Failure Probabilities in Computing and Data Transmission

Søren Asmussen

Aarhus University, Denmark

http://home.imf.au.dk/asmus

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SA, Fiorini, Lipsky, Rolski, Sheahan

Asymptotic total times for task that must restart after a failure occurs

Mathematics of Operations Research **2008/9**

SA, Lars N. Andersen

Failure recovery, parallel computing, and extreme values

Journal of Statistical Methodology and Applications

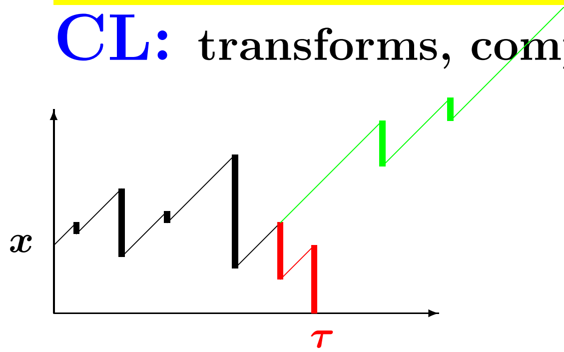
2 working papers (checkpointing, importance sampling)

Related work: Jelenkovic & Tan

Cramér-Lundberg Theory

$\psi(x) \sim C\mathrm{e}^{-\gamma x}$

CL: transforms, complex analysis



Classical expression P–K–B–B

Pollaczek-Khintchine-Beekman-Bowers

β Poisson int'y, unit premium rate

B claim distr'n, integrated tail B_I:

$$B_I(x) = \frac{1}{\mu}\int_0^x \overline{B}(y)\,dy\text{l}$$

$$\psi(u) \ = \ (1-\rho)\sum_{n=0}^{\infty} \rho^n \overline{B}_I^{*n}(u)$$

$$\rho = \beta\mu_B < 1$$

Geometric sum

Geometric Sums

$S_N = V_1 + \cdots + V_N$

$V_n \sim F$, N geom(ρ)

$Z(x) = P(S_N > x)$ solves

$$Z(x) = \rho\overline{F}(x) + \int_0^x Z(x-y)\,\rho F(\mathrm{d}y) \tag{1}$$

Defective renewal equation

Solution exponentially decaying

$\rho F(\mathrm{d}y)$ has mass $\rho < 1$

Choose γ as solution of

$$\int_0^\infty \mathrm{e}^{\gamma x}\,\rho F(\mathrm{d}x) = 1$$

Let

$F^*(\mathrm{d}x) = e^{\gamma x}\,\rho F(\mathrm{d}x)$

$Z^*(x) = e^{\gamma x} Z(x)$

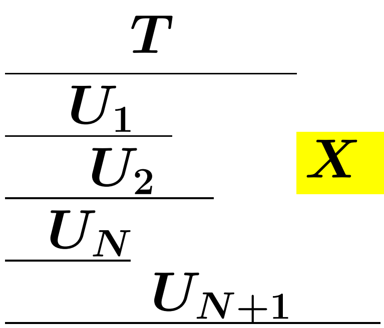
$z^*(x) = e^{\gamma x}\rho\overline{F}(x)$

Multiply (1) by $\mathrm{e}^{\gamma x}$

$Z(x) \sim C e^{-\gamma x}$

RESTART

T ideal job time $\sim$ F
(program time; file length; call center time)
U failure time $\sim$ G
(often exponential)
X total time $\sim$ H



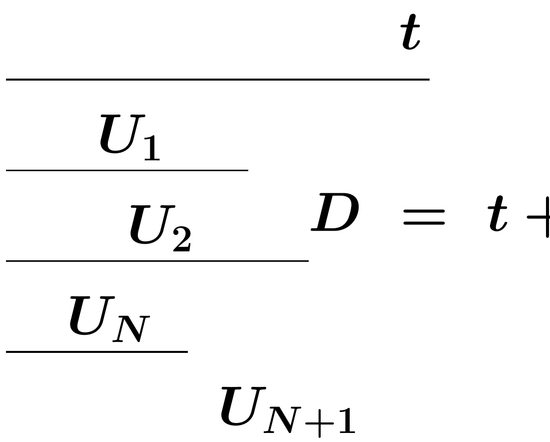
$$X = T + U_1 + \cdots + U_N$$

Target: tail $\overline{H}(x) = P(X > x)$

T bounded $\Rightarrow$ $\overline{H}(x) \approx \mathrm{e}^{-\gamma x}$

T unbd $\Rightarrow$ H heavy-tailed

$$T \equiv t$$



$$D = t + U_1 + \cdots + U_N$$

Geometric sum

Succes probability $\overline{G}(t)$

Summands: $U \mid U \leq t$

$\gamma(t)$ solution of $1 = \int_0^t e^{\gamma(t)u} g(u)\,du$

$$P(S > x) \sim C(t) e^{-\gamma(t)x}$$

$$\overline{H}(x) \sim e^{\gamma(t)t} C(t) e^{-\gamma(t)x}$$

$\gamma(t)$ solution of $1 = \int_0^t e^{\gamma(t)u} g(u)du$

$\gamma(t) \sim \mu\overline{G}(t)$ $\mu = 1/EU$

Crucial lemma:

$$\overline{H}(x) = \int_0^\infty \overline{H}_t(x) f(t)\, \mathrm{d}t$$
$$\sim \int_{t_0}^\infty \exp\{-\mu\overline{G}(t)x\} f(t)\, \mathrm{d}t$$

Purely analytical problem; non-trivial

Diagonal case $F = G$:

$y = \mu G(t)x \Rightarrow dy = \mu x f(t)\, dt$

$$\frac{1}{\mu x}\big(1 - \exp\{-\mu\overline{G}(t_0)x\}\big) \sim \frac{1}{\mu x}$$

General idea:

same substitution

assume F, G are connected s.t.

this leads to integral of

Abelian/Tauberian type.

$$f(t) \;=\; g(t)\overline{G}(t)^{\beta-1}L_0\big(\overline{G}(t)\big)$$

$$\overline{H}(x) \;\sim\; \frac{\Gamma(\beta)}{\mu^\beta}\frac{L_0(1/x)}{x^\beta}$$

f, g regularly varying
of form $L(t)/t^{1+\alpha}$
$\alpha_F, \alpha_G, L_F, L_G$

$$\overline{H}(x) \;=\; L_H(x)/x^{\alpha_H}$$

$\alpha_H = \alpha_F/\alpha_G,\, L_H(x) \;\sim\; c_H L'_H(x)$

$$c_H = \frac{\Gamma(\alpha_H)(\alpha_G+1)^{\alpha_H-1}}{\mu^{\alpha_H}}$$

$$L'_H(x) = \frac{L_F\big(x^{1/\alpha_G}\big)}{L_G^{\alpha_H}\big(x^{1/\alpha_G}\big)}.$$

f, g
of form $\exp\{-\lambda t^\eta\} t^{1+\alpha} L(t)$
$\lambda_F, \alpha_F, \lambda_G \alpha_G, L_F, L_G$

$$\overline{H}(x) = \log^{\kappa_H} x L_H(x)/x^{\alpha_H}$$

$\alpha_H = \lambda_F/\lambda_G, L_H(x) \sim c_H L'_H(x)$

$$c_H = \frac{\Gamma(\alpha_H)}{\mu^{\alpha_H} \lambda_G^{\omega_H + \alpha^H - 1}}$$
$$\omega_H = \alpha_F/\eta - \alpha_G \lambda_F/\lambda_G \eta$$
$$\kappa_H = \omega_H + (1 - 1/\eta)(\lambda_F/\lambda_G - 1)$$
$$L'_H(x) = \frac{L_F\left(\log x^{1/\eta}\right)}{L_G^{\alpha_H}\left(\log x^{1/\eta}\right)}.$$

F Gamma-like: $\overline{F}(x) \sim A x^\eta e^{-\delta x}$
$g(t) = \beta e^{-\beta t}$

$$\overline{H}(x) \sim \frac{A\Gamma(\delta/\beta)}{\beta^{\delta/\beta - 1 - \eta}} \frac{\log^\eta x}{x^{\delta/\beta}}$$

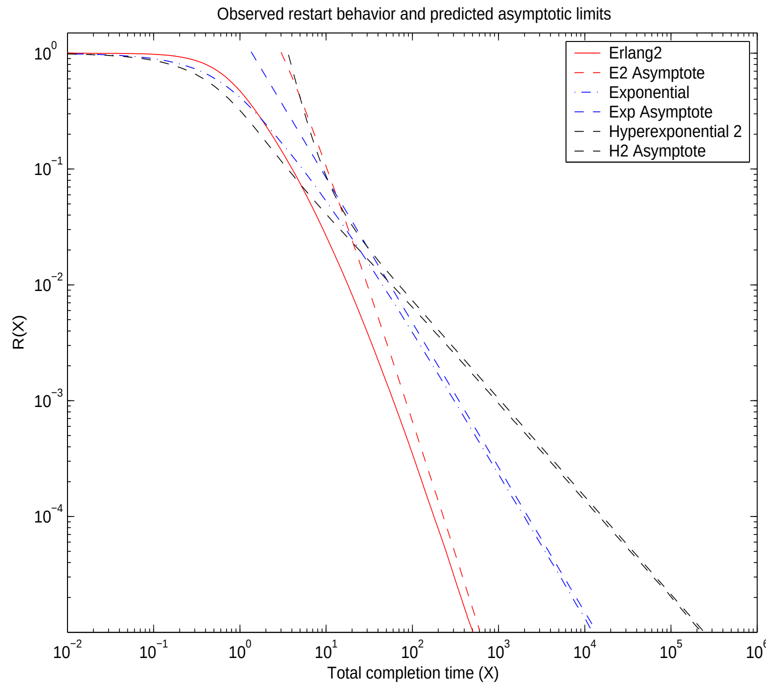


Figure 1:

