Cox Risk Processes and Ruin

Hanspeter Schmidli

University of Cologne

Risk Modelling in Insurance and Finance
in Honour of Jan Grandell's Birthday
Stockholm, 13th of June

1. The Classical Theory
   - The Cramér–Lundberg Model
   - The Sparre–Andersen Model

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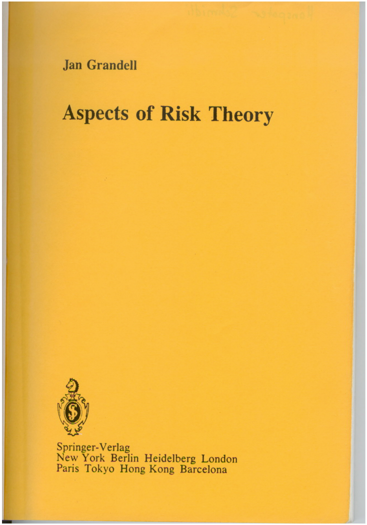
The Cramér–Lundberg Model

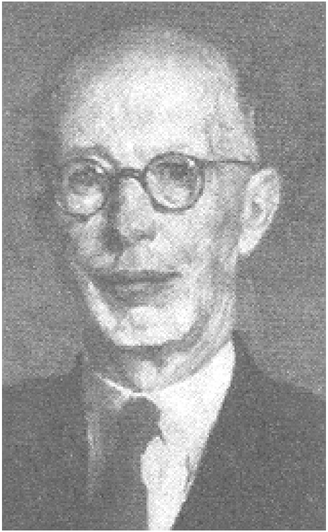

Jan Grandell

Aspects of Risk Theory

Springer-Verlag
New York Berlin Heidelberg London
Paris Tokyo Hong Kong Barcelona

Assumptions

We assume in this talk that all risk processes converge to ∞ and that all quantities defined are well-defined.



The Classical Risk Model

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i$$

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- $G(y)$: distribution function of Y_i, $G(0) = 0$
- $\mu_n = \mathbb{E}[Y_i^n]$, $\mu = \mu_1$, $M_Y(r) = \mathbb{E}[e^{rY}]$.

Change of Measure

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We let R be the strict positive solution to $\theta(r) = 0$.

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Under Q_r the model remains a Cramér–Lundberg model with intensity $\lambda_r = \lambda M_Y(r)$ and claim size distribution

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In particular, $Q_r[\liminf_{t\to\infty} X_t = -\infty] = 1$ for $\theta'(r) \ge 0$.

Ruin Probabilities

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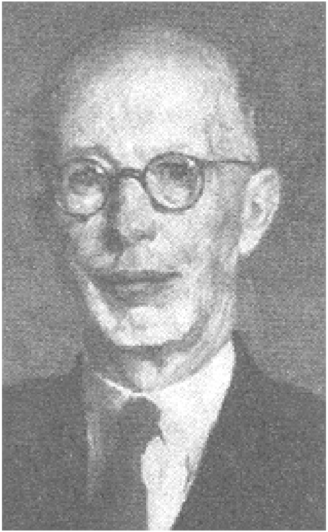
The Cramér–Lundberg Model



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From $X_\tau < 0$ we get

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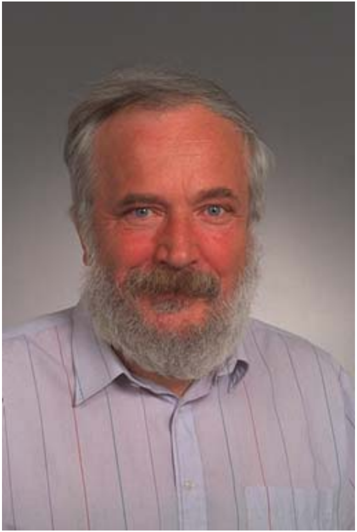

