5B1473 Elementary differential geometry, spring 2007

Homework problems 3 — To be handed in 10/4

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1 Show that real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd. Show that all Lie groups are orientable.

2 Prove that the wedge of two closed forms is closed and that the wedge of a closed and an exact form is exact.

3 Let the 2-torus T^2 be embedded in $\mathbb{R}^4$ as $T^2 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + w^2 = y^2 + z^2 = 1\}$. Give T^2 an orientation and compute $\int_{T^2} \omega$ where $\omega = xyz dw \wedge dy$.

4 Let ω be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega = \frac{1}{r^n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $\widehat{dx^i}$ is omitted and r is the distance to the origin. Show that ω is closed but not exact.

5 *Classical vector analysis.* Let U be an open subset of $\mathbb{R}^3$ and let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^3$. Define maps

$$\begin{aligned} \alpha_1 &: \mathcal{X}(U) \to \Omega^1(U); X \mapsto \langle X, \cdot \rangle, \\ \alpha_2 &: \mathcal{X}(U) \to \Omega^2(U); X \mapsto \omega(X, \cdot, \cdot), \\ \alpha_3 &: C^\infty(U) \to \Omega^3(U); f \mapsto f\omega, \end{aligned}$$

where $\omega = dx \wedge dy \wedge dz$ and $C^\infty(U)$, $\mathcal{X}(U)$, and $\Omega^k(U)$ denote the spaces of functions, vector fields, and k-forms on U.

Compute the maps α_i in the standard bases. Compute α_2, α_3 in terms of α_1 and the Hodge star (exercise 4.5, p. 76).

Show that the inner and cross products on $\mathbb{R}^3$ are related to the exterior product through

$$\begin{aligned} \alpha_1(X) \wedge \alpha_1(Y) &= \alpha_2(X \times Y) \\ \alpha_1(X) \wedge \alpha_2(Y) &= \alpha_3(\langle X, Y \rangle). \end{aligned}$$

Show that the classical operators grad, div, rot are related to the exterior derivative d through

$$\begin{aligned} df &= \alpha_1(\operatorname{grad} f), \\ d(\alpha_1(X)) &= \alpha_2(\operatorname{rot} X), \\ d(\alpha_2(X)) &= \alpha_3(\operatorname{div} X). \end{aligned}$$

Show how the classical theorems of Gauss and Stokes can be deduced from Stokes theorem for forms.