5B1466, Fourier Analysis, KTH spring 2006.

Brief notes from Lecture 4.

Inner product spaces. Let us briefly recall the inner products in $\mathbb{R}^N$ and $\mathbb{C}^N$.

We start with *the real case* and write $V = \mathbb{R}^N$. If $a = (a_0, a_1, ..., a_{N-1})$, $b = (b_0, b_1, ..., b_{N-1})$, where $a_j, b_j \in \mathbb{R}$, then the inner product (dot product) of a and b is

$$(a, b) = a \cdot b = \sum_{j=0}^{N-1} a_j b_j, \quad a, b \in V.$$

The important properties are

$$\begin{array}{ll} \text{(i)} & (\lambda a + \mu b, c) = \lambda(a, c) + \mu(b, c), \quad \lambda, \mu \in \mathbb{R},\ a, b, c \in V; \\ \text{(ii)} & (b, a) = (a, b), \quad a, b \in V; \\ \text{(iii)} & (a, a) = \sum_{j=0}^{N-1} a_j^2 = |a|^2 > 0 \quad \text{if} \quad a \in V, a \neq 0. \end{array}$$

In the *complex case*, $a = (a_0, a_1, ..., a_{N-1})$, $b = (b_0, b_1, ..., b_{N-1})$, where $a_j, b_j \in \mathbb{C}$, the complex inner product is

$$(a, b) = \sum_{j=0}^{N-1} a_j \overline{b_j}, \quad a, b \in V,$$

where this time $V = \mathbb{C}^N$. Properties (i)-(iii) become

$$\begin{array}{ll} \text{(i)} & (\lambda a + \mu b, c) = \lambda(a, c) + \mu(b, c), \quad \lambda, \mu \in \mathbb{C},\ a, b, c \in V; \\ \text{(ii)} & (b, a) = \overline{(a, b)}, \quad a, b \in V; \\ \text{(iii)} & (a, a) = \sum_{j=0}^{N-1} |a_j|^2 = |a|^2 > 0 \quad \text{if} \quad a \in V, a \neq 0. \end{array}$$

Consider now

$$(f, g) := \frac{1}{2\pi} \int_{\pi}^{\pi} f(\theta)\overline{g}(\theta)\, d\theta, \quad \text{and}\, ||f|| = ||f||_2 = \sqrt{(f, f)}. \tag{1}$$

Clearly properties (i) and (ii), in the complex case, hold. (iii) also holds if we only consider continuous functions. (f, g) is the inner product in the space $L^2(-\pi,\pi)$, the square integrable functions on the interval $(-\pi,\pi)$. For a general function in this space (iii) has to be interpreted to mean that $f=0$ *almost everywhere*.

Two functions f and g (in $L^2(-\pi,\pi)$) are *orthogonal* if $(f,g)=0$. We have seen before that for $e_n(\theta)=e^{2\pi in\theta}$, we have

$$(e_n,e_m)=\frac{1}{2\pi}\int_{\pi}^{\pi} e^{2\pi i(n-m)\theta}d\theta=\delta_{nm},$$

where the Kronecker symbol δ_{nm} equals 1 when $n=m$, and 0 otherwise. This means that $(e_n)_{n=-\infty}^{\infty}$ is an *orthonormal system*. All functions are *orthogonal*: $(e_n,e_m)=0$ if $n\neq m$, and of unit *norm* ("length"): $||e_n||=1$ for all $n\in\mathbb{Z}$. To explain this somewhat, we shall consider the following example. It is really about the Fourier transform on $\mathbb{Z}_N$, which will be examined more closely later on in the course.

A very fundamental example: Let $N>0$ be an integer and ω a complex number. Then $1+\omega+\omega^2+...+\omega^{N-1}=(1-\omega^N)/(1-\omega)$ if $\omega\neq 1$ and $=N$ for $\omega=1$. If also $\omega^N=1$, we get

$$\frac{1}{N}(1+\omega_j+\omega_j^2+...+\omega_j^{N-1})=\delta_{0j},$$

where we have denoted by ω_j the jth root of unity:

$$\omega_j=\omega_{j,N}:=\exp\left\{j\frac{2\pi i}{N}\right\},\qquad j=0,1,...,N-1.$$

Denote by $e_j=e_{jN}$ the vector $(1,\omega_j,\omega_j^2,...,\omega_j^{N-1})\in\mathbb{C}^N$. The inner product of e_j and e_k is ($(j-k)l$ is counted modulo N)

$$(e_j,e_k)=\frac{1}{N}\sum_{l=0}^{N-1}\omega_j^l\overline{\omega_k}^l=\frac{1}{N}\sum_{l=0}^{N-1}\omega_1^{jl}\omega_1{}^{-kl}=\frac{1}{N}\sum_{l=0}^{N-1}\omega_{j-k}^l=\delta_{j.k},$$

as above. Writing

$$\frac{1}{N}\sum_{l=0}^{N-1}\omega_j^l=\frac{1}{2\pi}\sum_{l=0}^{N-1}\exp\left\{j\frac{2\pi li}{N}\right\}\frac{2\pi}{N}=\frac{1}{2\pi}\sum_{l=0}^{N-1}\exp\left\{ij\theta_l\right\}\Delta\theta_l,$$

which is a Riemann sum for the integral $\frac{1}{2\pi}\int_0^{2\pi}e^{il\theta}\,d\theta$.