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Analogously,

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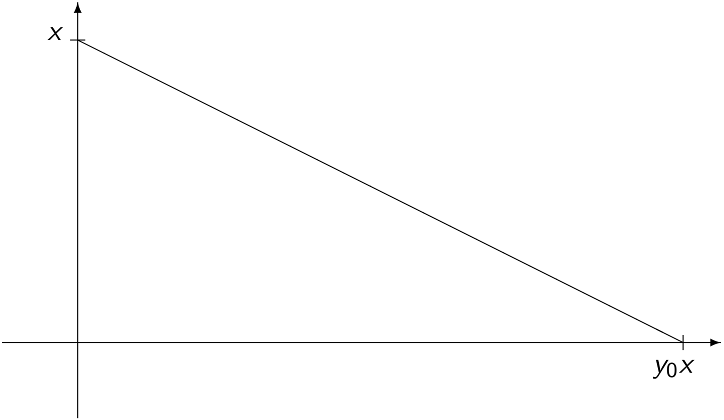
Moreover,

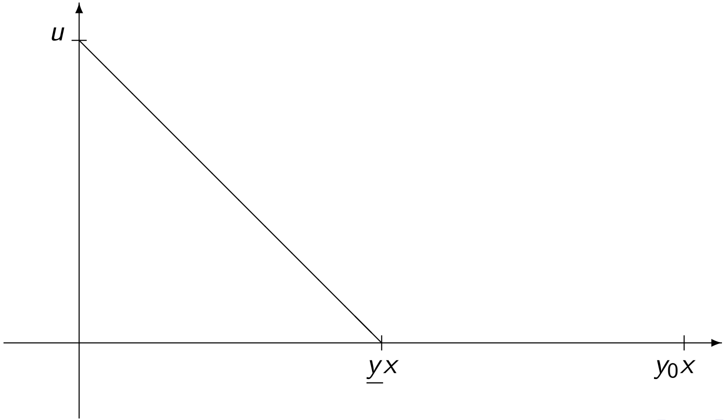
$$\frac{\tau}{x} \xrightarrow{\mathrm{P}} y_0$$

on $\{\tau < \infty\}$.

Path given $\{\tau < \infty\}$



Path given $\{\tau \le \underline{y}x\}$



The Cramér–Lundberg Approximation

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By considering the ladder times, the function $f(x) = \mathbb{E}_R[\mathrm{e}^{RX_\tau} \mid X_0 = x]$ fulfils a renewal equation. By the key renewal theorem we get

$$\lim_{x\to\infty} \psi(x)\mathrm{e}^{Rx} = \frac{c - \lambda\mu}{\lambda M_Y'(R) - c}\;.$$

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- $\lambda = (\mathbb{E}[T_i - T_{i-1}])^{-1}, \qquad M_T(r) = \mathbb{E}[e^{r(T_i - T_{i-1})}]$.

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Let $A_t = T_{N_t+1} - t$ be the time to the next claim. Then $\{X_t, A_t\}$ is a Markov process.

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We denote by R the positive solution to $\theta(r) = 0$. I.e., the solution to $M_Y(r)M_T(-cr) = 1$.

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It follows again by a renewal approach that

$$\lim_{x\to\infty} \psi(x)\mathrm{e}^{Rx} = C \; ,$$

where $C \ge 0$ is some constant. If $\theta'(R) < \infty$ then $C > 0$.

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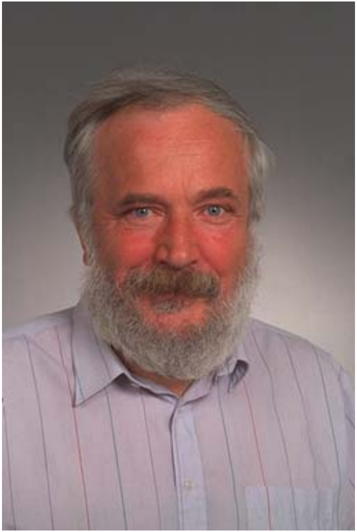
If L_i has a Gamma distribution then N_1 is negative binomially distributed.

The Björk–Grandell Model



The Björk–Grandell Model

Let $\{L_i, \sigma_i)\}$ be a sequence of iid random vectors with distribution function $F(\ell, s)$, where $L_i \ge 0$ and $\sigma_i > 0$. We denote by $S_i = \sum_{k=1}^{i} \sigma_k$. On the interval $[S_{i-1}, S_i)$ let $\{X_t\}$ behave like a classical risk model with claim intensity L_i and claim size distribution $G(y)$.

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We let $\lambda_t = L_i$ and $A_t = S_i - t$ if $S_{i-1} \le t \le S_i$. Then $\{(X_t, \lambda_t, A_t)$ is a Markov process.

