KTH Matematik

Exam 2
5B1309 Algebra g.k.
13 Februari, 2006

- time:**10:15-11:00**
- No books, no notes admitted.
- You are expected to give a motivation to all your answers. A simple Yes or No will not be accepted.
- Minimum 3 points are required to pass this partial exam. The bonus points will cover the second part of the final exam and will contribute to the final grade in accordance to the rules set for the course.

(1) (3 pts) (motivate your answer!) List all the abelian groups G of cardinality $|G| = 36$, up to isomorphism.

G will be isomorphic to a product of $\mathbb{Z}_n$. Using the fact that $\mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ if and only if $gcd(p, q) = 1$, and that $|G| = 2^2 3^2$, the possibilities are:
(a) $\mathbb{Z}_4 \times \mathbb{Z}_9$
(b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$
(c) $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
(d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

(2) (3 pts) Consider the following set of matrices:

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a, b \in \mathbb{R}, a \neq 0\}.$$

It is a group under matrix multiplication. Show that

$$N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R}\} \subset G$$

is a normal subgroup of G.
(a) N is a subgroup.

(i)

$$A = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \in N, AB = \begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} \in N$$

(ii)

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N$$

(iii)

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, A^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \in N$$

(b) N is normal. Let

$$A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G, H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in N,$$

then

$$AHA^{-1} = \begin{pmatrix} 1 & ah \\ 0 & 1 \end{pmatrix} \in N.$$

(3) (3 pts) (motivate your answer!) Determine the factor group:

$$\mathbb{Z}_4 \times \mathbb{Z}_{10} / < (2, 4) >$$

Since $ord(2, 4) = mcm(ord(2), ord(4)) = mcm(2, 5) = 10$, the cardinality $|\mathbb{Z}_4 \times \mathbb{Z}_{10}/ < (2, 4) > | = 40/10 = 4$. It follows that $\mathbb{Z}_4 \times \mathbb{Z}_{10}/ < (2, 4) >$ is an abelian group of cardinality 4 and thus isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

We see that $< (2, 4) >= \{(0, 2i), (2, 2i); i = 0, 1, 2, 3, 4\}$. It follows that for any $(n, m) \in \mathbb{Z}_4 \times \mathbb{Z}_{10}$:

$$((n, m) < (2, 4) >)^2 = \{(2n+0, 2(i+m), (2, 2i) : i = 0, 1, 2, 3, 4\} =< (2, 4) >$$

Therefore every element has order 2. In Z_4 the generator has order 4 so necesserely is:

$$\mathbb{Z}_4 \times \mathbb{Z}_{10}/ < (2, 4) > \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Good luck!