Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2006
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Homework assignment 3

This exercise set is due October 31, 2006

The Adding Machine (Misiurewicz) The purpose of this set of exercises is to construct and analyze a continuous map of I which has exactly one periodic point of period 2^j for each j and no other periodic points. The construction of the map relies on the notion of the double of a map, a topic discussed in class and in Robinson, Example 10.1.17, p. 375-376. Start with $f_0 = 1/3$. Let $f_1(x)$ denote the double of $f_0(x)$, i.e. $f_1(x)$ is obtained from $f_0(x)$ by the following procedure:

1. $f_1(x) = \frac{1}{3} f_0(3x) + \frac{2}{3}$ if $0 \leq x \leq 1/3$.
2. $f_1(2/3) = 0$; $f_1(1) = 1/3$.
3. f_1 is continuous and linear on the intervals $1/3 \leq x \leq 2/3$ and $2/3 \leq x \leq 1$.

That is, the graph of f_1 is obtained from f_0 as shown in Figure 10.1.9, Robinsson p. 376. Inductively, we define f_{n+1} to be the double of f_n. Finally let $F(x) = \lim_{n\to\infty} f_n(x)$.

1. Prove that $f_{n+j}(x) = f_n(x)$ for all $j \geq 1$ and $x \geq 1/3^n$. Conclude that if we define $F(0) = 1$, then $F(x)$ is a well-defined continuous map of I.
2. Prove that $f_n(x)$ has a unique periodic orbit 2^j for each $j \leq n$. Prove that each of these periodic orbits is repelling, if $j < n$.
3. Prove that $f_n(x)$ has no other periodic orbits.
4. Prove that $F(x)$ has a unique periodic orbit of period 2^j for each j and no other periodic orbits. Show that this periodic orbit is repelling.

   Recall the construction of the Cantor Middle-Third set. Let $A_0 = (\frac{1}{3}, \frac{2}{3})$ be the middle third of the unit interval I. Let $I_0 = I \setminus A_0$. Let $A_1 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ be the middle third of the two intervals in I_0. Let $I_1 = I_0 \setminus A_1$. Inductively, let A_n denote the middle third of the intervals in I_{n-1} and let $I_n = I_{n-1} \setminus A_n$. Finally, let

$$I_\infty = \bigcap_{n\geq 0} I_n.$$

I_∞ is the classical middle-thirds Cantor set.

5. Show that the periodic points of period 2^j for F lie in the union of intervals wich comprise A_j.
6. Prove that $F(I_n) = I_n$.
7. Prove that if $x \in A_n$ and x is not periodic, then there exists $k > 0$ such that $F^k(x) \in I_n$.
8. Prove that I_∞ is invariant under F.
9. Prove that, if $x \notin I_\infty$ and x is not periodic, then the orbit of x tends to I_∞ or eventually lies in I_∞.

Thus all of the non-periodic points for F are attracted to the set I_∞. Thus to understand the dynamics of F, it suffices to understand the dynamics of F on I_∞. For each point $p \in I_\infty$, we attach an infinite sequence of 0's and 1's, $S(p) = (s_0 s_1 s_2 \ldots)$, according to the rule: $s_0 = 1$ if p belongs to the left compontent of I_0; $s_0 = 0$ if p belongs to the right component. Note that this is slighly different from our coding for the quadratic map! Now p belongs to some component of I_{n-1}, and I_n is obtained by removing the middle third of this interval. Therefore we may set $s_n = 1$ if p belongs to the left hand interval in I_n an $s_n = 0$ otherwise.

Let Σ_2 be the set of all sequences of 0's and 1's. Define the adding machine $A : \Sigma_2 \to \Sigma_2$ by $A(s_0 s_1 s_2 \ldots) = (s_0 s_1 s_2 \ldots) + (100\ldots) \bmod 2$, i.e. A is obtained by adding 1 mod 2 to s_0 and carrying the result. For example, $A(110\overline{110}\ldots) = (001\,110\,\overline{110}\ldots)$ and $A(11\overline{1}\ldots) = (00\overline{0}\ldots)$.

10. Let d be the usual distance of Σ_2. Prove that $S : I_\infty \mapsto \Sigma_2$ is a topological conjugacy between F on I_∞ and A on Σ_2.
11. Prove that A has no periodic points.
12. Prove that every orbit of A is dense in Σ_2.

Since Σ_2 has no proper closed invariant subsets under A, Σ_2 is an example of a *minimal set.*