Change of Measure

Consider the function

$$\varphi(\vartheta, r) := \mathbb{E}[\exp\{[L(M_Y(r) - 1) - \vartheta - cr -]\sigma\}] \,.$$

We let $\theta(r)$ be the unique solution to $\varphi(\theta(r), r) = 1$ and R be the strictly positive solution to $\theta(r) = 0$.

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The process

$$\{\mathrm{e}^{-r(X_t - x) + (\lambda_t(M_Y(r) - 1) - cr - \theta(r))A_t - \theta(r)t}\}$$

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is a martingale. We define the measure

$$Q_r[A] = \mathbb{E}[\mathrm{e}^{-r(X_t - x) + (\lambda_t(M_Y(r)-1) - cr - \theta(r))A_t - \theta(r)t}; A] \;.$$

Change of Measure

Under the new measure the process is again a Björk–Grandell model with

$$Q_r[Y \le x] = \frac{1}{M_Y(r)} \int_0^x \mathrm{e}^{ry} \,\mathrm{d}G(y) \,,$$

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$$Q_r[L \le \ell, \sigma \le s] = \int_0^\ell \int_0^s \mathrm{e}^{[l(M_Y(r)-1)-cr-\theta(r)]w} F(\mathrm{d}l, \mathrm{d}w)$$

The claim intensity is $\lambda_t M_Y(r)$.

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The claim intensity is $\lambda_t M_Y(r)$.
The drift is $-\theta'(r)$.

The Ruin Probabilities

We can express the ruin probabilities as

$$\psi(x) = \mathbb{E}_R[\mathrm{e}^{RX_\tau - (\lambda_\tau (M_Y(R)-1) - cR)A_\tau}]\mathrm{e}^{-Rx} \ .$$

The Ruin Probabilities

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The finite time ruin probability becomes

$$\psi(x;t) = \mathbb{E}_r[\mathrm{e}^{r(X_\tau - x) - \{\lambda_\tau(M_Y(r)-1)-cr-\theta(r)\}A_\tau + \theta(r)\tau}; \tau \le t]\mathrm{e}^{-rx} \;.$$

The Assumption

We need the following assumption:
Suppose there is a constant $B > 0$ such that

$$\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{\{L(M_Y(r)-1)-cr-\theta(r)\}(\sigma-v)} \mid \sigma > v, L] \geq B \quad (a.s.). \qquad (\mathscr{A}_r)$$

for all $v \geq 0$.

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for all $v \geq 0$.

The condition is fulfilled if $\mathbb{E}_{\mathbb{P}}[\sigma - v \mid \sigma > v, L = \ell] < \infty$ for all $\ell \leq (M_Y(r) - 1)^{-1}(cr + \theta(r))$.

Lundberg Inequalities

Under the assumption $(\mathscr{A}_R)$ one has

$$\limsup_{x\to\infty} \psi(x)\mathrm{e}^{Rx} < \infty \; .$$

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Under the assumption $(\mathscr{A}_{\bar{R}})$ one has

$$\limsup_{x\to\infty} (\psi(x) - \psi(x;\bar{y}x))\mathrm{e}^{\bar{R}x} < \infty\ .$$

Here $\underline{R}$ and $\bar{R}$ are defined as above.

Cramér–Lundberg Approximation

Also here we get under $(\mathscr{A}_{\underline{R}})$ and $(\mathscr{A}_{\bar{R}})$ the limit

$$\frac{\tau}{x} \stackrel{\mathrm{P}}{\longrightarrow} y_0$$

on $\{\tau < \infty\}$.

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on $\{\tau < \infty\}$.

If $(\mathscr{A}_r)$ holds for some $r > R$ then there is a constant $C > 0$ such that

$$\lim_{x \to \infty} \psi(x) \mathrm{e}^{Rx} = C \; .$$

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The Markov-Modulated Model

The Markov-Modulated Model

Let $\{J_t\}$ be a Markov chain with state space $\{1, 2, \ldots, \mathscr{J}\}$ and intensity matrix $\boldsymbol{\eta} = (\eta_{ij})$. On $\{J_t = i\}$ the process $\{X_t\}$ behaves like a classical model with intensity L_i and claim size distribution G_i. We denote by $\{\pi_i\}$ the stationary distribution of $\{J_t\}$.

