Pricing and hedging of mortality-linked securities

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Overview of research

- Pricing and hedging of mortality-linked cash flows
  - Derivatives (e.g. forwards, bonds and swaps) linked to the mortality of a certain population
  - Insurance portfolios, pension fund management

  in incomplete markets
- Stochastic modelling of risk factors
  - Mortality
  - Liabilities
  - Assets
- Numerical techniques
  - Integration quadratures
  - Numerical optimization

Mortality-linked instruments

The value of liabilities depends essentially on

- **Probability distribution**: description of future development of claims and investment returns, both involving significant uncertainties
- **Risk preferences**: the level of risk at which assets should cover liabilities
- **Hedging strategy**: investment strategy for the given capital

Mortality-linked instruments

- Denote by $S_{x,t} \in [0,1]$ the proportion of survivors in cohort $x \in X \subset \mathbb{N}$ at times $t = 0, 1, \dots, T$ (*Survivor index*)
- (Annuity) survivor bond: coupon payments proportional to $S_{x,t}$ at times $t = 0, 1, \dots, T$ in exchange for an initial payment V_0
- Survivor forward: exchange of an amount of $S_{x,T}$ for a fixed payment F at the moment T
- Survivor swap: exchange of a cash flow proportional to S_t for a fixed cash flow $\bar{S}_t$ at times $t = 0, 1, \dots, T$
- Pension fund management: insurance claims c_t that depend on $S_t = S_{x,t}$ as well as consumer price and pension indices
- Other examples and variants (e.g. zero-coupon bond with terminal payment $S_{x,T}$)

Mortality-linked instruments

The following market model is used for the pricing:

- A finite set J of liquid assets (e.g. bonds, equities, ...), cash account indexed by $j = 0$
- Return of asset j over period $[t-1, t]$ is denoted by $R_{t,j}$,
- The amount of wealth invested in asset j at time is t $h_{t,j}$
- $S_t = (S_{x,t})_{x \in X}$, $R_t = (R_{t,j})_{j \in J}$ and $h_t = (h_{t,j})_{j \in J}$ are the vectors of survivor indices, returns and investments, respectively
- $(S_t)_{t=0}^T$, $(R_t)_{t=0}^T$, $(h_t)_{t=0}^T$ are adapted stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=1}^T, \mathbb{P})$
- $\mathbb{P}$ reflects the investor's views on the future development of the stochastic factors

Mortality-linked instruments

The pricing problem for the issuer of a survivor bond can be formulated as

$$\begin{aligned}
\text{minimize} \quad & \sum_{j\in J} h_{0,j} \quad \text{over } h \in \mathcal{N} \\
\text{subject to} \quad & \sum_{j\in J} h_{t,j} = \sum_{j\in J} R_{t,j} h_{t-1,j} - S_{x+t,t} \quad t = 1,\dots,T \\
& h_{t,j} \in D_t,\, t = 1,\dots,T \\
& \sum_{j\in J} h_{T,j} \in \mathcal{A}.
\end{aligned}$$

- $\mathcal{N}$ denotes the $\mathbb{R}^J$ -valued investment strategies, adapted to the filtration $(\mathcal{F})_{t=1}^{T}$
- $D_t(\omega) \in \mathbb{R}^J$ is the set of feasible investment strategies at time t and state ω
- $\mathcal{A} \subset L^0(\Omega, \mathcal{F}_T, \mathbb{P})$ is an acceptance set that quantifies the decision maker's preferences about the terminal wealth

Mortality-linked instruments

- Acceptance set $\mathcal{A} = \{X \in L^0 \mid X \geq 0 \ \mathbb{P} - a.s.\}$ corresponds to superhedging
- $\mathcal{A} = \{X \in L^0 \mid \mathbb{P}(X \geq 0) \geq \delta\}$ corresponds to quantile hedging
- $\mathcal{A} = \{X \in L^0 \mid Eu(X) \geq u(0)\}$, where u is a utility function, corresponds to efficient hedging in the sense of Föllmer and Leukert
- $\mathcal{A} = \{X \in L^0 \mid \rho(X) \leq 0)\}$, where ρ is a convex risk measure, corresponds to risk measure pricing

In general, analytical solutions to the pricing problem are not available. For some $\mathcal{A}$ numerical solutions can be sought, e.g. with integration quadratures and stochastic optimization methods.

