KTH Matematik

5B1309, Algebra
Exam 2
Februari 19, 2007

- *Exam's time:* **16:00-17:00.**
- No books or notes can be used
- **All you write has to be motivated**
- At least 3 points are required to pass this exam.

(1) (3 p.) Find the maximal order of a cyclic subgroup of $\mathbb{S}_{12}$.
Let $\sigma \in \mathbb{S}_{12}$ be of type $n_1^{k_1}...n_t^{k_t}$ (i.e (123)(45)(67)(89101112) is og type $2^2 35$.), where $k_1 n_1 + ...k_t n_t = 12$.

$$ord(\sigma) = l.c.m(n_1, ..., n_t)$$

Then the answer is given by

$$max\{l.c.m((n_1, ..., n_t), k_1 n_1 + ...k_t n_t = 12, k_i \geq 1, n_1 \geq 1\},$$

which is $l.c.m(3, 4, 5) = 60$.

(2) (4 p.) Let $G = \mathbb{S}_n$, for some $n \in \mathbb{Z}_+$ and fix some i between 1 and n. Let:

$$G_i = \{\sigma \in G, \sigma(i) = i.\}.$$

(a) Show that G_i is a subgroup of G.
Let $I_n = \{1, 2, ..., n\}$. The symmeric group $\mathbb{S}_n$ acts on I_n by the action:

$$(\sigma, i) = \sigma(i).$$

G_i is the stabilizer of the element i and therefore a group.

(b) Show that there are n distinct left-cosets of G_i. Moreover we know that:

$$[G : G_i] = |Orb(i)|$$

where $Orb(i) = \{\sigma(i), \sigma \in \mathbb{S}_n\} = I_n$.

(c) Show that G_i is not a normal subgroup.
For $\tau \in \mathbb{S}_n$ let $k = \tau^{-1}(i)$ and $\tau(i) = j$.

$$G_i \tau = \{\alpha = \sigma\tau, \alpha(k) = i\}.$$

$$\tau G_i = \{\beta = \tau\sigma, \beta(i) = j\}.$$

They are different in general, for example: $i = 1, n = 7, \sigma = (1)(2...7), \tau = (1\,2...7), \tau^{-1}(1) = 7, \tau(1) = 2$. Since $\tau\sigma(7) = 3 \neq 1$, $\tau\sigma \notin \tau G_1$.

(3) (2 p.) List all non isomorphic abelian groups of order 100 and of order 145.
A finite abelian group $G, |G| = n$, (up to isomorphism) can be decomposed uniquely as:

$$G = \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_t}$$

where $n_1 \cdot n_1 ... \cdot n_t = n$ and n_i / n_{i-1}, $i = 2, ..., t$. Since $100 = 2^2 5^2$ we have that the possibilities are:
(a) $n_1 = 100, G = \mathbb{Z}_{100}$.
(b) $n_1 = 2^2 5, n_2 = 5, G = \mathbb{Z}_{20} \times \mathbb{Z}_5$.
(c) $n_1 = 25^2, n_2 = 2, G = \mathbb{Z}_{50} \times \mathbb{Z}_2$.
(d) $n_1 = 10, n_2 = 10, G = \mathbb{Z}_{10} \times \mathbb{Z}_{10}$.
Since $145 = 5 \cdot 29$ the only abelian group is

$$G = \mathbb{Z}_{29} \times \mathbb{Z}_5 \cong \mathbb{Z}_{145}.$$