The Markov-Modulated Model

Let

$$\mathbf{S}(r) = \begin{pmatrix} L_1(M_1(r)-1) & 0 & \cdots & 0 \\ 0 & L_2(M_2(r)-1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{\mathscr{J}}(M_{\mathscr{J}}(r)-1) \end{pmatrix}$$

and

$$\boldsymbol{\Theta}(r) = \eta + \mathbf{S}(r) - cr\mathbf{I}\ .$$

The Martingale

We have

$$\mathbb{E}[\mathrm{e}^{-r(X_t-x)}\mathbb{1}_{\{J_t=j\}} \mid J_0 = i] = (\mathrm{e}^{t\boldsymbol{\Theta}(r)})_{ij} \; .$$

The Martingale

We have

$$\mathbb{E}[\mathrm{e}^{-r(X_t-x)}\mathbb{1}_{\{J_t=j\}} \mid J_0=i] = (\mathrm{e}^{t\mathbf{\Theta}(r)})_{ij} \;.$$

Let $\theta(r)$ be the eigenvalue of $\mathbf{\Theta}(r)$ with the largest real part and $\mathbf{g}(r)$ be the corresponding eigenvector ($g_i(r) > 0$). Then

$$\frac{g_{J_t}(r)}{\mathbb{E}[g_{J_0}(r)]}\mathrm{e}^{-r(X_t-x)-\theta(r)t}$$

is a martingale

The Change of Measure

Define as before

$$Q_r[A] = \mathbb{E}\Bigl[\frac{g_{J_t}(r)}{\mathbb{E}[g_{J_0}(r)]}\mathrm{e}^{-r(X_t-x)-\theta(r)t}; A\Bigr] .$$

The Change of Measure

Define as before

$$Q_r[A] = \mathbb{E}\Big[\frac{g_{J_t}(r)}{\mathbb{E}[g_{J_0}(r)]} \mathrm{e}^{-r(X_t - x) - \theta(r)t}; A\Big] .$$

Under Q_r the process $\{(X_t, J_t)\}$ remains a Markov modulated risk model with claim intesities $L_i M_i(r)$, claim size distribution

$$Q_t[Y \le y \mid J = i] = (M_i(r))^{-1} \int_0^y \mathrm{e}^{rz} \, \mathrm{d}G_i(z)$$

and intensity matrix $\operatorname{diag}((g_i(r))^{-1})\eta \operatorname{diag}(g_i(r))$.

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and intensity matrix $\mathrm{diag}((g_i(r))^{-1})\eta \, \mathrm{diag}(g_i(r))$. In particular, the drift is $-\theta'(r)$.

The Ruin Probabilities

Let R be the strictly positive solution to $\theta(r) = 0$. Then

$$\psi(x) = \mathbb{E}_{\mathbb{P}}[g_{J_0}(R)]\mathbb{E}_R\Bigl[\frac{e^{RX_\tau}}{g_{J_\tau}(R)}\Bigr]e^{-Rx} ,$$

$$\psi(x;t) = \mathbb{E}_{\mathbb{P}}[g_{J_0}(r)]\mathbb{E}_r\Bigl[\frac{e^{rX_\tau+\theta(r)\tau}}{g_{J_\tau}(r)};\tau \le t\Bigr]e^{-rx} .$$

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Lundberg inequalities and Cramér–Lundberg approximation follow as before.

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Cox Model with Piecewise Constant Intensities

Let $\{(J_i, \sigma_i)\}$ be some Markov chain (with infinite state space), where $\sigma_i > 0$. We define $S_i = \sum_{j=1}^{i} \sigma_i$. On $[S_{i-1}, S_i)$ the process $\{X_t\}$ behaves like a classical model with intensity $L(J_i) \geq 0$ and claim size distribution $G(y; J_i)$.