Mortality-linked instruments

The pricing problem for the issuer of a survivor forward can be formulated as

$$\begin{aligned} \text{minimize} \quad & F \\ \text{subject to} \quad & \sum_{j \in J} h_{t,j} = \sum_{j \in J} R_{t,j} h_{t-1,j} \\ & \sum_{j \in J} h_{0,j} = 0 \\ & h_{t,j} \in D_t, t = 1, \dots, T \\ & \sum_{j \in J} h_{T,j} + F - S_T \in \mathcal{A}. \end{aligned}$$

Mortality-linked instruments

Pricing problem for the issuer of a survivor swap can be formulated as

$$
\begin{aligned}
\text{minimize} \quad & \alpha \\
\text{subject to} \quad & \sum_{j \in J} h_{t,j} = \sum_{j \in J} R_{t,j} h_{t-1,j} + \alpha \bar{S}_t - S_t \quad t = 1, \ldots, T \\
& h_{0,j} = 0 \\
& h_{t,j} \in D_t, t = 1, \ldots, T \\
& \sum_{j \in J} h_{T,j} \in \mathcal{A}.
\end{aligned}
$$

- Finding the minimum acceptable rate, when fixed cash flows are a proportion of a forecast survival rate $\bar{S}_t$

Mortality-linked instruments

- The problem of determining the minimum initial capital required for acceptable hedging of pension liabilities can be formulated as

$$\begin{aligned}
\text{minimize} \quad & \sum_{j \in J} h_{0,j} \quad \text{over } h \in \mathcal{N} \\
\text{subject to} \quad & \sum_{j \in J} h_{t,j} = \sum_{j \in J} R_{t,j} h_{t-1,j} - c_t \quad t = 1, \ldots, T \\
& h_{t,j} \in D_t, t = 1, \ldots, T \\
& \sum_{j \in J} h_{T,j} \in \mathcal{A}
\end{aligned}$$

- The claims c_t depend on $(S_{x,t})_{x \in X}$ as well as the consumer price index

Stochastic modelling

- Modelling (the investor's view of) the probability distribution $\mathbb{P}$
  - Population dynamics $(S_t)_{t=1}^T$
  - Asset returns $(R_t)_{t=1}^T$
  - Other relevant information (inflation, GDP,...)

The mortality model

- Population dynamics described by a mortality model
- Several existing stochastic models for mortality (e.g. Lee&Carter, 1992)
- We propose a general discrete-time framework
  - Flexible but relatively simple
  - Incorporates population-specific characteristics and user preferences
  - Robust in calibration
  - Allows for a choice of easily interpretable risk factors

The mortality model

- Let $E(x,t)$ be the size of population aged $[x, x+1)$ (cohort) at the beginning of year t
- Objective: model the values of $E(x,t)$ over time $t = 0, 1, 2, \dots$ for a given set $X \subset \mathbb{N}$ of ages
- Assume the conditional distribution of $E(x+1, t+1)$ given $E(x,t)$ is binomial:

$$E(x+1, t+1) \sim \text{Bin}(E(x,t), p(x,t))$$

where $p(x,t)$ is the probability that an individual aged x and alive at the beginning of year t is still alive at the end of that year

The mortality model

- We reduce the dimensionality of $p(., t)$ by modelling the logistic probabilities by

$$\operatorname{logit} p(x,t) := \ln\left(\frac{p(x,t)}{1-p(x,t)}\right) = \sum_{i=1}^{n} v_i(t)\phi_i(x),$$

  where $\phi_i(x)$ are user-defined *basis functions* across cohorts, and $v_i(t)$ stochastic *risk factors* that vary over time
- In other words, $p(x,t) = p_{v(t)}(x)$, where $v(t) = (v_1(t), \ldots, v_n(t))$, and $p_v : X \to (0,1)$ is the parametric function defined for each $v \in \mathbb{R}^n$ by

$$p_v(x) = \frac{\exp\left(\sum_{i=1}^{n} v_i\phi_i(x)\right)}{1+\exp(\sum_{i=1}^{n} v_i\phi_i(x))}$$

- Modelling the logit transforms instead of $p(x,t)$ directly guarantees that $p(x,t) \in (0,1)$.

