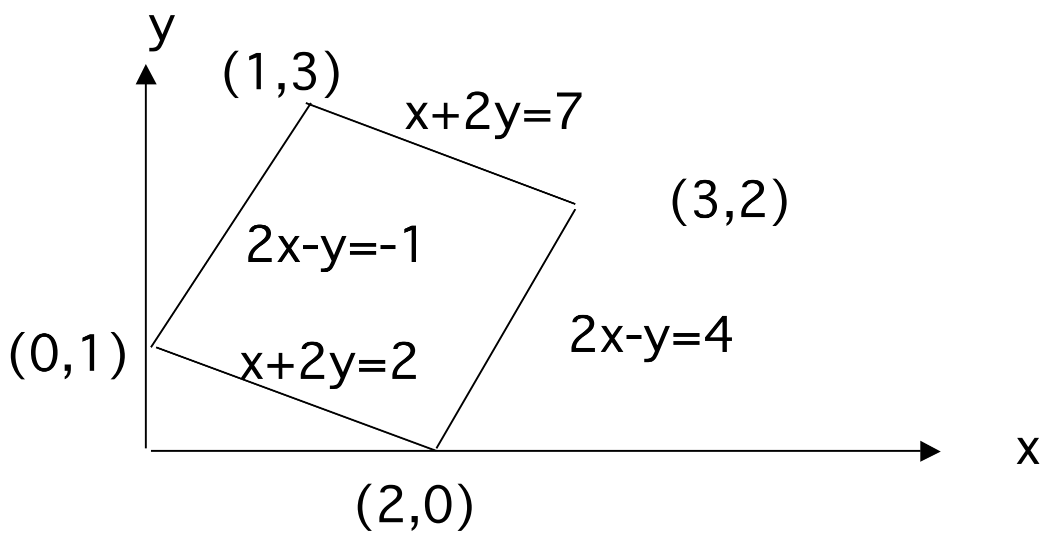
921.e.

$$V = \iint_D (2x^2 - 2y^2 + 3xy)\,dxdy$$

D är fyrhörningen med hörnen

$(0,1), (1,3), (2,0)$ och $(3,2)$.



Integranden $2x^2 - 2y^2 + 3xy = (2x - y)(x + 2y)$.

$$\begin{cases} u = 2x - y \\ v = x + 2y \end{cases} \quad \frac{d(u,v)}{d(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$dxdy = \left| \det\left(\frac{d(x,y)}{d(u,v)}\right) \right| dudv = \frac{dudv}{\left| \det\left(\frac{d(u,v)}{d(x,y)}\right) \right|} = \frac{dudv}{5}$$

$$D_{uv} = \left\{(u,v) : -1 \le u \le 4,\ 2 \le v \le 7\right\}$$

$$V = \iint_{D_{uv}} uv \frac{dudv}{5} = \frac{1}{5} \int_{u=-1}^{4} udu \int_{v=2}^{7} vdv = \frac{1}{5} \frac{16-1}{2} \frac{49-4}{2} = \frac{135}{4}$$

921.g.

$$V = \iint_D \frac{1}{x^2} \ln \frac{y}{x} dxdy$$

$$D = \left\{ (x,y) : 1 \le x + y \le 2,\ 1 \le \frac{y}{x} \le 2 \right\}$$

$$\begin{cases} u = x + y \\ v = \dfrac{y}{x} \end{cases} \qquad \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{vmatrix}$$

$$dxdy = \left| \det\left(\frac{d(x,y)}{d(u,v)}\right) \right| dudv = \frac{dudv}{\left| \det\left(\frac{d(u,v)}{d(x,y)}\right) \right|} = \frac{dudv}{\frac{x+y}{x^2}}$$

$$D_{uv} = \left\{ (u,v) : 1 \le u \le 2,\ 1 \le v \le 2 \right\}$$

$$\frac{1}{x^2} dxdy = \frac{dudv}{u}$$

$$V = \iint_{D_{uv}} \ln v \frac{du\,dv}{u} = \int_{u=1}^{2} \frac{du}{u} \int_{v=1}^{2} \ln v\,dv$$

$$V = \{\ln 2\} \left(\left[v \ln v \right]_{v=1}^{2} - \int_{v=1}^{2} dv \right) = \ln 2(2 \ln 2 - 1)$$

921.j.

