# UNDERSTANDING THE DIAGONALIZATION PROBLEM

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Abstract. These notes are additional material to the course 5B1307, given fall 2003. The style may appear a bit coarse and consequently the student is encouraged to read these notes through a writing pencil and with a critical mind.

## §1.- Linear Maps

**1.1. Linear maps.** A map $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear map if

$$T(aX + bY) = aT(X) + bT(Y)$$

for all scalars $a$ and $b$, and all vectors $X$ and $Y$.

*1.1.1.* Note that $T(0) = 0$ for a linear map $T$.

**1.2. Example.** The zero map $T : \mathbf{R}^n \to \mathbf{R}^m$ that maps any vector $X$ to the zero vector 0 is linear. Any other constant map is not linear.

**1.3. Example.** When $n = m$ we have the identity map $T : \mathbf{R}^n \to \mathbf{R}^n$ that maps any vector $X$ to itself. The identity map is denoted with id or $\mathrm{id}_{\mathbf{R}^n}$.

**1.4. Example.** The map $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by sending an arbitrary vector $X = (x, y, z, w)$ to

$$T(X) = (2x - z, x + y + z, 2w - 5y)$$

is linear.

**1.5. Example.** Let $L$ be a given line in $\mathbf{R}^2$ that passes through the origin. The map $T : \mathbf{R}^2 \to \mathbf{R}^2$ that maps a vector $X$ to its reflection through the line $L$ is linear (draw some pictures to get a feeling for this particular map).

**1.6. Coordinate matrix.** We will make the notion of linear maps more concrete, and in fact classify the possible linear maps. To do so we introudce the following. Let $\beta = \{F_1, \dots, F_n\}$ be a fixed basis of $\mathbf{R}^n$. Any vector $X \in \mathbf{R}^n$ is then a linear combination

$$X = \sum_{i=1}^{n} a_i F_i.$$

The linear combination is unique, up to order, and consequently we define the *coordinate matrix* $[X]_\beta$, or $[X]$ for short, as the matrix

$$[X]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

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*1.6.1.* Note that a fixed vector $X$ have different coordinate matrices for different bases.

**1.7. Matrix representation.** Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map, and let $\beta = \{F_1, \dots, F_n\}$ be a basis of $\mathbf{R}^n$, and let $\gamma$ be a basis of $\mathbf{R}^m$. Furthermore, we let $A$ be the $(m \times n)$-matrix

$$A = Mat(T, \beta, \gamma) = [\, [\, T(F_1)\,]_\gamma \quad [\, T(F_2)\,]_\gamma \quad \cdots \quad [\, T(F_n)\,]_\gamma \,].$$

Look now carefully and understand the notation above; You apply the linear map $T$ to the vector $F_1$, and obtain a vector $T(F_1)$ in $\mathbf{R}^m$. That particular vector $T(F_1)$ is expressed in terms of the basis $\gamma$, yielding an $(m \times 1)$-matrix $[T(F_1)]_\gamma$ that will constitute the first column of $A$. Then you repeat the procedure with $F_2, \dots, F_n$.

We say that $A = Mat(T, \beta, \gamma)$ is the *matrix representation* of $T$ with respect to the bases $\beta$ and $\gamma$. If both $\beta$ and $\gamma$ are the standard bases, then we say that $A$ is the standard matrix representing $T$.

**Theorem 1.8.** *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map, and $\beta$ a basis for $\mathbf{R}^n$, and $\gamma$ a basis for $\mathbf{R}^m$. For any vector $X \in R^n$, we have the following formulae*

$$[\, T(X)\,]_\gamma = Mat(T, \beta, \gamma)[X]_\beta.$$

*Proof.* Let $\beta = \{F_1, \dots, F_n\}$ be a basis of $\mathbf{R}^n$ and write the vector$X = \sum_{i=1}^n a_i F_i$. As the map $T$ is linear we have that

$$T(X) = \sum_{i=1}^n a_i T(F_i).$$

Let now $A = Mat(T, \beta, \gamma)$. By taking brackets the above expression reads

$$[T(X)]_\gamma = A \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

As the matrix $[\, a_1 \quad a_2 \quad \cdots \quad a_n \,]^t$ is the coordinate matrix $[X]_\beta$ we have proven the formula.