The mortality model

- Vector $v(t)$ of risk factors is modelled as a stochastic process, based on historical values, expert opinions, or both
- Historical values of $v(t)$ are constructed by maximum likelihood estimation, maximization problem is concave with very mild assumptions
- Selection of basis functions determines characteristics of the model
- Certain desired properties of $p(x, t)$, e.g. continuity or smoothness across cohorts, are achieved by corresponding choices of $\phi_i(x)$
  - Incorporation of user preferences and/or population-specific characteristics
- Appropriate choice of basis functions assigns interpretations to risk factors
- Concrete interpretations facilitate the modelling of risk factors, which is advantageous the engineering of mortality-linked instruments

Example: Modelling Finnish mortality

- We consider the mortality of Finnish males aged 18-100 years
- Data consists of annual values of $E(x, t)$, covering years 1900-2007 [1]
- A model with three parameters and three piecewise linear basis functions is fitted into the data
- We present simulations for future population dynamics and a simple survival bond hedging example

[1] Source: Human mortality database, www.mortality.org

Example: Modelling Finnish mortality

- The model:

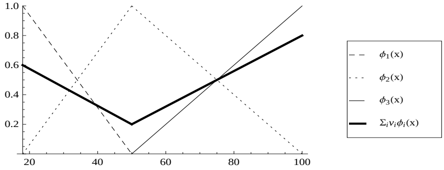
$$\operatorname{logit} \, p(x,t) = v_1(t)\phi_1(x) + v_2(t)\phi_2(x) + v_3(t)\phi_3(x),$$

where basis functions are piecewise linear:

$$\phi_1(x) = \begin{cases} 1 - \frac{x-18}{32} & \text{for } x \le 50 \\ 0 & \text{for } x \ge 50, \end{cases} \quad \phi_2(x) = \begin{cases} \frac{1}{32}(x-18) & \text{for } x \le 50 \\ 2 - \frac{x}{50} & \text{for } x \ge 50, \end{cases}$$

$$\phi_3(x) = \begin{cases} 0 & \text{for } x \le 50 \\ \frac{x}{50} - 1 & \text{for } x \ge 50. \end{cases}$$

- The linear combination now also piecewise linear:



- Values of $v_i(t)$ points on the $\operatorname{logit} p(x,t)$ curve: a natural interpretation

Example: Modelling Finnish mortality

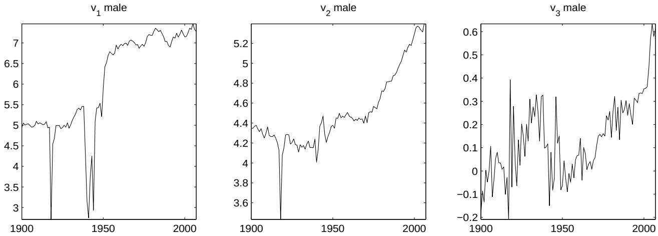


Figure: Estimated parameter values (logit survival probabilities for 18-, 50- and 100-year-olds).

Example: Modelling Finnish mortality

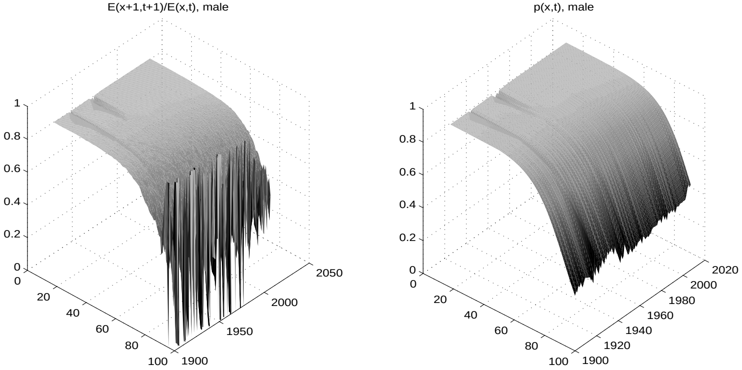


Figure: Estimated values of $p(x, t)$ vs. $\frac{E(x+1,t+1)}{E(x,t)}$ for three-factor model

