5B1473 Elementary differential geometry, spring 2007

Homework problems 1 — To be handed in 13/2

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1 Lemma 1.1.10 in the course book tells us that each smooth atlas on a topological manifold M is contained in a unique maximal smooth atlas. Show that two smooth atlases for M determine the same maximal smooth atlas if and only if their union is an atlas.

2 Prove the statements made in Example 1.2.8. 4) in the course book:

1. The quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ has rank n at all points.
2. The restriction $q|_{S^n} : S^n \to \mathbb{R}P^n$ is a local diffeomorphism, but not a bijection, so it is not a global diffeomorphism.

3 Let $0 < b < a$. Show that the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$(a - \sqrt{x^2 + y^2})^2 + z^2 = b^2$$

is a smooth submanifold. Show that it is diffeomorphic to the torus $T^2 = S^1 \times S^1$.

4 The real symplectic group $Sp(n, \mathbb{R})$ is the set of $2n \times 2n$ real matrices A such that $A^T J A = J$, where J is given in block form as

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix. Show that $Sp(n, \mathbb{R})$ is a differentiable submanifold of $\mathbb{R}^{(2n)^2}$ and determine its dimension. Is $Sp(n, \mathbb{R})$ compact?

Comment: The symplectic form ω on $\mathbb{R}^{2n}$ is given by $\omega(X, Y) = \langle JX, Y \rangle$, where $\langle X, Y \rangle$ is the inner product and vectors $X, Y \in \mathbb{R}^{2n}$ are written as columns. Just as the orthogonal group consists of matrices preserving the inner product, we have that $Sp(n, \mathbb{R})$ consists of matrices A preserving the symplectic form, that is $\omega(AX, AY) = \omega(X, Y)$ for all vectors X, Y.