5B1466, Fourier Analysis, KTH spring 2006.

Brief notes from Lecture 1. In addition to ordinary Fourier series and Fourier integrals, we shall study the Fourier transform on $\mathbb{Z}_n$, integers modulo n, and more general examples of FT on *abelian groups*. If time permits, we will also study FT on *finite groups*.

Among the applications of FT we mention theory of numbers, probability, the beautiful interplay with analytic functions, e.g. in the Payley-Wiener theorems.

Historically, FT arose when trying to solve differential equations of mathematical physics.

Wave equation. In 1+1 dimensions (time and space) this is the linear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where u represents the amplitude (height) of the wave, and $c > 0$ is a physical constant (often $= 1$ below). A particular type of solution is given by *travelling waves* $G(x-ct)$, travelling to the right as t increases, or $F(x+ct)$ travelling to the left. We simply translate the same function. Here, F and G are arbitrary functions with continuous derivatives of order two (F, $G \in C^2$). By linearity $F(x+ct) + G(x-ct)$ is also a solution.

A classical result, due to d'Alembert, is that given a solution u (in C^2) to the wave equation, satisfying $u(x,0) = f(x)$ and $\partial u/\partial t(x,0) = g(x)$, where $f \in C^2$ and $g \in C^1$ are given, we can find F and G so that $u(x,t) = F(x+ct) + G(x-ct)$. In fact, the formula (due to Lagrange)

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t) + \int_0^{x+t} g(y)\,dy - \int_0^{x-t} g(y)\,dy \right)$$

holds. It is a rather straight-forward consequence of the variable transformation $\xi = x+t$, $\eta = x-t$. (With $v(\xi,\eta) = u(x,t)$ we get $\partial^2 v/\partial\xi\partial\eta = 0$. It is easy to integrate this equation to $v = F(\xi) + G(\eta)$.....)

Standing waves. The idea is a string moving up and down as time evolves, with both ends fixed. Suppose we make the Ansatz $\phi(x)\psi(t)$ (separation of variables) for a solution to the wave equation. Differentiating and rewriting, we obtain $\phi''(x)/\phi(x) = \psi''(t)/\psi(t)$. Since the left-hand side is a function of x and the right-hand side is a function of t, they must be constant. We get the equations $\phi''(x) = \lambda\phi(x)$ and $\psi''(t) = \lambda\psi(t)$. Only the case $\lambda < 0$ is of interest. Then we get sine and cosine functions. If we require $\phi(0) = \phi(\pi) = 0$, corresponding to $\lambda = -m^2$, where $m = 1, 2, \ldots.$, we obtain the elementary solutions

$$u_m = (A_m \cos mt + B_m \sin mt) \sin mx,$$

where A_m and B_m are constants. By superposition we get (convergence questions are postponed) the solution

$$u(t,x) = \sum_{m=1}^{\infty} u_m(x,t) = \sum_{m=1}^{\infty} (A_m \cos mt + B_m \sin mt) \sin mx,$$

so that $u(x,0) = \sum_{m=1}^{\infty} A_m \sin mx = f(x)$.

The basic question that arises is: given f (vanishing at 0 and π), can we find numbers A_m such that $f(x) = \sum_{m=1}^{\infty} A_m \sin mx$? In what sense? How do we find the A_m?

Well, assume the formula holds, and integrate $f(x) \sin nx$:

$$\int_0^{\pi} f(x) \sin nx \, dx = \int_0^{\pi} \sum_{m=1}^{\infty} A_m \sin mx \sin nx \, dx = \sum_{m=1}^{\infty} A_m \int_0^{\pi} \sin mx \sin nx \, dx.$$

The last integral vanishes if $m \neq n$ and equals $\pi/2$ if $n = m$. Hence

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

A_n is what is now known as the nth *Fourier sine coefficient* for f.

More generally, we could consider a series $f(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=0}^{\infty} B_m \cos mx$, or, using Euler's formula $e^{iy} = \cos y + i \sin y$, $f(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}$, where the coefficients a_m may be complex numbers. Then $a_m = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-imx} \, dx$.

Heat equation. In $2+1$ dimensions (variables (x,y) and t), it is

$$c\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u,$$

where $c > 0$ is a physical constant and u is the temperature. Δ is the *Laplacian*. The equation $\Delta u = 0$ is called Laplace's equation, and a solution is said to be *harmonic* in the open domain considered. Clearly harmonic functions solve the heat equation. They are *steady state solutions*.

We shall consider the *Dirichlet problem* for the unit disk $D = \{x^2 + y^2 < 1\}$, with boundary $C = \{x^2 + y^2 = 1\}$, the unit circle. Given a suitable function f on C, the problem is to find a function u which is harmonic in D and coincides with f on C. In polar coordinates r and θ, $u(1, \theta) = f(\theta)$.

Expressing the Laplacian in polar coordinates: $\Delta = \partial^2/\partial r^2 + \frac{1}{r}\partial/\partial r + \frac{1}{r^2}\partial/\partial\theta$, one can separate the variables r and θ. This leads to elementary solutions

$$u_m(r, \theta) = r^{|m|}e^{im\theta}, \qquad m = 0, \pm 1, \pm 2, ...$$

By linearity and superposition we obtain a solution of the form

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta},$$

where a_m are constants. Putting $r = 1$ we get (formally)

$$u(1, \theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta).$$

Again we see the need to represent f as a *Fourier series*.

Comparison with the wave equation. If we identify the point (x, y) with the complex number $z = x + iy$, then $z = re^{i\theta}$ using polar coordinates. Then $r^{|m|}e^{im\theta} = (re^{i\theta})^m = z^m$, if $m \geq 0$, and $r^{|m|}e^{im\theta} = \overline{z}^{-m}$ if $m < 0$. Hence we may write

$$u(x, y) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} a_{-m}\overline{z}^m = F(x + iy) + G(x - iy).$$