*1.7.1. Remark.* Thus, when having fixed bases for $\mathbf{R}^n$ and $\mathbf{R}^m$, the whole business of linear maps is reduced to matrices and matrix multiplication. A linear map gives a matrix, and aa matrix determines a linear map - which you verify. However linear maps are not exactly the same as matrices, but linear maps with fixed bases $\beta$ and $\gamma$ are.

**1.8. Example.** Consider the the line $L = \{(x, y) \mid 2x - y = 0\}$, and the linear operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by reflection through $L$. We will represent that operator with a matrix. We chose the standard basis $\beta = \gamma = \{E_1, E_2\}$, where $E_1 = (1, 0)$, and $E_2 = (0, 2)$. Now, in order to find the standard matrix of $T$, we need to find the reflections of $E_1$ and $E_2$. By drawing accurate pictures you see that $T(5, 0) = (-3, 4)$ such that, by linearity,

$$T(E_1) = \frac{1}{5}(-3, 4).$$

Similarily you find that $T(E_2) = \frac{1}{5}(4,3)$. We consequently get that

$$A = [\,[T(E_1)]_\gamma \quad [T(E_2)]_\gamma\,] = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Hence, for any vector $X = (x,y) = x \cdot E_1 + y \cdot E_2$ we have that

$$[T(X)]_\gamma = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_\beta = \begin{bmatrix} \frac{-3x+4y}{5} \\ \frac{4x+3y}{5} \end{bmatrix}_\gamma.$$

**1.9. Composition.** Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map and $S : \mathbf{R}^m \to \mathbf{R}^k$ be another. The composition of $T$ and $S$ is denoted by $S \circ T : \mathbf{R}^n \to \mathbf{R}^k$ and is defined by taking a vector $X \in \mathbf{R}^n$ to

$$S \circ T(X) := S(T(X)).$$

Note that unless $k = n$, then we have not defined the composition of $T \circ S$.

*1.9.1. Remark.* The composition $S \circ T$ is to be read from the right to the left! First you apply $T$, thereafter $S$.

**Theorem 1.10.** *Let $\beta, \beta'$ and $\beta''$ be bases of $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^k$, respectively. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map, and let $A = Mat(T, \beta, \beta')$ be the matrix representing $T$. Let $S : \mathbf{R}^m \to \mathbf{R}^k$ be another linear map, represented by $B = Mat(S, \beta', \beta'')$. Then we have that the matrix representing the composition $S \circ T$ with respect to the bases $\beta$ and $\beta''$ is*

$$Mat(S \circ T, \beta, \beta'') = BA.$$

*Proof.* Left as an exercise for the reader.

*1.10.1.* Pay attention to which order the two matrices $A$ and $B$ are to be multiplied.

## §2.- The diagonalization problem

**2.1. Change of basis.** We now rally into a complicated part of linear algebra, the subject of basis. But why care of other bases than the standard basis as the others seem so complicated. For motivation we redo the Example 1.8.

**2.2. Example.** As in Example 1.8, we let $L$ be the line cut out by the equation $2x - y = 0$, and we let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be given as the reflection through that line. We will represent the linear map $T$ with respect to another basis than the standard basis. Let $F_1$ be any point on the line $L$, and different from $(0,0)$, say $F_1 = (1,2)$. And let $F_2 = (-4,2)$. If you draw the picture you will see that $F_2$ lies on a line $N$ that passes through origin, and which is perpendicular to $L$. Instead of our particular $F_2$, we could have chosen any point on $N$, different from $(0,0)$.

You convince yourself that $F_1$ and $F_2$ are linearly independent (they do not lie on the same line), and consequently $\beta = \{F_1, F_2\}$ is a basis. We shall now find the matrix $A = Mat(T, \beta, \beta)$ representing $T$. To do so we need to find the reflections of $F_1$ and $F_2$, and thereafter express the reflections $T(F_1)$ and $T(F_2)$ as linear combination of $F_1$ and $F_2$. However, this is not at all hard as

$$T(F_1) = F_1 \quad \text{and} \quad T(F_2) = -F_2,$$

and consequently we have that the matrix we are looking for is

$$D = [\,[T(F_1)_\beta][T(F_2)]_\beta\,] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

*2.2.1. The bottom line.* The point with the two examples 1.8 and 2.2 is that they show different representations of the same problem. The matrix $B$ of Example 2.2 is much more easy to work with than the matrix $A$ of Example 1.8.