When f and g are orthogonal, i.e. $(f, g) = 0$, we get (Pythagoras)

$$\|f\|^2 + g\|^2 = \|f\|^2 + \|g\|^2$$

because the LHS is $(f + g, f + g) = (f, f) + (f, g) + (g, f) + (g, g) = \|f\|^2 + 2\,\mathrm{Re}\,(f, g) + \|g\|^2$, and $(f, g) = 0$.

Consider now a trigonometric polynomial, i.e. a function on the form $\sum_{n=N}^{M} c_n e_n$, i.e. a *finite* linear combination of e_ns. Then

$$\Big|\Big| \sum_{n=N}^{M} c_n e_n \Big|\Big|^2 = \sum_{n=N}^{M} |c_n|^2. \tag{2}$$

because the LHS is

$$\left(\sum_{n=N}^{M} c_n e_n, \sum_{m=N}^{M} c_m e_m \right) = \sum_{n=N}^{M} \sum_{m=N}^{M} c_n \overline{c_m} (e_n, e_m) = \text{RHS}.$$

The Fourier coefficients of a general function f can be written

$$\hat{f}(n) = (f, e_n) \qquad n \in \mathbb{Z}.$$

We shall prove the following extension of (2):
Parseval's formula: Let f be a continuous and 2π-periodic function. Then

$$\|f\|^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2. \tag{3}$$

Proof. Write

$$S_N = \sum_{|n| \le N} \hat{f}(n) e_n = \sum_{|n| \le N} (f, e_n) e_n$$

for the partial sum of the Fourier series of f. Then

$$\widehat{S}_N(n) = \hat{f}(n), \quad |n| \le N, \quad \text{and} \quad \widehat{S}_N(n) = 0, \quad |n| > N.$$

This implies that $f - S_N$ and S_N are orthogonal:

$$(f - S_N, S_N) = \sum_{|n| \le N} \overline{\hat{f}(n)} (f - S_N, e_n) = \sum_{|n| \le N} \overline{\hat{f}(n)} (\hat{f}(n) - \widehat{S}_N(n)) = 0$$

as we just saw. The orthogonality implies that $||f||^2 = ||f - S_N||^2 + ||S_N||^2$. Hence

$$\sum_{|n|\le N} |\hat{f}(n)|^2 = ||S_N||^2 \le ||f - S_N||^2 + ||S_N||^2 = ||f||^2$$

Letting $N \to \infty$ we get *Bessel's inequality*

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 \le ||f||^2.$$

Introduce the (continuous) function

$$F(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\overline{f}(-t)\,dt.$$

This is the convolution $f * g$, where $g(t) = \overline{f}(-t)$. The Fourier coefficients of g are $\hat{g}(n) =$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f}(-t)e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f}(s)e^{ins}\,ds = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ins}\,ds} = \overline{\hat{f}(n)}.$$

Hence $\hat{F}(n) = \hat{f}(n)\hat{g}(n) = \hat{f}(n)\overline{\hat{f}(n)} = |\hat{f}(n)|^2$. By Bessel's inequality the Fourier series of F converges uniformly to F:

$$F(\theta) = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 e^{in\theta}.$$

Parseval's formula follows upon putting $\theta = 0$.

The calculations in the proof above lead to

$$\lim_{N\to\infty} ||f - S_N||_2 = 0,$$

i.e. the Fourier series converges to the continous function f in the L^2-norm. Indeed,

$$||f - S_N||^2 = \sum_{|n|>N} |\hat{f}(n)|^2$$

which tends to zero as $N \to \infty$ by Bessel's inequality.

The meaning of Parseval's formula: The formula shows that that the Fourier transform on the circle, or torus $\mathbb{T}$, is a norm-preserving mapping, an *isometry* between the inner product spaces $L^2(\mathbb{T}) = L^2(-\pi, \pi)$ and $L^2(\mathbb{Z})$, all doubly infinite sequences $(c_n)_{-\infty}^{\infty}$ of complex numbers such that $\sum |c_n|^2 < \infty$. (In the book, this space is denoted $l^2(\mathbb{Z})$.) $\mathbb{T}$ and $\mathbb{Z}$ are (abelian) *groups*. Parsevals formula, may be written

$$||f||_{L^2(\mathbb{T})} = ||\hat{f}||_{L^2(\mathbb{Z})}.$$

More generally, the formula

$$||f||_{L^2(G)} = ||\hat{f}||_{L^2(\hat{G})}$$

holds, when G is an abelian group and $\hat{G}$ its dual group. We will return to this later on.

The Dirichlet kernel revisited. We now return to the pointwise convergence of the Fourier series, already touched upon in Lecture 2. Assume that f is a continuous 2π-periodic function, which in addition is *differentiable* at a point θ_0. We shall prove that then

$$\lim_{N\to\infty} S_N(\theta_0) = f(\theta_0).$$

Without loss of generality we may assume $\theta_0 = 0$. We use the formulae deduced in Lecture 2:

$$\begin{aligned} S_N(0) - f(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N + 1/2)t}{\sin t/2} f(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (F(t) \cos Nt + G(t) \sin Nt)\, dt, \end{aligned}$$

where F and G are absolutely integrable functions. By the Riemann-Lebesgue lemma the RHS tends to 0 as $N \to \infty$.