5B1473 Elementary differential geometry, spring 2007

Homework problems 4 — To be handed in 1/5

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1 Suppose M and N are oriented compact manifolds of dimension m and that $f : M \to N$ is a smooth map such that $\int_M f^*\omega \neq 0$ for some m-form ω on N. Show that f must be surjective.

(*hint:* See section 5.5 in the book.)

2 A manifold M is called simply connected if every piecewise smooth map $S^1 \to M$ can be extended to a piecewise smooth map $D^2 \to M$. Show that $H^1_{\mathrm{dR}}(M) = 0$ if M is simply connected.

3 For integers p, q define maps $f_{pq} : S^1 \to T^2$ by

$$S^1 \ni (\cos t, \sin t) \mapsto (\cos pt, \sin pt, \cos qt, \sin qt) \in S^1 \times S^1.$$

Compute the induced maps on H^1 and show that f_{pq} and $f_{p'q'}$ are not homotopic if $(p, q) \neq (p', q')$.

4 A symplectic form on a manifold M of dimension $2n$ is a closed two-form ω such that $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ is never zero. Show that ω^n is not exact. Show that $H^{2k}_{\mathrm{dR}}(M^{2n}) \neq 0$ for $k = 0, \ldots, n$ if M^{2n} has a symplectic form. Show that S^2 is the only sphere which has a symplectic form.

5 Compute $H^k_{\mathrm{dR}}(M)$, $k = 0, \ldots, 3$ where M is $\mathbb{R}^3$ with the x-axis and the point $(0, 0, 1)$ removed. Find explicit forms representing basis vectors of $H^k_{\mathrm{dR}}(M)$.