**2.3. Change of basis matrix.** Let $\beta$ and $\gamma$ be two bases for $\mathbf{R}^n$. The matrix representation $P = P(\mathrm{id}, \beta, \gamma)$ of the identity map $\mathrm{id} : \mathbf{R}^n \to \mathbf{R}^n$ with respect to the bases $\beta$ and $\gamma$ is the *change of basis matrix* from $\beta$ to $\gamma$.

**2.4. Example.** The two vectors $F_1 = (1, 2)$ and $F_2 = (1, 0)$ form a basis $\beta$ for the plane $\mathbf{R}^2$. The formulae for the change of basis matrix from $\beta$ to the standard basis $\gamma = \{E_1 = (1, 0), E_2 = (0, 1)\}$ is

$$P = [\,[\mathrm{id}(F_1)]_\gamma \quad [\mathrm{id}(F_2)]_\gamma\,].$$

As we have $\mathrm{id}(F_1) = F_1 = 1 \cdot E_1 + 2 \cdot E_2$ and $\mathrm{id}(F)2) = F_2 = 1 \cdot E_1$ we get

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

**2.5. Example.** We use the notation from Example 2.4. We will describe the change of basis matrix from the standard basis to the basis $\beta = \{F_1, F_2\}$. We have

$$E_1 = 1 \cdot F_2 \quad \text{and} \quad E_2 = \frac{1}{2} \cdot F_1 - \frac{1}{2} \cdot F_2,$$

and consequently we obtain that the change of basis matrix is

$$Q = [\,[\mathrm{id}(E_1)]_\beta \quad [\mathrm{id}(E_2)]_\beta\,] = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

*2.5.1.* Note that the matrix $Q$ of Example 2.5 equals the inverse of $P$; the matrix of Example 2.4.

**Proposition 2.6.** *If $P$ is the change of basis matrix from a basis $\beta$ to $\gamma$, then $P^{-1}$ is the change of basis matrix from $\gamma$ to $\beta$.*

*Proof.* The result is a consequence of the composition Theorem 1.10.

**2.7. The equation $A = P^{-1}DP$.** Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map and let $\mathrm{id} : \mathbf{R}^n \to \mathbf{R}^n$ denote the identity map. We clearly have that the two compositions $\mathrm{id} \circ T = T \circ \mathrm{id}$, or equivalently that the diagram

$$\begin{tikzcd} \mathbf{R}^n \arrow[r, "T"] \arrow[d, "\mathrm{id}"'] & \mathbf{R}^n \arrow[d, "\mathrm{id}"] \\ \mathbf{R}^n \arrow[r, "T"] & \mathbf{R}^n \end{tikzcd}$$

is commutative. We fix now a basis $\beta$ for $\mathbf{R}^n$ and let $A = Mat(T, \beta, \beta)$ denote the matrix representing $T$ with respect to $\beta$. If $\gamma$ is another basis, then we let $D = Mat(T, \gamma, \gamma)$ represent $T$ with respect to $\gamma$ and we let $P = Mat(\mathrm{id}, \beta, \gamma)$ denote the base change matrix from $\beta$ to $\gamma$.

In our diagram above we think of having fixed the basis $\beta$ on the top row, and $\gamma$ on the bottom row. Theorem 1.10 combined with the identity $\mathrm{id} \circ T = T \circ \mathrm{id}$ now yield the identity of matrices $PA = DP$. Or equivalently that

$$A = P^{-1}DP \quad \text{and } PAP^{-1} = D. \tag{2.7.1}$$

*2.7.1. Remark.* There is no point to remember the expression 2.7.1, but you should remember the set up and the diagram above.

**2.8. The diagonalization problem.** The problem of diagonalization is the following. Given a $(n \times n)$-matrix $A$, does there exist an invertible matrix $P$ such that $PAP^{-1}$ is a diagonal matrix. The way to understand the problem is to think of the matrix $A$ as one representation of a linear map $T$ with respect to some basis. Can we find another basis such that the matrix representation becomes a diagonal matrix?

