5B1473 Elementary differential geometry, spring 2007

Homework problems 2 — To be handed in 6/3

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1 Identify $\mathbb{R}^4$ with the quaternions as $(x, y, z, w) \leftrightarrow x + iy + jz + kw$ and identify S^3 with the unit quaternions. For $p \in S^3$ compute the tangent space T_pS^3 and the derivative of the diffeomorphism f_p given by $f_p(q) = pq$.

(*reminder:* The quaternions are defined by $i^2 = j^2 = k^2 = ijk = -1$.)

2 Use the previous problem to find three everywhere linearly independent vector fields on S^3 of the form $(f_p)_*(\alpha)$ for $\alpha \in T_1S^3$. Show that TS^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$. Show that all integral curves of the vector fields are circles.

(*hint:* Make sense of the expression $e^{t\alpha}$ to find the integral curves.)

3 Find a smooth vector field on S^2 which is zero at exactly one point.

4 Let $V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$. Find coordinates near the point $(1, 0)$ for which V_1, V_2 are the coordinate vector fields.

5 Let $V_1 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $V_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$. Compute the flows ρ_t^1, ρ_t^2 of V_1, V_2 and verify that they do not commute by finding explicit times t_1, t_2 such that $\rho_{t_1}^1 \circ \rho_{t_2}^2 \neq \rho_{t_2}^2 \circ \rho_{t_1}^1$.

6 Let M be a connected smooth manifold and let $p, q \in M$. Show that there is a diffeomorphism $f : M \to M$ with $f(p) = q$.

(*hint:* First prove the statement for p, q in the unit ball of $\mathbb{R}^n$ by constructing a compactly supported vector field which flows p to q.)