KTH Matematik

5B1309, Algebra
Exam 1
Februari 5, 2007

- *Exam's time:* **16:00-17:00.**
- No books or notes can be used
- **All you write has to be motivated**
- At least 3 points are required to pass this exam.

(1) (4 p.) Consider the following set:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, 0 = [0]_3, a, b, c \in \mathbb{Z}_3, \det(A) = 1 \right\}$$

(a) Show that $(G, \cdot)$ is a group, where $\cdot$ is the matrix-multiplication. Because $\det(A) = ac = 1 (in \mathbb{Z}_3)$ it is $a = 1, b = 1$ 0r $a = 2, b = 2$. It follows that G contains 6 elements:

$$G = \left\{ I_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \right\}$$

The operation is closed , in fact if $A, B \in G$ then $AB \in G$ (show the multiplication). It is associative because the matris multiplication is associative. The identity is given by I_3. regarding the inverse, denoting by $-a$ the inverse of a in $\mathbb{Z}_3$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -a \\ 0 & 2 \end{pmatrix}$$

(b) Show that $(A, \cdot)$ is isomorphic to $(\mathbb{Z}_6, +)$. By keep multiplying one shoes that

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^6 = I_3$$

and the lower powers are never equal to the identity. Since G contains an element of order 6 it must be equal to the cyclic group generated by that element. So G is a cyclic group of order 6 and thus isomorphic to $(\mathbb{Z}_6, +)$.

(2) (2 p.) Consider the following permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 1 & 7 & 5 & 8 & 4 & 9 & 10 & 6 \end{pmatrix}$$

(a) Find the cycle decomposition of σ and σ^2.
$\sigma = (123)(47)(5)(68910), \sigma^2 = (132)(4)(7)(5)(69)(810)$.

(b) Say if σ is even or odd.
They are both even.

(3) (3 p.) Let G be a finite group with an even number of elements. Show that G contains an element of order 2 (i.e. there is $x \in G$, such that $x^2 = 1$).
Recall that by definition $ord(1_G) = 1$. Let $G = 2k$ and $S = \{1_G \neq x \in G, ord(x) \neq 2\}$, and $T = \{x \in G, ord(x) = 2\}$. It is

$$G = S \cup T \cup \{1_G\}.$$

Moreover since G and T are disjoint and do not contain the identity it is

$$|G| = |S| + |T| + 1.$$

Observe that for every element $x \in S$ it must be $x \neq x^{-1} \in S$ because $ord(x) \neq 2$. This shows that the cardinality of S is even, say $|S| = 2m$. Then $|G| = 2k = 1 + 2m + |T|$, which implies that T is not empty and thus there is an element of order two.