**2.9. Example.** Consider the matrix $A$ of Example 1.8. The matrix

$$A = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

we know from Example 2.2 is diagonalizable. Because the matrix $A$ was the representation of a linear map $T$ that we saw in Example 2.2. had a representation by a diagonal matrix. Thus if $P$ is the base change matrix from the standard basis (that gave the matrix $A$) to the basis $\beta$ in Example 2.2, then we obtain that $D = PAP^{-1}$.

Let us in this example verify the statement above. Thus we need to find the matrix $P = Mat(\mathrm{id}, \beta', \beta)$, where $\beta'$ is the standard basis. One way to find $P$ is actually to first find $P^{-1}$; the base change matrix from $\beta$ to $\beta$. It is easy to read off the coordinate matrices for the vectors $F_1$ and $F_2$ in the standard basis. We have $F_1 = (1, 2)$ and $F_2 = (-4, 2)$ such that

$$P^{-1} = Mat(\mathrm{id}, \beta, \beta) = \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix},$$

hence

$$P = (P^{-1})^{-1} = \begin{bmatrix} \frac{2}{10} & \frac{4}{10} \\ -\frac{2}{10} & \frac{1}{10} \end{bmatrix}.$$

Finally one checks that

$$PAP^{-1} = \begin{bmatrix} \frac{2}{10} & \frac{4}{10} \\ -\frac{2}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D.$$

**2.10. Iterations.** One reason for the search for diagonal matrices $D$ is the fact that it is easy to compute iterations $D^r$; you simply get a diagonal matrix where the diagonal elements are the diagonal elements of $D$ raised to the power $r$. Imagine that you are given a matrix $A$ and want to compute $A^r$, for some given postive integer $r$. Assume furthermore that you have solved the diagonalization problem and found a matrix $P$ such that $PAP^{-1} = D$ is diagonal. As we have that $A = P^{-1}DP$ we get that

$$A^r = (P^{-1}DP)^r = P^{-1}D^rP. \tag{2.10.1}$$

To compute $A^r$ we nee to compute the product of the three matrices $P^{-1}, D^r$ and $P$.

**2.11. Example.** Returning to the matrix $A$ of Example 2.8. Say we want to compute, for some reason, $A^{205}$. From the Example 2.8 we have solved the diagonalization problem such that $A = P^{-1}DP$, and such that

$$A^{205} = P^{-1}D^{205}P = P^{-1}DP = A.$$

As surprisingly simple solution! But what have we actually done - we have done 205 reflections through a given fixed line $L$ in $\mathbf{R}^2$. Two, and any even number of reflections acts as the identity, i.e. nothing happens. Thus any odd number of reflections, as 205, equals one reflection.

## §3.- Eigenvectors and the Cayley-Hamilton Theorem

**3.1. Eigen -values and -vectors.** We have so far stated the diagonalization problem, but not yet attacked it. We observe the following simple, but important fact. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map, and $\beta$ is a basis such that the matrix representation of $T$ is a diagonal matrix then we have for any vector $F$ of the basis $\beta$ that $T(F) = \lambda F$, for some scalar $\lambda$. We are thus lead to the following definition.

**3.2. Definition.** Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. If there exists a non-zero vector $X$ such that $T(X) = \lambda X$ for some scalar $\lambda$ then we say that $X$ is an eigenvector and that $\lambda$ is the corresponding eigenvalue.

**3.3.** Note that the identity $T(X) = \lambda X$ is equivalent with $(\lambda \cdot \mathrm{id} - T)(X) = 0$. We are thus looking for the kernel of the linear map $S = \lambda \cdot \mathrm{id} - T$. If we fix a basis $\beta$ for $\mathbf{R}^n$ and denote by $A$ the matrix representing $T$, then the matrix $\lambda 1_n - A$ represents the linear map $S = (\lambda \cdot \mathrm{id} - T)$ -where $1_n$ is the identity matrix. Furthermore we have that $\lambda$ is an eigenvalue if there is a non-trivial solution to $(\lambda \cdot 1_n - A)[X]_\beta = 0$. A matrix has non-trivial solution or a non-trivial kernel if and only if its determinant is zero. Consequently the eigenvalues of $T$ are precisely those values $\lambda$ such that

$$\det(\lambda 1_n - A) = 0.$$

**3.4. Definition.** Let $A$ be an $(n \times n)$-matrix. The characteristic polynomial of $A$ is a polynomial in $\lambda$ defined as

$$c_A(\lambda) = \det(\lambda \cdot 1_n - A).$$

*3.4.1. Remark.* Note that if $A$ is a matrix representing a linear map $T$, then the zeros of the characteristic polynomial $c_A(\lambda)$ are precisely the eigenvalues of $T$.

