KTH Matematik

5B1309, Algebra
Exam 3
March 19, 2007

- *Exam's time:* **16:00-17:00.**
- No books or notes can be used
- **All you write has to be motivated**
- At least 3 points are required to pass this exam.

(1) (3 p.) How many groups of cardinality 33 are there, up to isomorphism?

(Hint for the solution) Using Sylow Theorem one sees that any group of cardinality 33 has a unique subgroup of order 3, $H \cong \mathbb{Z}_3$, and a unique subgroup of order 11, $K \cong \mathbb{Z}_{11}$. Because they are normal (since they are unique) they commute with each other and thus the set HK is a subgroup of cardinality 33 (since $K \cap H = \{0\}$, because they hace orders which are coprime). It follows that $G = HK$ and that $f : HK \to H \times K$ defined by $f(hk) = (h, k)$ is an isomorphism. The answer is then that there is only one group (up to isomorphism): $\mathbb{Z}_{11} \times \mathbb{Z}_3$.

(2) (2 p.) Let A, B be commutative rings. Let $Hom(A, B)$ denote the set of ring homomorphisms between A and B. Similarly $Hom(A[x], B)$ denotes the set of ring homomorphisms between the polynomial ring $A[x]$ and B. Sow that the map:

$$T : Hom(A[x], B) \to B \times Hom(A, B)$$

defined by $T(\phi) = (\phi(x), \phi|_A)$, defines a bijection of sets.

(Hint for the solution)<u>injectivity</u>: Assume that $T(\phi) = T(\psi)$, i.e. $\phi|_A = \psi|_A$ and $\phi(x) = \psi(x)$ then, since ϕ and ψ are ring-homomorphisms, for every $p(x) = a_0 + a_1x + ... + a_nx^n \in A[x]$

$$\phi(p(x)) = \phi(a_0)+\phi(a_1)\phi(x)+...+\phi(a_n)\phi(x)^n = \psi(a_0)+\psi(a_1)\psi(x)+...+\psi(a_n)\psi(x)^n = \psi(p(x)).$$

which means that $\phi = \psi$. <u>surjectivity</u>: For every $(b, f) \in B \times Hom(A, B)$ the morphism:

$$\phi(a_0 + a_1x + ...a_nx^n) = f(a_0) + f(a_1)b + ... + f(a_n)b^n$$

is a ring homomorphism in $Hom(A[x], B)$ such that $T(\phi) = (b, f)$.

(3) (4 p.) Consider the ring $\mathbb{Z}[2i] = \{a + 2bi \in \mathbb{C} \,|\, a, b \in \mathbb{Z}\}$. (Recall that $i^2 = -1$.)

(a) (1 p) Show that $\mathbb{Z}[2i]$ is a subring of $\mathbb{C}$. It is a subgroup: the inverse $-(c+2di) = (-a-2di)$ and $(a+2bi)+(-(c+2di)) = (a-c)+2(b-d)i \in \mathbb{Z}[2i]$. It is closed under multiplication: $(a+2bi)(c+2di) = (ac-4bd)+2(bc+ad)i \in \mathbb{Z}[2i]$.

(b) (2 p) Let $(2i)$ be the principal ideal generated by the element $2i$. Show that

$$\frac{\mathbb{Z}[2i]}{(2i)} \cong \mathbb{Z}_4 \text{ as rings.}$$

(Hint for the solution)Consider the surjective map of rings $\phi : \mathbb{Z}[2i] \to \mathbb{Z}_4$ defined by $\phi(a+2bi) = [a]_4$. Because for every $\alpha, \beta \in \mathbb{Z}$, $2i(\alpha+2\beta i) = -4\beta+2i\alpha$, the ideal $(2i) = \{a+2bi | a/a\}$. Therefore $Ker(\phi = (2i)$ and thus $\frac{\mathbb{Z}[2i]}{(2i)} \cong \mathbb{Z}_4$ as rings.

(c) (1 p) Is $(2i)$ a prime ideal? Since $\mathbb{Z}_4$ is not an I.D. the ideat $(2i)$ is not prime.

Do not forget to register for the final exam. The deadline is: 2007-03-25 at 24:00.

Good Luck!