Example: Modelling Finnish mortality

- The vector of risk factors $v(t)$ is modelled as a stochastic process
- Three-dimensional random walk with a drift is fitted into the estimated values of $v(t)$ for the years 1960-2007 (an even drift)
- Survival probabilities $p(x,t)$ and cohort sizes $E(x,t)$ for Finnish males were simulated for 30 years into the future by simulating the process v with the Monte Carlo method (sample size 10000)
- Population size $E(x+1,t+1)$ in each cohort was approximated by its expected value $E(x,t)p(x,t)$

Example: Modelling Finnish mortality

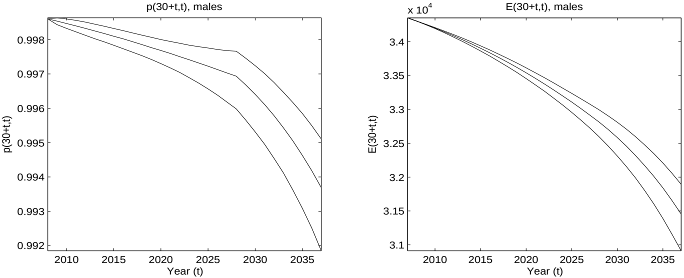


Figure: Medians and 90% confidence intervals for living probabilities $p(.,t)$ and cohort sizes $E(.,t)$. Cohort aged 30 in 2007.

Example: Modelling Finnish mortality

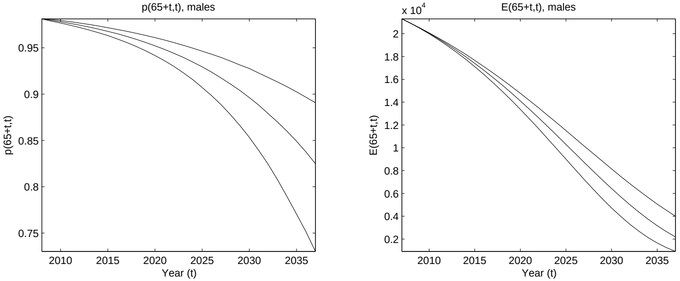


Figure: Medians and 90% confidence intervals for living probabilities $p(.,t)$ and cohort sizes $E(.,t)$. Cohort aged 65 in 2007.

Asset returns

- Modelling relevant asset returns
- Pension insurance liabilities typically hedged with bonds and equities
- Dependencies between assets and liabilities are essential in construction of hedging strategies
- Bonds, inflation-linked securities, equities in pharmaceutical or healthcare sectors,...

Numerical methods for the hedging problem

- Generally, analytical solutions are not available
- Numerical methods are needed to solve the optimization problem
- The infinite-dimensional space of feasible investment strategies can be approximated by a finite-dimensional subspace spanned by a finite set of basis strategies (Galerkin method)
- Linear combinations of basis strategies can be optimized using integration quadratures and numerical optimization techniques

Example: survivor bond

- We consider a survivor bond with coupons S_t and no terminal payment
- S_t is now the survival index of 65-year-old Finnish males, and $T = 30$ years
- Mortality model risk factors $v(t)$ modelled as a random walk with a drift as before
- We consider only one asset and model its returns as

$$\ln R_t = \mu + \sigma \epsilon_t,$$

  where $\mu = \sigma = 0.06$ (mean and standard deviation of annual returns are approx. 6%)
- Acceptance set is defined by means of risk measure $CV@R$ with risk level 85% (risk measure pricing)
- Monte Carlo method with 10000 simulations is employed to obtain the $CV@R$ value of the terminal wealth for a given initial capital
- Minimum initial capital computed with a simple line search

Example: survivor bond

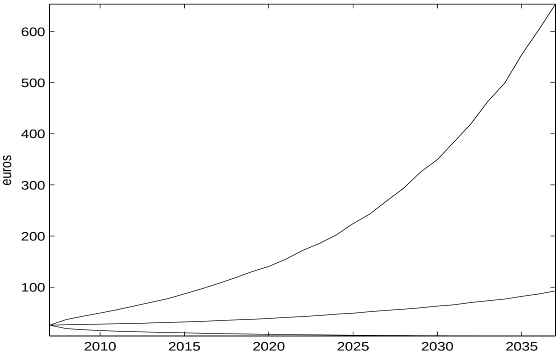


Figure: Evolution of the 10%, 50%, and 90% quantiles of the seller's total capital V_t when initial capital $V_0 = 25.9€$ corresponds to $CV@R_{85\%}$

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