5B1466, Fourier Analysis, KTH spring 2006.

Brief notes from Lecture 2. Suppose f is a continuous 2π-periodic function, and define its *Fourier coefficients* by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Interpreting θ as an angle, we realise that equivalently, f may be considered as a function on the unit circle.

Remarks: 1. If the period is T, the definition is changed to $\hat{f}(n) = \frac{1}{T}\int_{-T/2}^{T/2} f(\theta)e^{-2\pi in\theta/T}\, d\theta$ for $n \in \mathbb{Z}$.
2. For $\hat{f}$ to be defined it is sufficient to assume that $\|f\|_1 = \int_{-\pi}^{\pi} |f(\theta)|\, d\theta$ is finite. Indeed, $|\hat{f}(n)| \le \|f\|_1$ for any n.

We shall show

Proposition.

$$\lim_{r\to 1^-} \sum_{n=-\infty}^{\infty} \hat{f}(n)r^{|n|}e^{in\theta} = f(\theta), \quad \text{for all } \theta. \tag{1}$$

Proof. Denote by $u(r,\theta)$ the sum on the left. We first show that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)f(t)\, dt, \quad \text{where } P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$

the Poisson kernel for the unit disk, encountered in §1. For fixed $0 \le r < 1$, the series defining $u(r,\theta)$ is *uniformly convergent*, which implies that we may interchange integration and summation:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt\right)e^{in\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-t)}\right)f(t)\, dt$$

For $t = 0$, the sum in parentheses may be written as (we use the geometric series formula $1 + z + z^2 + z^3 + ... = 1/(1-z)$, if $|z| < 1$)

$$\sum_{N=0}^{\infty}(re^{i\theta})^n + \sum_{n=1}^{\infty}(re^{-i\theta})^n = \frac{1}{1-re^{i\theta}} + re^{-i\theta}\frac{1}{1-re^{-i\theta}}.$$

This is easily seen to give the explicit formula for $P(r,\theta)$.

Now, $P(r,\cdot)$ satisfies three important properties, viz.

$$\text{(i) } P(r,\theta) \geq 0;\ \text{(ii) } \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1;\ \text{(iii) } \forall\delta>0:\ \lim_{r\to 1^-}\int_{\delta<|\theta|\leq\pi} P(r,\theta)\,d\theta = 0.$$

The explicit formula for P shows (i) (for $0 \leq r < 1$). (ii) is proved using the absolutely fundamental formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\,d\theta = \begin{cases} 1, n \neq 0 \\ 0, n = 0 \end{cases}$$

together with uniform convergence, as above:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}\,d\theta = \sum_{n=-\infty}^{\infty} r^{|n|}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\,d\theta\right) = 1,$$

because the only term that contributes corresponds to $n = 0$.

To deduce (iii), we write $P(r,\theta) = (1-r^2)/((1-r\cos\theta)^2 + r^2\sin^2\theta)$. If $\pi/2 \leq |\theta| \leq \pi$ this is clearly $\leq 1-r^2$. whereas it is $\leq (1-r^2)/(r^2\sin^2\delta)$ if $\delta \leq |\theta| \leq \pi$. In both cases it is at most a constant times $1-r^2$. (iii) follows.

To show that $u(r,\theta) \to f(\theta)$ as $r \to 1^-$, we assume $\theta = 0$, without loss of generality. By property (ii), and $P(r,\cdot)$ being even, we get

$$u(r,0) - f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t)(f(t)-f(0))\,dt$$

Choose $\varepsilon > 0$. There is a $\delta > 0$ such that $|f(t) - f(0)| < \varepsilon/2$ whenever $|t| < \delta$. Split the integral into $\int_{|t|<\delta}$ and $\int_{\delta\leq|t|\leq\pi}$. Since $P \geq 0$ (property (i)), we get

$$|u(r,0)-f(0)| = \frac{1}{2\pi}\int\limits_{|t|<\delta} P(r,t)|f(t)-f(0)|\,dt + \frac{1}{2\pi}\int\limits_{\delta\leq|t|\leq\pi} P(r,t)|f(t)-f(0)|\,dt = A+B.$$

Here $A < \varepsilon/2$ by (ii) and the choice of δ. Clearly $|f(t) - f(0)| < 2M$, where M is the maximum of $|f|$. By (iii) we can choose r_0 so that $\frac{1}{2\pi}\int\limits_{\delta\leq|t|\leq\pi} P(r,t)\,dt < \varepsilon/(4M)$ whenever $r_0 < r < 1$.