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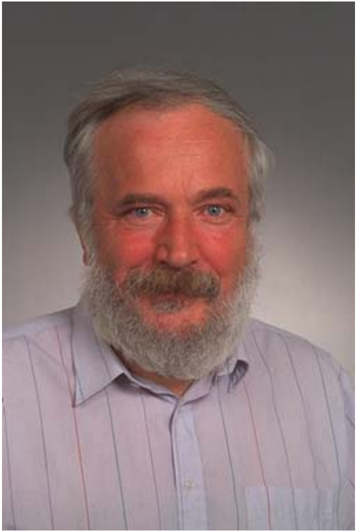
Let $\{(J_i, \sigma_i)\}$ be some Markov chain (with infinite state space), where $\sigma_i > 0$. We define $S_i = \sum_{j=1}^{i} \sigma_i$. On $[S_{i-1}, S_i)$ the process $\{X_t\}$ behaves like a classical model with intensity $L(J_i) \geq 0$ and claim size distribution $G(y; J_i)$.

Under additional conditions (for ergodicity) the Lundberg inequalities and the Cramér–Lundberg approximation holds.



Diffusion Intensities

Let $\{Z_t\}$ be a diffusion process following the stochastic differential equation

$$\mathrm{d}Z_t = b(Z_t)\,\mathrm{d}W_t + a(Z_t)\,\mathrm{d}t$$

for some Brownian motion $\{W_t\}$. The claim number process $\{N_t\}$ is a compound Poisson process with rate $\{\ell(Z_t)\}$.

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The process $\{g(Z_t)\mathrm{e}^{-r(X_t-x)-\theta(r)t}\}$ is a martingale if

$$\tfrac{1}{2}b^2(z)g''(z) + a(z)g'(z) + [\ell(z)(M_Y(r)-1) - \theta - cr-]g(z) = 0\;.$$

The Change of Measure

Consider the measure

$$Q_r[A] = \frac{\mathbb{E}[g(Z_T)\mathrm{e}^{-r(X_T-x)-\theta(r)T}; A]}{\mathbb{E}[g(Z_0)]} .$$

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The process $(\{X_t, Z_t\})$ remains a Cox model with claim size distribution

$$Q[Y \le x] = (M_Y(r))^{-1}\int_0^x \mathrm{e}^{ry}\,\mathrm{d}G(y)\;,$$

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claim intensity $\ell(Z_t)M_Y(r)$, and generator of the diffusion

$$\tilde{\mathfrak{A}}f = \frac{ga + b^2 g'}{g} f' + \tfrac{1}{2} b^2 f'' .$$

The Ruin Probabilities

The drift of $\{X_t\}$ is again $-\theta'(r)$.

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$$\psi(x;t) = \mathbb{E}_{\mathbb{P}}[g(Z_0)]\mathbb{E}_R\Bigl[\frac{\mathrm{e}^{rX_\tau+\theta(r)\tau}}{g(Z_\tau)};\tau \le t\Bigr]\mathrm{e}^{-rx} \;.$$

The Lundberg Inequalities

If $g(z)$ is bounded away from zero we obtain as before

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$$\psi(x) - \psi(x; \bar{y} x) < \frac{\mathbb{E}_{\mathbb{P}}[g(Z_0)]}{\inf g(z)} \mathrm{e}^{-\bar{R} x} \;,$$

where $\underline{R}$ and $\bar{R}$ are defined as before. If $\underline{y} < y_0 = 1/\theta'(R)$ $(\bar{y} > Y_0)$ then $\underline{R} > R$ $(\bar{R} > R)$.

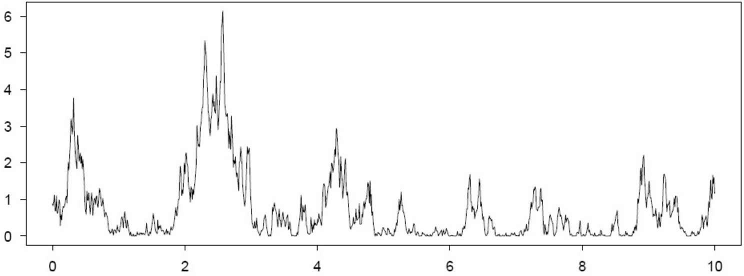
The Cramér–Lundberg Approximation

If we further suppose that $\{Z_t\}$ is Harris recurrent, then there is a constant C such that

$$\lim_{x \to \infty} \psi(x) e^{Rx} = C \; .$$

Ornstein–Uhlenbeck Intensities

Let $a(z) = -az$, $b(z) = b > 0$ and $\ell(z) = z^2$.



