5B1466, Fourier Analysis, KTH spring 2006.

Brief notes from Lecture 3.

Convolution: Suppose f is a Riemann (absolutely) integrable 2π-periodic function, $\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\, d\theta < \infty$. The *convolution* of two such functions, f and g, is defined by

$$f * g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t) g(t)\, dt.$$

Then

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \tag{1}$$

because the LHS $\leq \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(\theta - t)|\, d\theta |g(t)|\, dt =$ RHS. A simple change of variables shows that $f * g = g * f$, so it does not matter in which order two functions are convolved. The following very important, easily proved, formula shows how convolution and multiplication correspond (under the Fourier transform on the circle):

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n), \quad n \in \mathbb{Z}.$$

We have already seen that the Poisson integral is the convolution

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\, dt, \quad P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Similarly the partial sum of the Fourier series is the convolution with the Dirichlet kernel:

$$S_N(f)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta - t) f(t)\, dt, \quad D_N(\theta) = \frac{\sin((2N+1)\theta/2)}{\sin(\theta/2)}.$$

Regularity: Generally speaking, convolution improves the regularity. For instance, if we convolve a differentiable function with an integrable one, the result is a differentiable function. If one of them is C^k (continuous derivatives of order $\leq k$), the convolution is C^k.

Example: Let f be the (2π-periodic) function which is identically one on the interval $[-1/2, 1/2]$ and zero on the rest of $[-\pi, \pi]$. Clearly f is not continuous, but $f * f$ is continuous. Its graph is the isosceles triangle with base $[-1, 1]$ and height one, and it is zero outside this interval.

It is proved in the book that $f*g$ is continuous when f and g are integrable and bounded. (Prop. 3.1. (v).)

Approximation with regular functions. This is a very important idea which permeates all of Fourier analysis.

Approximation by continuous functions. We first prove that given f, absolutely integrable on a finite interval, I say, there is a sequence of continuous functions g_n such that $\int_I |f-g_n|\,dx \to 0$ as $n \to \infty$. Equivalently, given $\varepsilon > 0$, there is a continuous function g with $||f-g|| = \int_I |f-g|\,dx < \varepsilon$.

Since f is absolutely Riemann integrable, there is a step function ϕ with $||f-\phi||_1 < \varepsilon/2$. That ϕ is a step function means that there is a partition of I such that ϕ is constant on each (half-open, say) subinterval. If we consider such a rectangle-shaped part, it is clear that we can tilt the sides in the rectangle so that the area difference is $< \varepsilon/(2n)$, where n is the number of subintervals. Doing this for each subinterval, we obtain a continuous function g such that $||\phi - g|| < \varepsilon/2$. Then

$$||f-g|| = ||(f-\phi)+(\phi-g)|| \le ||f-\phi|| + ||\phi-g|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as asserted.

Approximation by infinitely differentiable functions. Let f be a given 2π-periodic function with $||f||_1 < \infty$. For fixed $0 \le r < 1$, the series $P_r * f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$ is infinitely differentiable w.r.t. θ (uniform convergence). We shall show that given $\varepsilon > 0$ we have $||P_r * f - f||_1 < \varepsilon$.

First, we choose a continuous function g with $||f-g||_1 < \varepsilon/3$. Write

$$\begin{aligned} ||P_r * f - f||_1 &= ||P_r * (f-g) + (P_r * g - g) - (f-g)||_1 \\ &\le ||P_r * (f-g)||_1 + ||P_r * g - g||_1 + ||f-g||_1 \end{aligned}$$

The third term on the second line is less than $\varepsilon/3$. This holds for the first term as well, because by Eq. (1), $||P_r * (f-g)||_1 \le ||P_r||_1 ||f-g||_1$ and $||P_r||_1 = 1$. The second term becomes small by choosing r close enough to 1. We know that $P_r * g(\theta) \to g(\theta)$ uniformly as $r \to 1^-$, i.e. $\max_\theta |P_r * g(\theta) g(\theta)| \to 0$ as $r \to 1^-$. Hence there is an r_0 such that it is $< \varepsilon/3$ whenever $r_0 \le r < 1$. Clearly the second term is less than $\max_\theta |P_r * g(\theta) - g(\theta)|$. Thus all terms on the last line are $< \varepsilon/3$, so that $||P_r * f - f||_1 < \varepsilon$.

The following is an important application of the above approximation results. It is known as the *Riemann-Lebesgue lemma.*

Lemma 1 *Let f be periodic and absolutely integrable. Then $\hat{f}(n) \to 0$ as $|n| \to \infty$.*

Proof. We assume that the period is 2π and write $||f||_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\, d\theta$, as usual. By assumption $||f||_1 < \infty$. Given $\varepsilon > 0$. We must show that there is an N such that $|n| > N$ implies $|\hat{f}(n)| < \varepsilon$. Choose a periodic C^1 function g with $||f - g||_1 < \varepsilon/2$. Then, partially integrating, we get (for $n \neq 0$)

$$\hat{g}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} g'(\theta)\frac{e^{-in\theta}}{-in}\, d\theta.$$

Hence

$$|\hat{g}(n)| \leq \frac{||g'||_1}{|n|} < \frac{\varepsilon}{2}, \quad \text{if} \quad |n| > N,$$

for some N. Writing $\hat{f}(n) = (\hat{f}(n) - \hat{g}(n)) + \hat{g}(n) = \widehat{(f-g)}(n) + \hat{g}(n)$, the lemma follows upon using $|\widehat{(f-g)}(n)| \leq ||f-g||_1 < \varepsilon/2$.