*3.4.2. Remark.* Note that the degree of the characteristic polynomial $c_A(\lambda)$ equals $n$; the number of rows and columns of $A$.

**3.5. Example.** Consider the matrix

$$A = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

The characteristic polynomial $c_A(\lambda)$ is

$$c_A(\lambda) = \det \begin{bmatrix} \lambda + \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5}\lambda & -\frac{3}{5} \end{bmatrix} = \lambda^2 - 1.$$

Consequently the eigenvalues are $\lambda = 1$ and $\lambda = -1$.

**Lemma 3.6.** *Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. The characteristic polynomial $c_A(\lambda)$ is independent of the matrix $A$ representing $T$.*

*Proof.* Let $\beta$ be a basis and $A = Mat(T, \beta, \beta)$, and $\gamma$ another basis with $D = Mat(T, \gamma, \gamma)$. We need to show that their characteristic polynomials $c_A(\lambda) = c_D(\lambda)$ are equal. Let $P$ be the base change matrix from the base $\beta$ to $\gamma$. As explained in (2.7.1) we then obtain the equality $D = PAP^{-1}$. Consequently we have that

$$\begin{aligned} c_D(\lambda) &= \det(\lambda 1_n - D) = \det(\lambda 1_n - PAP^{-1}) \\ &= \det(P(\lambda 1_n - A)P^{-1}) = \det(P) c_A(\lambda) \det(P^{-1}) = c_A(\lambda). \end{aligned}$$

And we have proven our claim.

Because of the above lemma we talk about the characteristic polynomial $c_T(\lambda)$ of a linear map $T : \mathbf{R}^n \to \mathbf{R}^n$ without specifying a matrix $A$ that represents the map $T$.

**3.7. Example.** We have that the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

has characteristic polynomial $c_D(\lambda) = \lambda^2 - 1$. Equal to the characteristic polynomial $c_A(\lambda)$ of the matrix $A$ of Example 3.4. Indeed, both matrices represent the same linear operator $T$ described in Example 1.8 and Example 2.2.

**Theorem 3.8.** *[The Cayley-Hamilton Theorem] Let $A$ be a $(n \times n)$-matrix and $c_A(\lambda)$ its characteristic polynomial. Then we have that the matrix $c_A(A)$ is the zero-matrix, that is $c_A(A) = 0$.*

*Proof.* Recall that if $M$ is any $(n \times n)$-matrix and $C(M)$ its conjugate, then we have the formulae $\det(M) 1_n = M\mathrm{C}(M)$. Thus with $M = \lambda 1_n - A$ we get the following equation

$$c_A(\lambda) 1_n = (\lambda 1_n - A)\mathrm{C}(M) \tag{3.8.1.}$$

By taking out the variable $\lambda$ we can write the conjugate matrix of $\lambda 1_n - A$ as

$$\mathrm{C}(M) = C_0 + \lambda C_1 + \cdots + \lambda^{n-1} C_{n-1}.$$

In the above expression the matrices $C_0, \ldots, C_{n-1}$ contain no positive powers of $\lambda$, simply scalars. Multiplying the above expansion of the conjugate with $M = \lambda 1_n - A$ yields the following expression of matrices

$$\lambda C_0 - AC_0 + \lambda^2 C_1 - \lambda AC_1 + \cdots + \lambda^n C_{n-1} - \lambda^{n-1} AC_{n-1}. \tag{3.8.2}$$

In the expression (3.8.2) we substitute $\lambda$ with $A$ and obtain zero. As (3.8.2) is the right hand side of (3.8.1) we get that $c_A(A)1_n$ is zero. Thus $c_A(A)$ is zero, and we have proven the statement.

## §4.- Eigenspaces

We continue with our attack on the diagonalization problem.