The function $g(z)$

Trying $g(z) = \kappa e^{kz^2}$ we find

$$k = \frac{a - \sqrt{a^2 - 2b^2(M_Y(r) - 1)}}{2b^2}\;,$$

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$$\theta(r) = \frac{a - \sqrt{a^2 - 2b^2(M_Y(r) - 1)}}{2} - cr .$$

Choosing κ such that $\mathbb{E}[g(Z_0)] = 1$ gives

$$\kappa = \sqrt{\frac{a + \sqrt{a^2 - 2b^2(M_Y(r) - 1)}}{2a}} .$$

The Diffusion Under Q_r

For the diffusion we find

$$\begin{aligned}\tilde{\mathfrak{A}}f(z) &= \frac{-\kappa \mathrm{e}^{kz^2} az + b^2 2kz\kappa \mathrm{e}^{kz^2}}{\kappa \mathrm{e}^{kz^2}} f'(z) + \tfrac{1}{2} b^2 f''(z) \\ &= -(a - 2kb^2) z f'(z) + \tfrac{1}{2} b^2 f''(z)\,.\end{aligned}$$

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The Sparre–Andersen Risk Model

Let $B(x)$ be the ladder height distribution. Then for $\rho = \psi(0)$

$$\psi(x) = (1-\rho)\sum_{n=1}^{\infty} \rho^n (1 - B^{*n}(x))\;.$$

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If $B(x)$ is subexponential we find

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In the Sparre–Andersen Risk model
$1 - B(x) \sim \mu^{-1} \int_x^\infty (1 - G(y)) \,\mathrm{d}y$, provided $\mu^{-1} \int_0^x (1 - G(y)) \,\mathrm{d}y$ is subexponential.

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The Björk–Grandell Model

If for large initial capital ruin occurs then the surplus will not recover until the next change of the intensity provided $\sigma_k L_k$ is not too large.

The Björk–Grandell Model

If for large initial capital ruin occurs then the surplus will not recover until the next change of the intensity provided $\sigma_k L_k$ is not too large.
Suppose there is a $\delta > 0$ such that $\mathbb{E}[e^{\delta\sigma L}] < \infty$. Then

$$\psi(x) \sim \frac{\mathbb{E}[\sigma L]}{c\mathbb{E}[\sigma] - \mathbb{E}[\sigma L]\mu} \int_x^\infty (1 - G(y))\,\mathrm{d}y\,,$$

provided $\mu^{-1}\int_0^x (1 - G(y))\,\mathrm{d}y$ is subexponential.

The Markov-Modulated Model

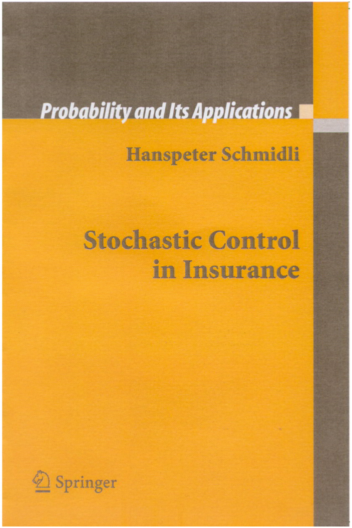
Let $H(x)$ be a distribution such that the integrated tail is a subexponential distribution. Suppose that $(1 - G_i(x)) \sim a_i(1 - H(x))$ and that $a = \sum_{i=1}^{\mathscr{J}} \pi_i L_i a_i > 0$. Then we find

$$\psi_i(x) \sim \frac{a}{c - \sum_{i=1}^{\mathscr{J}} \pi_i L_i \mu_i} \int_x^\infty (1 - H(y))\,\mathrm{d}y\;.$$

Ornstein–Uhlenbeck Intensity

For Ornstein–Uhlenbeck intensities regeneration is fast. Thus also here

$$\begin{aligned}\psi(x) &\sim \frac{b^2/(2a)}{c-\mu b^2/(2a)}\int_x^\infty (1-G(y))\,\mathrm{d}y \\ &= \frac{b^2}{2ac-\mu b^2}\int_x^\infty (1-G(y))\,\mathrm{d}y\;.\end{aligned}$$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Proportional Reinsurance

Consider the classical model and suppose the insurer can buy proportional reinsurance with retention level $\{b_t\}$. Then

$$X_t^b = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} b_{T_i-} Y_i \;,$$

where $c(b)$ is the part of the premium left for the insurer. We look for $\psi(x) = \inf_b \mathbb{P}[\inf_t X_t^b < 0]$.