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Crude Monte Carlo
Computationally heavy
Importance sampling

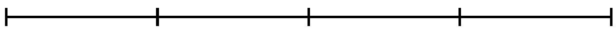
4 examples of each of F, G:
LT Weibull
exponential
HT Weibull
power

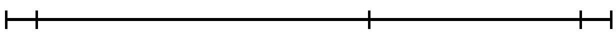
. $\overline{F}(t)$ / $\overline{G}(u)$	e^{-t^2}	e^{-t}	$e^{-t^{1/2}}$	$\frac{1}{t^\alpha}$
e^{-u^2}	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$e^{-\log^{1/4} x}$	$\frac{1}{\log^{\alpha/2} x}$
e^{-u}	$e^{-\log^2 x}$	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$\frac{1}{\log^{\alpha} x}$
$e^{-u^{1/2}}$	$e^{-\log^4 x}$	$e^{-\log^2 x}$	$\frac{1}{x}$	$\frac{1}{\log^{2\alpha} x}$
$\frac{1}{u^\alpha}$	$e^{-x^{\frac{2}{2+\alpha}}}$	$e^{-x^{\frac{1}{1+\alpha}}}$	$e^{-x^{\frac{1/2}{1/2+\alpha}}}$	$\frac{1}{x}$

Constants omitted $e^{-c\log^{1/2} x}$; $\frac{1}{x} = e^{-\log x}$
In some corners even $\log\log$ asymptotics

Fragmentation

K parts





Parallel computing:

$$X = \max(X_1, \ldots, X_K)$$

Checkpointing:

$$X = X_1 + \cdots X_K$$

Parallel computing

$T_1, \ldots, T_K$ i.i.d. Gamma(α_K, λ)
Exponential(μ) failures

$\alpha_K \equiv K$: $\mathbb{P}(X_i > x) \sim c \log^{\alpha-1}/x^r$,
$r = \lambda/\mu$

$$\frac{L(K)}{K^{\beta}} X \rightarrow \text{Frechet}$$

$\alpha_K \rightarrow \infty$:

$$X \approx e^{r\alpha_K}$$

$$\frac{\log X}{r\alpha_K} \rightarrow c$$

Checkpointing

K segments of lengths $h_1, \ldots, h_K$
Checkpoints $t_0 = 0, t_1 = h_1, t_2 = h_1 + h_2, \ldots$

A: T is deterministic, $T \equiv t$, and the checkpoints are deterministic and equally spaced, $t_1 = t/K$, $t_2 = 2t/K, \ldots, t_{K-1} = (K-1)t/K$. Equivalently, $h_k = t/K$.

B: T is deterministic, $T \equiv t$, and the checkpoints are deterministic but not equally spaced, for simplicity $h_k \neq h_\ell$ for $k \neq \ell$.

C: T is deterministic, $T \equiv t$, and the checkpoints are random. More precisely, the set $\{t_1, \ldots, t_{K-1}\}$ is the outcome of $K-1$ i.i.d. uniform r.v.'s on $(0,t)$. That is, $t_1 < \cdots < t_{K-1}$ are the order statistics of $K-1$ i.i.d. uniform r.v.'s on $(0,t)$.

D: T is random and the checkpoints equally spaced, $h_k \equiv h$. Thus, $K = \lceil T/h \rceil$ is random

E: T is random and the checkpoints are given by $t_k = t'_k T$ for a deterministic set of constants $0 = t'_0 < t'_1 < \ldots < t'_{K-1} < 1$.