**Lemma 4.1.** *Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map, and let $X_1, \ldots, X_k$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the vectors $X_1, \ldots, X_k$ are linearly independent.*

*Proof.* We prove the statement by induction on the number $k$ of eigenvectors. If $k = 1$ there is nothing to prove as one non-zero vector is linearly independent. Assume the statement is valid for $k$ number of eigenvectors, we shall prove the statement for $k + 1$ number of eigenvectors. Given a linear combination

$$t_1 X_1 + \cdots + t_{k+1} X_{k+1} = 0. \tag{4.1.1}$$

In order to show that the vectors $X_1, \ldots, X_{k+1}$ are linearly independent we need to show that $t_1 = \cdots = t_{k+1} = 0$. We apply the operator $T$ to the expression (4.1.1). On one hand we get that $T(0) = 0$ since $T$ is linear. On the other hand we get, since the $X_i$'s are eigenvectors that

$$0 = T(t_1 X_1 + \cdots + t_{k+1} X_{k+1}) = t_1 \lambda_1 X_1 + \cdots + t_{k+1} \lambda_{k+1} X_{k+1}. \tag{4.1.2}$$

We manipulate furthermore the expression (4.1.1) above by multiplication with the number $\lambda_{k+1}$ and obtain that

$$0 = t_1 \lambda_{k+1} X_1 + \cdots + t_{k+1} \lambda_{k+1} X_{k+1}. \tag{4.1.3}$$

We then subtract the expression (4.1.1) from the expression (4.1.3), zero minus zero, and get after collecting the terms that

$$0 = t_1(\lambda_1 - \lambda_{k+1}) X_1 + + \cdots t_k(\lambda_k - \lambda_{k+1}) X_k.$$

However, in the latter expression we only have $k$ number of eigenvectors (corresponding to distinct eigenvalues). By the induction hypothesis these are linarly independent, and consequently the coefficients $t_i(\lambda_i - \lambda_{k+1}) = 0$, for $i = 1, \ldots, k$. As the eigenvalues are supposed to be distinct it follows that $t_1 = \cdots = t_k = 0$. And finally by (4.1.1) we then also have that $t_{k+1} = 0$, and the vectors are linearly independent.

**Theorem 4.2.** *Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. Assume that $T$ has $n$-distinct eigen values. Then there exists a basis $\beta$ of $\mathbf{R}^n$ consisting entirely of eigenvectors. Furthermore, the matrix representation of $T$ with respect to $\beta$ is then a diagonal matrix.*

*Proof.* Let $\beta = \{X_1, \ldots, X_n\}$ be eigenvectors corresponding to the $n$-distinct eigenvalues that we assume $T$ posesses. By Lemma (4.1) the vectors $\beta$ are linearly independent, and since there are $n$ of them they form a basis. This proves the first claim of the theorem, and the second follows from the definition of eigenvectors.

*4.2.1.* Note that when in the situation as of the theorem, then the entries of the diagonal matrix are precisely the eigenvalues of $T$.

The above Theorem does not solve the diagonalization problem entirely, but partially. It is possible to diagonalize even though if the number of distinct eigenvalues does not equal the dimension of the vector space. An example of that is the identity operator. What we have is the following

**Theorem 4.3.** *The following two statements are equivalent for a linear map $T : \mathbf{R}^n \to \mathbf{R}^n$.*

(1) $\mathbf{R}^n$ *has a basis of eigen vectors.*
(2) $T$ *is representable by a diagonal matrix.*

*Proof.* Clear, isn't it.

**4.4. Example - Multiplicity.** Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear map, whose matrix representation with respect to some basis is

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

You may check that the characteristic polyomial $c_A(\lambda) = (\lambda - 1)(\lambda - 2)^2$. Hence there are only two distinct eigen values, namely $\lambda = 1$ and $\lambda = 2$, whereas the dimension of the vector space is $n = 3$. We note that the root $\lambda = 2$ appears twice in the factorization of $c_A(\lambda)$ and we say that $\lambda = 2$ has multiplicity two. The other root $\lambda = 1$ has multiplicity 1.

**4.5. Eigenspaces.** Let $\lambda$ be an eigen value of $T$. The kernel

$$ES_\lambda := \mathrm{Null}(\lambda\,\mathrm{id} - T)$$

is the eigenspace corresponding to $\lambda$.