The Hamilton–Jacobi–Bellman Equation

$\psi(x)$ solves then

$$\inf_b c(b)\psi'(x) + \lambda\Bigl[\int_0^\infty \psi(x - by)\,\mathrm{d}G(y) - \psi(x)\Bigr] = 0\;.$$

The Hamilton–Jacobi–Bellman Equation

$\psi(x)$ solves then

$$\inf_b c(b)\psi'(x) + \lambda\Bigl[\int_0^\infty \psi(x - by)\,\mathrm{d}G(y) - \psi(x)\Bigr] = 0\;.$$

The optimal strategy is $b_t = b^*(X_t^b)$, where $b(x)$ minimises the left hand side of the equation.

Optimal Investment

Suppose the insurer can invest into a risky asset

$$Z_t = Z_0 \exp\{(m - \tfrac{1}{2}\sigma^2) + \sigma W_t\}\ .$$

Using a strategy $\{A_t\}$ the surplus fulfils

$$\mathrm{d}X_t^A = (c + mA_t)\,\mathrm{d}t + \sigma A_t\,\mathrm{d}W_t - \mathrm{d}S_t\ .$$

We again want to minimise $\psi(x) = \inf_A \mathbb{P}[\inf_t X_t^A < 0]$.

The Hamilton–Jacobi–Bellman Equation

Then $\psi(x)$ fulfils

$$\begin{aligned} 0 &= \inf_A \tfrac{1}{2}\sigma^2 A^2 \psi''(x) + (c + mA)\psi'(x) \\ &\quad + \lambda\Bigl[\int_0^\infty \psi(x-y)\,\mathrm{d}G(y) - \psi(x)\Bigr] \end{aligned}$$

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Note that $\psi(x)$ becomes convex.

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Optimal Reinsurance: the Lundberg Bound

For a constant reinsurance strategy $b_t = b$ the Lundberg exponent $R(b)$ is the solution to

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We find $\psi(x) < \mathrm{e}^{-Rx}$.

Optimal Reinsurance: the Martingale

Let $\theta(b) = \lambda(M_Y(bR) - 1) - c(b)R \ge 0$. Then

$$M_t = \exp\Bigl\{-R(X_t - x) - \int_0^t \theta(b^*(X_s))\,\mathrm{d}s\Bigr\}$$

is a martingale.

Optimal Reinsurance: the Change of Measure

Define the measure $Q_R[A] = \mathbb{E}[M_T; A]$. Then $\{X_t\}$ is a PDMP with

intensity: $\lambda M_Y(Rb(x))$,

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The drift of the process is negative.

Optimal Reinsurance: the Ruin Probability

The function $\psi(x)$ can be expressed as

$$\psi(x) = \mathbb{E}_R\Bigl[\exp\Bigl\{RX_\tau + \int_0^\tau \theta(b^*(X_s))\,\mathrm{d}s\Bigr\}\Bigr]\mathrm{e}^{-Rx} \ge \mathbb{E}_R[\mathrm{e}^{RX_\tau}]\mathrm{e}^{-Rx} .$$

As for the classical model it follows that $\psi(x) \ge \underline{C}\mathrm{e}^{-Rx}$.

Optimal Reinsurance: the lim sup

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$$\int_0^x (g(x-z) - g(x))(1 - G(z/b^*))\mathrm{e}^{Rz}\,\mathrm{d}z \geq \int_x^\infty (1 - G(z/b^*))\mathrm{e}^{Rz}\,\mathrm{d}z g(x) - \delta(0)(1 - G(x/b^*))\mathrm{e}^{Rx}\,.$$

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We conclude that for x large enough, $g(x)$ stays close to $\limsup g(x)$ on an arbitrary long interval.
Thus also for x large enough, $f(x)$ stays close to $\limsup f(x)$ on an arbitrary long interval.