Model A:

$$\mathbb{P}(X > x) \sim c_K x^{K-1} \mathrm{e}^{-x\gamma(t/K)}$$

Let $U_1, \ldots, U_m \geq 0$ be independent r.v.'s such that $\mathbb{P}(U_i > x) \sim C_i x^{\alpha_i - 1} \mathrm{e}^{-\eta x}$ for some $\eta > 0$, $\alpha_1, \ldots, \alpha_m > 0$ and some $C_1, \ldots, C_2$. Then

$$\mathbb{P}(U_1 + \cdots + U_m > x) \ \sim \ C x^{\alpha - 1} \mathrm{e}^{-\eta x},$$

where $\alpha = \alpha_1 + \cdots + \alpha_m$, $C = C_1 \Gamma(\alpha_1) \cdots C_n \Gamma(\alpha_m) / \lambda \Gamma(\alpha)$

Model B:

$$\mathbb{P}(X > x) \sim c \mathrm{e}^{-x\gamma(h^*)}$$

$$h^* = \max(h_1, \ldots, h_K$$

Let $U_1, U_2 \geq 0$ be independent r.v.'s such that $\mathbb{P}(U_1 > x) \sim C \mathrm{e}^{-\eta x}$ and $\mathbb{P}(U_2 > x) = \mathrm{o}\big(\mathrm{e}^{-\eta x}\big)$ for some $\eta > 0$. Then

$$\mathbb{P}(U_1 + U_2 > x) \ \sim \ C \mathbb{E} \mathrm{e}^{\eta U_2} \mathrm{e}^{-\eta x}.$$

Model C

Assume that the checkpoints are $0, t$ and $K-1$ i.i.d. uniform r.v.'s on $(0,t)$. That is, the checkpoints are $0 = t_0 < t_1 < \cdots < t_{K-1} < t_K = t$ where $t_1 < \cdots < t_{K-1}$ are the order statistics of $K-1$ i.i.d. uniform r.v.'s $R_1, \ldots, R_{K-1}$ on $(0,t)$. Then

$$\mathbb{P}(X^* > x) \sim C_7(t)\frac{\mathrm{e}^{-\gamma(t)x}}{x^{K-1}}$$

$$\begin{aligned}
&\mathbb{P}(h_{k^*} > y) \\
&\quad = (K-1)(1-y)^{K-2} \\
&\qquad -\binom{K-1}{2}(1-2y)^{K-2} \\
&\qquad + \cdots + (-1)^{i-1}\binom{K-1}{i}(1-iy)^{K-2}
\end{aligned}$$

Fisher 1929

$$\begin{aligned}
&\mathbb{P}(h_{k^*} \in \mathrm{d}y) \\
&\quad = (K-1)\mathbb{P}(R_1 \in \mathrm{d}y,\, k^* = 1) \\
&\quad = \mathbb{P}(R_1 \in \mathrm{d}y, R_2 > y, \ldots, R_{K-1} > y) \\
&\quad = \frac{K-1}{t^{K-1}}(1-y)^{K-2}\,.
\end{aligned}$$

Model D

Assume that the distribution of T is regularly varying, $\mathbb{P}(T > t) = L(t)/x^{\alpha}$ with $\alpha > 0$ and $L(\cdot)$ slowly varying. Then

$$\mathbb{P}(X > x) \sim \mathbb{P}\big(T > xh/m_1(h)\big) \quad (1)$$

$$\sim \frac{m_1(h)^{\alpha} L(x)}{h^{\alpha} x^{\alpha}} \quad (2)$$

Asmussen, Klüppelberg & Sigman 1999

Assume $f(t) \sim ct^{\alpha}\mathrm{e}^{-\lambda t}$ as $t \to \infty$ where $-\infty < \alpha < \infty$. Then $\mathbb{P}(X > x) \sim C_7\mathrm{e}^{-\gamma_2 x}$ where $\gamma_2 = \gamma_2(h)$ is the solution of $\widehat{H}_{[}\gamma_2] = \mathrm{e}^{\lambda h}$ and $C_7 = \ldots$

Let $U_1, U_2, \ldots > 0$ be i.i.d. and define $S_n = U_1 + \cdots + U_n$. Let $N \in \mathbb{N}$ be an independent r.v. such that $\mathbb{P}(N \geq n) \sim cn^{\alpha-1}\rho^n$ with $\alpha > 0$ and $0 < \rho < 1$. Assume that a solution $\theta > 0$ of $\rho\mathbb{E}\mathrm{e}^{\theta U} = 1$ exists with the additional property $\mathbb{E}\mathrm{e}^{\theta' U} < \infty$ for some $\theta' > \theta$. Then $\mathbb{P}(S_N > x) \sim C_9 x^{\alpha-1}\mathrm{e}^{-\theta x}$, where $\theta > 0$ is the solution of $\rho\mathbb{E}\mathrm{e}^{\theta U} = 1$

and

$$\boldsymbol{C_9 = \frac{c(1-\rho)}{\theta m_1(\theta)^{\alpha}}} \quad \textbf{where} \quad \boldsymbol{m_1(\theta) = \rho\mathbb{E}\big[U\mathrm{e}^{\theta}}$$