KTH Matematik

Exam 3
5B1309 Algebra g.k.
26 Februari, 2006

(1) (3 pts) (motivate your answer!) Is a group G of cardinality $|G| = 30$ simple? $|G| = 3 \cdot 5 \cdot 2$.

Let N_i be the number of i-Sylow subgroups. Then $N_2 = 1, 3, 5, 15$ and $N_3 = 1, 10$, $N_5 = 1, 6$. If one of them is one than the corresponding Sylow subgroups, being unique, will be normal in G and thus G will not be simple.

The intersection of two distinct i-Sylow subgroups (i=1,2,3) is necesserely only the identity element, since the cardiality has to devide the prime i.

Assuming that $N_3 = 10, N_5 = 6$ G would hane 20 distinct elements of order 3 and 24 distinct elements of order 5, which is impossible.

We conclude that G has a normal subgroup of cardinality 3 or 5 and thus it is NON SIMPLE.

(2) (3 pts) (motivate your answer!) Find the solutions of the modular equation:

$$15x \equiv_{12} 27$$

Since the $g.c.d(15, 12) = 3$ and 3 devides 27, the equation has 3 distinct solutions.

We know that $[x]_1 5$ is a solution if and only if $[x]_4$ is the unique solution of

$$5x \equiv_4 9$$

Thus $[x]_4 = [5]_4^{-1}[9]_4 = [1]_4^{-1}[1]_4 = [1]_4$.

It follows that $[1]_{12}, [1 + 4]_{12}, [1 + 8]_{12}$ are the three solutions.

(3) (3 pts) Let R be a ring. An element $a \in R$ is an *idempotent element* if $a^2 = a$.
Show that the only idempotent elements in an integral domain are 0 and 1.

Assume that a is an idempotent element:

$$a^2 = a \Leftrightarrow a \cdot a + (-a \cdot 1_R) = 0_R \Leftrightarrow a \cdot (a + (-1_R)) = 0_R$$

Since R does not have zero-divisors it is $a = 0_R$ or $a + (-1_R) = 0_R$ and thus $a = 0$ or $a = 1$.