*4.5.1. A geometric approach.* If $X \in \mathbf{R}^n$ is a non-zero vector then $L(X) = \mathrm{Span}(X)$ is the line through $X$ and origin. Note that if $X$ is an eigenvector of $T$, then any point on $L(X)$ is also an eigenvector. Thus, one way to visualize teh eigenspaces are to look for lines that are invarian under the action of $T$; If there is a line (through origo) that is mapped to itself by $T$, not necessarily pointvise, then points on the line are eigenvectors.

**4.6. Example.** Let $L \subset \mathbf{R}^2$ be a line through the origin, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ denote the reflection through that particular line. What are the eigenvalues and eigenvectors of $T$? We look geometrically at the linear map $T$ and realize that the line $L$ itself is invariant under the action of $T$. Actually, for any $X \in L$ we have that $T(X) = X = 1 \cdot X$, thus points on $L$ are eigenvectors with eigenvalue $\lambda = 1$.

Furthermore, we realize that the line $N$ that passes through origin and is perpendicular to $L$ is invariant. Because for any $X \in N$ we have

$$T(X) = -X,$$

that is an eigenvector with eigenvalue $\lambda = -1$. No other lines are invariant under $T$.

**Lemma 4.7.** *Let $\lambda = e$ be an eigenvalue of multiplicity $m$. Then $dim ES_\lambda \leq m$.*

*Proof.* Let $d = \dim ES_{\lambda=e}$, and let $\{X_1, \ldots, X_d\}$ be a basis of $ES_{\lambda=e} \subseteq \mathbf{R}^n$. We extend that basis to a basis $\beta = \{X_1, \ldots, X_d, Y_1, \ldots, Y_{n-d}\}$ of $\mathbf{R}^n$. The matrix representation of $T$ with respect to $\beta$ then is of the block form

$$M = Mat(T, \beta, \beta) = \begin{bmatrix} e1_d & B \\ 0_{n-d} & A \end{bmatrix},$$

where $1_d$ is the $(d \times d)$ identity matrix, $0_{n-d}$ is the zero matrix of sice $(n-d \times d)$. The matrices $B$ and $A$ we do know very little about, and we do not care for the moment. Because of the zero block $0_{n-d}$ we get that the characteristic polynomial $c_M(\lambda) = (\lambda - e)^d c_A(\lambda)$. But then we have that the multiplicity of the root $\lambda = e$ in the characteristic polynomial $c_M(\lambda)$ of $T$ has multiplicity at least $d$, and we have proven our claim.

**4.8. Example.** We continue the Example 4.4. We have found the eigenvalues and will here describe the eigenspaces. Everything will be with respect to the basis $\beta = \{F_1, F_2, F_3\}$ that we have assumed fixed. The eigen space corresponding to $\lambda = 1$ is given by the matrix equation

$$(1 - A)[X]_\beta = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

The solutions to this equation is of the form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

where $t$ is an arbitrary scalar. Thus a basis for the eigenspace $ES_1$ is the vector $X_1 = -2F_1 + F_2 + F_3$.

The eigenspace corresponding to $\lambda = 2$ is given by the matrix equation

$$(2 - A)[X]_\beta = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} [X]_\beta = 0.$$

The solutions are of the form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix},$$

where $s$ and $t$ are arbitrary scalars. A basis for $ES_2$ is then given by the two vectors $X_2 = -F_1 + F_3$ and $Y_2 = F_2$.

*4.8.1. Remark.* In this example we have that the dimension of the eigen spaces equals the multiplicities of the corresponding eigenvalues.

*4.8.2. Remark.* Note that in the Example 4.6 the space $\mathbf{R}^3$ has a basis of eigen vectors, for instance $\{X_1, X_2, Y_2\}$ and consequently the operator $T$ is diagonalizable.

**4.9. Orthogonal diagonalization.** In the last section we will give a complete answer to those operators that are not only diagonalizable, but orthogonally diagonalizable. One problem with diagonalization is the need of computing the inverse $P^{-1}$ of a change of basis matrix $P$. The computation of the inverse is in general hard or at least time consuming, but some matrices have a nice formula for the inverse. We say that a matrix $P$ is orthogonal if $P$ is invertible and its inverse is its transpose, $P^{-1} = P^t$.