Optimal Reinsurance: the Cramér–Lundberg Approximation

Let $\xi = \limsup f(z)$. Choose $\beta > 0, \varepsilon > 0$ and x_0, such that $f(x) > \xi - \varepsilon$ for $x \in [x_0 - \beta, x_0]$. Then for $T = \inf\{t : X_t < x_0\}$

$$f(x) = \mathbb{E}_R\Bigl[f(X_T)\exp\Bigl\{\int_0^T \theta(b^*(X_s))\,\mathrm{d}s\Bigr\}\Bigr]$$

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f(x) &= \mathbb{E}_R\Bigl[f(X_T)\exp\Bigl\{\int_0^T \theta(b^*(X_s))\,\mathrm{d}s\Bigr\}\Bigr] \\
&\geq \mathbb{E}_R[f(X_T)]
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Therefore, $\lim f(x) = \liminf f(x) = \xi$.

Optimal Reinsurance: Asymptotics of the Strategy

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Thus $\limsup f'(x) = 0$.
Therefore $\theta(b^*(x)) \to 0$. If b^* is unique, then $\lim b^*(x) = b^*$.

Optimal Investment: the Lundberg Bound

For a constant investment strategy $A_t = A$ let $R(A)$ be the Lundberg coefficient, that is the solution to

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Note that $R = R(A^*)$ for $A^* = m/(\sigma^2 R)$.

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Optimal Investment: the Cramér–Lundberg Approximation

Analogously to before we get

$$\lim_{x\to\infty} \psi(x)\mathrm{e}^{Rx} = \xi$$

for some $\xi \in (0,1)$,

Optimal Investment: the Cramér–Lundberg Approximation

Analogously to before we get

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Optimal Investment: Subexponential Claims

Suppose that $G \in \mathcal{S}^*$; i.e., that

$$\lim_{x\to\infty} \int_0^x \frac{(1-G(z))(1-G(x-z))}{1-G(x)} \,\mathrm{d}z = 2\mu \;.$$

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Let $\kappa = 2\sigma^2\lambda/m^2$. Assume that

$$\lim_{y\to\infty} \ell(y) = \lim_{y\to\infty} \frac{G'(y)}{1-G(y)} = 0 \;.$$

Optimal Investment: the Function $g(x)$

Let $g(x) = -\psi'(x)/(1 - G(x))$.

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It follows, as expected, that $\lim_{x\to\infty} g(x) = 0$.

Optimal Investment: the Asymptotics

From the HJB equation we find that

$$\lim_{x\to\infty} \frac{\psi'(x)^2}{\psi''(x)(1-G(x))} = \frac{2\sigma^2\lambda}{m^2} = \kappa \; .$$

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Integration shows

$$\psi(x) \sim \kappa \int_x^\infty \frac{1}{\int_0^y \frac{1}{1-G(z)}\,\mathrm{d}z}\,\mathrm{d}y\;.$$

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By tail equivalence the results holds for all $G \in \mathcal{S}^*$.

Optimal Investment: Simpler Asymptotics

Suppose that $G \in \mathrm{MDA}(\exp\{-x^{-\alpha}\})$. Then

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Optimal Investment: Asymptotics of $A(x)$

We obtain the behaviour of $A(x)$

$$A(x) \sim \frac{m}{\sigma^2} \int_0^x \frac{1 - G(x)}{1 - G(z)} \, \mathrm{d}z \; .$$

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If $G \in \mathrm{MDA}(\exp\{-\mathrm{e}^{-x}\})$, then

$$\lim_{x \to \infty} A(x) a(x) = \frac{m}{\sigma^2} \,.$$

In particular, $A(x)/x$ tends to zero.

Optimal Reinsurance: MDA($\exp\{-x^{-\alpha}\}$)

Similar methods and technical considerations yield

$$\lim_{x\to\infty} \frac{\psi(x)}{\int_x^\infty (1-G(z))\,\mathrm{d}z} = \inf_b \frac{\lambda b^\alpha}{(c(b)-\lambda\mu b)^+}\;.$$

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Let b^* an argument where the inf is taken. If b^* is unique we get also

$$\lim_{x\to\infty} b(x) = b^*\;.$$

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Suppose that

$$\lim_{x\to\infty} \int_0^x \frac{(1-G(z))(1-G(x-z))}{1-G(x)} \,\mathrm{d}z = 2\mu$$

and that the distribution tail $1-G(x)$ is of rapid variation. Let $b_0 = \inf\{b : c(b) > \lambda\mu b\}$. Then for any $b > b_0$

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For the strategy we obtain that $\limsup_{x\to\infty} b(x) = b_0$.

Thank you for your attention