5B1466, Fourier Analysis, KTH spring 2006.

Brief notes from Lecture 5.

Hilbert spaces. An inner product space (Lecture 4) which is *complete* is called a *Hilbert space.*

Completeness means that the space contains all its limit points. Think of the rational numbers, which, when adding all their (finite) limits, yield the real numbers. In our case we could include all limits of continuous functions in the norm $||f||_2$, where $||f||_2^2 = (f, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$. The space obtained is denoted $L^2(-\pi, \pi)$, all square integrable functions on the interval $(-\pi, \pi)$.

We shall need the following variant of Parseval's formula:

$$(f, g) = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}. \tag{1}$$

It is a consequence of the identity

$$(u, v) = \frac{1}{4}\left(||u + v||^2 - ||u - v||^2 + i(||u + iv||^2 - ||u - iv||^2)\right),$$

which is valid for any complex inner product.

The isoperimetric inequality. This first application of Fourier series relates the length of a curve C in the plane to the area of the domain D inscribed. The proof goes back to Hurwitz 1901.

We have

$$l = \text{length of } C = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \, dt = \int_a^b |\dot{r}(t)| \, dt.$$

Here $\mathbf{r}(t) = (x(t), y(t))$, $a \leq t \leq b$, is a parametrisation of the C^1, say, curve. The length l is independent of the parametrisation. In particular, we may choose the arc-length s as parameter. Then $|\dot{r}(s)| = 1$ for $0 \leq s \leq l$.

Assume now that C is *closed*: $\mathbf{r}(0) = \mathbf{r}(l)$, and *simple*: the curve never intersects itself. Then the area of D is by definition

$$A = \text{area of } D = \frac{1}{2}\left|\int_C xdy - ydx\right| = \frac{1}{2}\left|\int_0^l (x(s)\dot{y}(s) - \dot{x}(s)y(s)) \, ds\right|.$$

We shall prove that

$$A \leq \frac{l^2}{4\pi} \quad \text{with equality only if } C \text{ is a circle.}$$

Rescaling, we may assume that $l = 2\pi$. Then we have to show that $A \le \pi$ with equality only if C is a circle. Consider the Fourier series for x and y:

$$x(s) \sim \sum_{-\infty}^{\infty} a_n e^{ins}, \quad y(s) \sim \sum_{-\infty}^{\infty} b_n e^{ins}.$$

Then

$$\dot{x}(s) \sim \sum_{-\infty}^{\infty} ina_n e^{ins}, \quad \dot{y}(s) \sim \sum_{-\infty}^{\infty} inb_n e^{ins}.$$

By Parseval's formula

$$\sum_{-\infty}^{\infty} |n|^2 \left(|a_n|^2 + |b_n|^2\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (|\dot{x}(s)|^2 + |\dot{y}(s)|^2)\, ds = 1,$$

since $|\dot{x}(s)|^2 + |\dot{y}(s)|^2 = |\mathbf{r}(s)|^2 = 1$ everywhere. By the general form of Parseval's formula (1) we find

$$\begin{aligned} A &= \pi \left| \sum_{-\infty}^{\infty} \left(a_n \overline{inb_n} - ina_n \overline{b_n}\right) \right| = \pi \left| \sum_{-\infty}^{\infty} \left(-2ina_n \overline{b_n}\right) \right| \\ &\le \pi \sum_{-\infty}^{\infty} |n|(|a_n|^2 + |b_n|^2) \le \pi \sum_{-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) = \pi, \end{aligned}$$

using the elementary inequality $2|a_n b_n| \le |a_n|^2 + |b_n|^2$. Thus $A \le \pi$. It remains to analyse when equality holds. If equality holds then this is the case in the last inequality: $\pi \sum_{-\infty}^{\infty} |n|(|a_n|^2 + |b_n|^2) \le \pi \sum_{-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2)$. Since $|n| < |n|^2$ if $|n| > 1$, we must have $a_n = b_n = 0$ for $|n| > 1$. Thus

$$x(s) = a_{-1}e^{-is} + a_0 + a_1 e^{is} \quad \text{and} \quad y(s) = b_{-1}e^{-is} + b_0 + b_1 e^{is}.$$

Since x and y are real-valued, $a_{-1} = \overline{a}_1$ and $b_{-1} = \overline{b}_1$ must hold. Furthermore, a_0 and b_0 are both real. We also know that $2(|a_1|^2 + |b_1|)^2 = 1$, and $|a_1| = |b_1|$ (from equality in $2|a_n b_n| \le |a_n|^2 + |b_n|^2$). Some further manipulation leads to $x(s) - a_0 = \cos(\alpha + s)$ and $y(s) - b_0 = \pm \sin(\alpha + s)$. Then $(x(s) - a_0)^2 + (y(s) - b_0)^2 = 1$, proving the claim.