**Theorem 4.10.** *Let $P$ be an $(n \times n)$-matrix. The following statements are equivalent.*

(1) *$P$ is invertible with inverse $P^{-1} = P^t$.*
(2) *The columns of $P$ are orthonormal.*
(3) *The rows of $P$ are orthonormal.*

*Proof.* We show that (1) is equivalent with (2), the proof of the equivalence of (1) and (3) is similar. Assume that the inverse of $P$ is its transpose $P^t$. We then have that $P^tP = 1_n$. But that means that the $i$'th row of $P^t$ dotted with the $j$'th column of $P$ equals the Kronecker $\delta_{i,j}$ which is zero if $i \neq j$, and 1 if $i = j$. As the $i$'th row of $P^t$ is precisely the $i$'th coloumn of $P$, we have that the coloumns of $P$ are orthonormal. Conversely, assume that the coloumns of $P$ are orthonormal. Take the transpose of $P$ and consider the product $P^tP$. As the coloumns are orthonormal the product $P^tP$ is the identity matrix, hence $P$ is invertible with inverse $P^{-1} = P^t$.

**Theorem 4.11.** *[Real Spectral Theorem] Let $A$ be an $(n \times n)$-matrix representing a linear map $T : \mathbf{R}^n \to \mathbf{R}^n$. The following are equivalent*

(1) *$A$ is orthogonally diagonalizable.*
(2) *$\mathbf{R}^n$ has an orthonormal basis of eigen vectors.*
(3) *$A$ is symmetric.*

*Proof.* Assume that 1) holds and let $P$ be an orthogonal matrix that diagonalizes $A$. We then have that $PAP^t = D$ is a diagonal matrix and that $A = P^tDP$. We then get that $A^t = P^tDP = A$, and the matrix $A$ is symmetric. Thus 1 implies 3).

We next show by induction on the size $n$ of the matrix $A$ that 3) implies 1). When $n = 1$ the case is clear, and we assume that 3) implies 1) for size $(n \times n)$ matrices. As the matrix $A$ is symmetric it has at least on real eigenvalue $\lambda_1$ ([??]), and we let $X_1$ be a corresponding eigenvector of length $|X_1| = 1$. We extend $X_1$ to a basis $\{X_1, Y_2, \ldots, Y_n\}$ of $\mathbf{R}^n$, a basis we can assume is orthonormal. The matrix $P$ with coloumns $X_1, Y_2, \ldots, Y_n$ is then orthogonal, and we have the following block form

$$P^tAP = P^{-1}AP = \begin{bmatrix} \lambda_1 & B \\ 0 & A_1 \end{bmatrix}.$$

Since $A$ is symmetric we have that $P^tAP$ is symmetric, and it follows that $B = 0$. The lower block $A_1$ is symmetric of size $(n \times n)$ and can consequently by the induction hypothesis be orthogonally diagonalized. Consequently there exists an orthogonal matrix $Q$ such that $Q^tA_1Q = D_1$ is diagonal. The matrix $P' = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ is clearly orthogonal and since the product of orthogonal matrices remains orthogonal we also have $PP'$ is orthogonal. We finally note that

$$\begin{aligned}(PP')^tA(PP') &= P'^tP^tAPP' \\ &= \begin{bmatrix} 1 & 0 \\ 0 & Q^t \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_1 \end{bmatrix}\end{aligned}$$

is a diagonal matrix, hence orthogonally diagonalizable.

We have left to prove the equality of 1) and 2). Let $P$ be an orthogonal matrix that diagonalizes $A$, and let $X_1, \dots, X_n$ denote its coloumns. It follows by Theorem 4.8 that the set $\{X_1, \dots, X_n\}$ is an orthonormal basis of $\mathbf{R}^n$ if and only if $P$ is an orthogonal matrix. Furthermore, we may think of $P$ as the base change matrix from $\{X_1, \dots, X_n\}$ to the standard matrix, and the set $\{X_1, \dots, X_n\}$ is a basis of eigenvectors if and only if $D = PAP^{-1}$ is diagonal. The two stated equalities show that 1) is equivalent with 2).