Final test, 5B1309, ALGEBRA G.K.

You can get a maximum of 65 points for this exam. To achieve this maximum you can either solve the first eight problems or the extra problem and some of the first problems so the points add up to 65.

The exam can be written in english. Du får också skriva i svenska.

You should include all the arguments and explanations.

Problem 1 Consider the following permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 7 & 5 & 2 & 8 & 1 & 9 & 10 & 6 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 6 & 5 & 8 & 10 & 2 & 1 & 9 & 4 & 3 \end{pmatrix}$$

(a) (**5 points**). Are they odd or even permutations?

(b) (**5 points**). Are σ and τ conjugate?

Problem 2 (**10 points**). What is a maximum possible order of an element in the permutation group S_{12}? Find two permutations in S_{12} with such an order.

Problem 3

(a) (**5 points**). Show that the rings $\mathbf{Z}/36$ and $\mathbf{Z}/9 \times \mathbf{Z}/4$ are isomorphic.

(b) (**5 points**). Identify the group of invertible elements in $\mathbf{Z}/36$.

Problem 4 (**10 points**). How many abelian groups (up to isomorphism) are there of order 36? How many are there of order 54?

Problem 5 (**5 points**). Use Sylow theorem to show that any group of order 160 is NOT simple.

Problem 6 (**5 points**). Recall that $\phi(n)$ denotes the number of non-zero divisors in the ring $\mathbf{Z}/n$. Let p be a prime number. Calculate $\phi(p^2)$.

Problem 7 (**5 points**). Use Fermat's theorem to find the remainder of 3^{47} when it is divided by 23.

Problem 8 Consider the polynomial $f = X^3 + 3X + 2$.

(a) (**3 points**). Is f irreducible in $\mathbf{Z}/4[X]$?

(b) (**7 points**). Is f prime in $\mathbf{Z}[X]$? Is f prime in $\mathbf{Q}[X]$? Is f irreducible in $\mathbf{R}[X]$?

Extra problem (**30 points**). Show that any group of order 1645 is cyclic (note that $1645 = 5 \cdot 7 \cdot 47$).