This means that also $B < \varepsilon/2$. The claim follows.

Remarks: 1. Essentially the same proof shows that $\max_\theta |u(r,\theta) - f(\theta)| \to 0$ as $r \to 1^-$, for f continuous. This is because f is a continuous function on a compact interval, and therefore uniformly continuous.
2. If f is just (absolutely) integrable ($||f||_1 < \infty$), the result is valid at each continuity point of f.

Definition: A sequence of 2π-periodic functions K_n, $n = 1, 2, ...,$ is called a sequence of *good kernels* if properties (i) and (ii) are fulfilled for each n, and if (iii) holds as $n \to \infty$ for each $\delta > 0$. (The main difference compared to $P(r,\theta)$ is that the continuous parameter r has been replaced by a discrete one.)

Denote by $S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}$ the partial sum for the Fourier series of f. It can be written

$$S_N(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(\theta - t) f(t)\, dt, \quad \text{where } D_N(\theta) = \frac{\sin((2N+1)\theta/2)}{\sin(\theta/2)}.$$

The *Dirichlet kernel* D_N fulfills (ii) and (iii), but it is *not positive.* To get pointwise convergence one needs to require more about f. Assume that f is differentiable at the origin. One can reason very much like we did for the Poisson kernel. However, the integral $\int_{|t|<\delta}$ becomes troublesome, since, loosely speaking, D_N is not absolutely integrable at $t = 0$, as $N \to \infty$. The extra requirement on f allows us to estimate the integral as

$$\int_{|t|<\delta} \left| \sin((2N+1)t/2) \frac{t}{\sin t/2} \frac{f(t) - f(0)}{t} \right| dt \le C \int_{|t|<\delta} |f'(\tau(t)|\, dt$$

which is as small as we like. We'll return to this in more detail in the next chapter.

If we take the mean-value of the D_n, we get the *Fejér kernel*

$$F_N = \frac{1}{N}(D_0 + D_1 + ... + D_{N-1}).$$

This is a sequence of good kernels. In fact,

$$F_n(\theta) = \frac{1}{N} \frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)},$$

showing that $F_N \geq 0$. The other criteria holds since they were already fulfilled by the Dirichlet kernel.

Uniqueness. Suppose f is a continuous 2π periodic function. It follows from Proposition 1, that if f satisfies $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(\theta) = 0$ for all θ. The Fourier coefficients determine f. A consequence is that *if* the Fourier coefficients of f fulfill

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty, \tag{2}$$

then f is equal to its Fourier series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}. \tag{3}$$

We shall see later, that this does not hold for all continuous functions.

A simple criterion for (2), and thereby (3) to hold, is that $\hat{f}(n) = O(1/n^2)$, which in its turn holds if f is twice differentiable.

A final remark: We have seen how the Fejér kernel, which has better convergence properties, was obtained from the Dirichlet kernel. as a *Cesàro* mean value. The Poisson kernel is obtained from using an Abel mean value. These mean value formations when trying to understand convergence of infinite series around 200 years ago. As an example consider the series $1-1+1-1+....$ It is clearly divergent since each partial sum is either 0 or 1. We form the Cesàro mean of the partial sums. It is clear that this goes to $1/2$ as $n \to \infty$, the partial sums alternating between 0 and 1. With Abel's method we replace $1-1+1-1+...$ by $1-r+r^2-r^3+.... = 1/(1+r)$, where $0 \leq r < 1$. As $r \to 1^-$ we get the limit $1/2$ also in this case.