KTH Matematik

5B1309, Algebra
Exam
March 31, 2007

- No books or notes can be used.
- **All you write has to be motivated!**
- At least 3 points from each part are required to pass this exam. Minimun 15 points is required for grade 4 and minimum 6 points in each part is required for grade 5.

PART 1

(1) (3 p.) Consider the following set:

$$G_a = \{A \in GL_n(\mathbb{R}) \text{ such that } \det(A) = a\}.$$

(a) For which $a \in \mathbb{R}$ is $(G_a, \cdot)$ a subgroup?($\cdot$ denotes the matrix multiplication). A subgroup has to contain the identity I_n, thus the only possibility is $a = 1$. Moreover for any $A, B \in G_1$, it is $\det(A \cdot B^{-1}) = \det(A)(\det(B))^{-1} = 1$. This shows that G_1 is a subgroup.

(b) For which $a \in \mathbb{R}$ is $(G_a, \cdot)$ a normal subgroup? We only have to check G_1. It is a normal subgroup since it is the kernel of

$$\det : (GL_n(\mathbb{R}), \cdot) \to (\mathbb{R} \setminus \{0\}, \cdot),$$

which is a homomorphism: $\det(A \cdot B) = \det(A) \cdot \det(B)$.

(2) (2 p.) Consider the following permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 1 & 7 & 2 & 8 & 4 & 3 & 10 & 6 \end{pmatrix}$$

(a) Find the cycle decomposition of σ^3.

$$\sigma^3 = (198510326)(47)$$

(b) Is $\sigma^3 \in A_{10}$ (the subgroup of even permutations)? σ^3 can be decomposed in the following transpositions:

$$(16)(12)(13)(110)(15)(18)(19)(47)$$

It follows that $\sigma^3 \in A_{10}$.

(3) (3 p.) Give an example of a nonabelian group such that each of its proper subgroups are cyclic. One example is $\mathbb{S}_3$. It is non-abelian and it has cardinality 6. All proper subgroups have cardinality 2 or 3, which are prime numbers, and hence they are cyclic.

PART 2

(1) (4 p.)(Motivate your answer!) Let $Aut(G)$ denote the group of automorphisms (isomorphisms $f : G \to G$), where the binary operation is the composition. Is $Aut(\mathbb{Z}_8)$ isomorphic to any of the following groups?

$$\mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{S}_3.$$

We know that $|Aut(\mathbb{Z}_8) = \phi(n)$ (the Euler number of 8, the numebr of positive integers smaller that 8 and coprime with 8). It follws that $|Aut(\mathbb{Z}_8)| = 4$. The only groups of cardinality 4 are $\mathbb{Z}_2 \times \mathbb{Z}_2$ or the cyclic one $\mathbb{Z}_4$. An element $f \in Aut(\mathbb{Z}_8)$ has to take 1 to a generator so they are f_1, f_2, f_3, f_4, where

$$f_1(1) = 1, f_2(1) = 3, f_3(1) = 5, f_4(1) = 7.$$

We see that $f_1^2(1) = 1, f_2^2(1) = 3 \cdot 3 = 1, f_3^2(1) = 25 = 1, f_4^2(1) = 49 = 1$ and thus $ord(f_i) = 2$ for $i = 1, 2, 3, 4$. Then it must be $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) (3 p.) Let G be a group of cardinality 21. Assume that there is an element $a \in Z(G)$ with $ord(a) = 3$, then show that G is cyclic.

Since $|Z(G)| \geq 3$ and $|Z(G)|/21$ it is $|Z(G)| = 3$ or 21. If $Z(G) = 21$ then G is abelian, and thus isomorphic to $\mathbb{Z}_{21}$, which is cyclic. We can than assume that

$$|Z(G)| = 3, Z(G) =< e > .$$

we have that $|G/Z(G)| = 7$ cyclic. Let $b \cdot Z(G)$ be a generator, i.e. $b^7 \in Z(G)$, and $b^6 \notin Z(G)$. Thus $ord(b) = 7$ in G. Let $H =< b >$. Since H commutes with $Z(G)$ $H \cdot Z(G)$ is a subgroup and it has the same cardinality as G. So it is

$$G = H \cdot Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_7.$$

The isomorphism is given by $\phi(e^i b^j) = (e^i, b^j)$. It follows that G is cyclic.

(3) (2 p.) How many non cyclic abelian groups of cardinality 245 are there? A finite abelian group $G, |G| = n$, (up to isomorphism) can be decomposed uniquely as:

$$G = \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_t}$$

where $n_1 \cdot n_1 ... \cdot n_t = n$ and n_i / n_{i-1}, $i = 2, ..., t$. Since $245 = 5 \cdot 7^2$ we have that the possibilities are:
(a) $n_1 = 245, G = \mathbb{Z}_{245}$.
(b) $n_1 = 35, n_2 = 7, G = \mathbb{Z}_{35} \times \mathbb{Z}_7$.
There is only one non cyclic:

$$G = \mathbb{Z}_{35} \times \mathbb{Z}_7.$$

PART 3

(1) (4 p.) Show that no group of cardinality 48 is simple. The number of 2-Sylow subgroups can be 1 or 3. If there is only one subgorup of order 16, then by Sylow theorem, it must be normal and thus the group is not simple.

Assume that that there are 3 subgroups of cardinality 16, P_1, P_2, P_3. Since as a set

$$P_1 \cdot P_2 = \frac{|P_1||P_2|}{|P_1 \cap P_2|},$$

we have that $|P_1 \cap P_2| = 8$. This implies that $[P_1 \cap P_2 : P_1] = 2$ for $i = 1, 2$, and thus $P_1 \cap P_2$ is normal in P_1 and P_2. This implies that the normalizer $N(P_1 \cap P_2)$ in G contains $P_1 \cup P_2$, subset which has more than 16 elements. At the same time

$$16 < |N(P_1 \cap P_2)|/|G| = 16 \cdot 3.$$

We can conclude that $N(P_1 \cap P_2) = G$ and thus $N(P_1 \cap P_2)$ is normal in G, which implies that G is not simple.

(2) (2 p.) Show that the ideal $(2, x)$ (ideal generated by the elements 2 and x) in the ring $\mathbb{Z}[x]$ is maximal. This ideal is the kernel of the ring-morphism

$$\phi : \mathbb{Z}[x] \to \mathbb{Z}_2, \ \phi(p(x)) = [p(0)]_2.$$

Since $\mathbb{Z}_2$ is a field the ideal is maximal.

(3) (3 p.) Let R be a commutative ring such that $R[x]$ is a $P.I.D.$ Show that R is a field.

Every non zero prime ideal is maximal in a PID. Morover since R is a subring of $R[x]$, R is an integral domain. The kernel of the ring homomorphism

$$\phi : R[x] \to R, \ \phi(p(x)) = p(0_R),$$

is the ideal (x), which is prime since R is an integral domain. It follows that (x) is maximal and hence the quotient (via ϕ) $R[x]/(x) \cong R$ is a field.

Good Luck!