$$V = \iint_D x^2 \sqrt{1 + x^4 y^4}\, dxdy$$

$$D = \left\{ (x,y) : xy < 1,\ 0 < y < x < 2y \right\}$$

$$\begin{cases} u = xy \\ v = \dfrac{x}{y} \end{cases} \quad \frac{d(u,v)}{d(x,y)} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ \dfrac{1}{y} & -\dfrac{x}{y^2} \end{bmatrix}$$

$$dxdy = \left| \det\left(\frac{d(x,y)}{d(u,v)}\right) \right| dudv = \frac{dudv}{\left| \det\left(\dfrac{d(u,v)}{d(x,y)}\right) \right|} = \frac{dudv}{\left| \dfrac{-2x}{y} \right|}$$

$$D_{uv} = \left\{(u,v):0 \le u \le 1,\ 1 \le v \le 2\right\}$$

$$V = \iint_{D_{uv}} uv\sqrt{1-u^4}\,\frac{dudv}{|-2v|} = \frac{1}{2}\int_{u=0}^{1} u\sqrt{1-u^4}\,du$$

$$V = \left[\begin{array}{ll} t = u^2 & u = 1 \Rightarrow t = 1 \\ dt = 2udu & u = 0 \Rightarrow t = 0 \end{array}\right] = \frac{1}{2}\frac{1}{2}\int_{t=0}^{1}\sqrt{1-t^2}\,dt$$

$$V = \frac{1}{2}\frac{1}{2}\frac{\pi\cdot 1^2}{4} = \frac{\pi}{16}$$

921.k.

$$M = \iiint_K \left(\frac{x}{y} + \frac{y}{x^3} + \frac{z^2}{xy^3} \right) dxdydz$$

$$K = \left\{ (x, y, z) : 1 \le x \le 2,\ \ x \le y \le 2x, y \le z \le 2y \right\}$$

$$\begin{cases} u = x \\ v = \dfrac{y}{x} \\ w = \dfrac{z}{y} \end{cases} \qquad \frac{d(u, v, w)}{d(x, y, z)} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{-y}{x^2} & \dfrac{1}{x} & 0 \\ 0 & \dfrac{-z}{y^2} & \dfrac{1}{y} \end{pmatrix}$$

$$dxdydz = \left| \det\left(\frac{d(x,y,z)}{d(u,v,w)}\right) \right| dudvdw = \frac{dudvdw}{\left| \det\left(\frac{d(u,v,w)}{d(x,y,z)}\right) \right|} = \frac{dudvdw}{\left|\frac{1}{xy}\right|}$$

$$K_{uvw} = \left\{(u,v,w) : 1 \le u \le 2,\ 1 \le v \le 2, 1 \le w \le 2\right\}$$

$$M = \iiint_{K_{uvw}} \left(\frac{1}{v} + \frac{v}{u^2} + \frac{w^2}{u^2 v}\right) u^2 v\, dudvdw$$

$$M = \iiint_{K_{uvw}} \left(u^2 + v^2 + w^2\right) du\,dv\,dw$$

$$M = \left\{\text{pga symmetri}\right\} = 3 \iiint_{K_{uvw}} u^2\,du\,dv\,dw$$

$$M = \int_{u=1}^{2} 3u^2\,du \cdot \int_{v=1}^{2} dv \cdot \int_{w=1}^{2} dw$$

$$M = 2^3 - 1 = 7$$

921.å.

$$M = \iiint_K \left(x^2 + y^2 - z^2\right) dxdydz$$

$$K = \left\{(x, y, z): x^2 + y^2 + z^2 \le 1\right\}$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$K_{r\theta\varphi} = \left\{(r, \theta, \varphi): 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le \varphi < 2\pi\right\}$$

$$\frac{d(x,y,z)}{d(r,\theta,\varphi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

$$\det\left(\frac{d(x,y,z)}{d(r,\theta,\varphi)}\right) = r\sin\theta\sin\varphi(r\sin\theta \; \sin\theta\sin\varphi + \cos\theta \; r\cos\theta\sin\varphi) + \\ + r\sin\theta\cos\varphi(+r\sin\theta \; \sin\theta\cos\varphi + \cos\theta \; r\cos\theta\cos\varphi)$$

$$\det\left(\frac{d(x,y,z)}{d(r,\theta,\varphi)}\right) = r^2\sin\theta\left\{\sin^2\theta + \cos^2\theta\right\} = r^2\sin\theta$$

$$dxdydz = \left| \det\left(\frac{d(x,y,z)}{d(r,\square,\square)}\right) \right| dr\,d\square\,d\square = r^2 \sin\square\, dr\,d\square\,d\square$$

$$x^2 + y^2 \square z^2 =$$

$$= (r\sin\square\cos\square)^2 + (r\sin\square\sin\square)^2 \square (r\cos\square)^2 =$$

$$= r^2\left(\sin^2\square \square \cos^2\square\right)$$

$$M = \square\square\square\square_{K_{r\square\square}} \left(\sin^2\square \square \cos^2\square\right) r^4 \sin\square\, dr\,d\square\,d\square$$

$$M = \frac{1}{5} 2\pi \int_{\varphi=0}^{\pi} (1 - 2\cos^2\varphi)\sin\varphi \, d\varphi$$

$$M = \left\{ \begin{array}{l} t = \cos\varphi, \quad dt = -\sin\varphi \, d\varphi \\ \varphi = 0 \to t = 1, \quad \varphi = \pi \to t = -1 \end{array} \right\} = \frac{2\pi}{5} \int_{t=1}^{-1} (1 - 2t^2)(-1) \, dt$$

$$M = \frac{4\pi}{15}$$