Weyl's equidistribution theorem. A sequence $(\xi_n)_{n=1}^{\infty}$ of numbers in the interval $[0, 1)$ is *equidistributed* if for every $0 \le a \le b < 1$, the number $N_n(a, b)$ of points $\xi_k \in [a, b]$, for $k \le n$ satisfies

$$\lim_{n\to\infty} \frac{N_n(a, b)}{n} = b - a.$$

Suppose we fix a number $\gamma \in [0,1)$ and look at the sequence $\gamma_n = n\gamma (\mathrm{mod}\, 1)$ $(n = 1, 2, ...)$, i.e. we only consider the part of $n\gamma$ that falls into our interval. For instance, if $\gamma = 2/7$, the sequence becomes
$(2/7, 4/7, 6/7, 1/7, 3/7, 5/7, 0, 2/7, 4/7, 6/7, ...)$.

This periodic behaviour occurs if and only if γ is a rational number. One could think of the γ_n as coming from a probability distribution on the points $0, 1/7, 2/7, ...6/7$, with equal probability $1/7$ for each point. If δ_a denotes a point mass at the point a, then the *invariant measure* is $\frac{1}{7}\sum_{k=0}^{6}\delta_{k/7}$, the equidistribution on the points $0, 1/7, 2/7, ...6/7$.

Suppose now that γ is irrational. We may form the probability distributions $\frac{1}{n}\sum_{k=1}^{n}\delta_{\gamma_k}$. Weyl's equidistribution theorem says that this sequence converge to the 'length', or rectangular distribution, on $[0,1)$, the invariant probability measure.

Theorem. For any rational γ, the sequence $(\gamma_n)_1^\infty$ is equidistributed in $[0,1)$.

Proof. If ϕ is a function on $[0,1)$, we denote by $M_n(\phi)$ the mean-value

$$M_n(\phi) := \frac{1}{n}\sum_{j=1}^{n}\phi(\gamma_j).$$

We must show that for any interval $[a,b] \subset [0,1)$,

$$\lim_{n\to\infty} M_n(\phi) = \int_0^1 \phi(t)\, dt \qquad (2)$$

when $\phi = 1_{[a,b]}$, the *indicator function* of $[a,b]$, which is 1 on $[a,b]$ and 0 otherwise. One can easily see that the indicator function ϕ can be approximated arbitrarily well by continuous functions: given $\varepsilon > 0$, there are continuous functions $\phi_\pm$ such that $\phi_- \le \phi \le \phi_+$ and $\int_0^1 |\phi_\pm - \phi|\, dt < \varepsilon$. Hence it is sufficient to show (2) when ϕ is a continuous function on $[0,1]$. We may view ϕ as a 1-periodic function. We know that the Cesàro means $\sigma_N(\phi) = F_N * \phi$ tends to ϕ uniformly, i.e. $\max_{0\le t\le 1}|\sigma_N(\phi)(t) - \phi(t)| \to 0$, as $N \to \infty$. The $\sigma_N(\phi)$ are *trigonometric polynomials*, i.e. finite linear combinations of e_k for $|k| \le N$, where $e_k(t) = \exp(2\pi i k t)$. Hence it suffices to prove (2) for any e_k, $k \in \mathbb{Z}$. Then $M_n(\phi) =$

$$M_n(e_k) = \frac{1}{n}\sum_{j=1}^{n} e_k(\gamma_j) = \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i j k\gamma} = \frac{1}{n}\sum_{j=1}^{n}\left(e^{2\pi i k\gamma}\right)^j = \frac{\omega(1-\omega^n)}{n(1-\omega)},$$

where $\omega = e^{2\pi i k\gamma} \neq 1$, since γ is irrational. Clearly, this tends to 0 as $n \to \infty$. The theorem follows.

The heat equation on the circle. We want to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R}, \quad u(0,x) = f(x),$$

where f is a given, continuous 2π-periodic function. We make the Ansatz

$$u(t,x) = \sum_{-\infty}^{\infty} c_n(t) e^{inx}.$$

We note that $t = 0$ yields $\sum c_n(0) e^{inx} = u(0,x) = f(x)$, so $c_n(0) = \hat{f}(n)$. Differentiating under the summation sign leads to

$$\sum_{-\infty}^{\infty} c_n'(t) e^{inx} = \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \sum_{-\infty}^{\infty} c_n(t)(in)^2 e^{inx} = -\sum_{-\infty}^{\infty} n^2 c_n(t) e^{inx},$$

so

$$c_n'(t) = -n^2 c_n(t) \Longrightarrow c_n(t) = c_n(0) e^{-n^2 t} = \hat{f}(n) e^{-n^2 t},$$

i.e.

$$u(t,x) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{-n^2 t}(t) e^{inx} = H_t * f(x),$$

where

$$H_t(x) = \sum_{-\infty}^{\infty} e^{-n^2 t}(t) e^{inx}, \quad t > 0,\ x \in \mathbb{R}.$$

This is a good kernel. It is related to *Jacobi's theta function*, to be encountered later.