Bahadur efficiency of a goodness-of-fit tests based on distribution characterizations.

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Introduction

In this talk we will consider two examples of goodness-of-fit tests: tests of normality and tests of exponentiality.

For each example we will

- Describe the limiting distributions of statistics under null hypothesis.
- Find their logarithmic large deviation asymptotics under null hypothesis.
- Calculate the local Bahadur efficiency of statistics under some common parametric alternatives.

Tests of normality based on Shepp property, and their efficiencies

- In 1964 L.Shepp discovered that if X and Y are two independent centred normal rv's with some variance $\sigma^2 > 0$, then the rv

$$k(X,Y) := 2XY/\sqrt{X^2+Y^2}$$

has again the normal distribution $\mathcal{N}(0,\sigma^2)$ (Shepp property.)

- J.Galambos and I.Simonelli proved in 2003, that the Shepp property implies the characterization of the normal law in a broad class of distributions. Consider the class $\mathcal{F}$, of df's F satisfying the conditions: i) $0 < F(0) < 1$ and ii) $F(x) - F(-x)$ is regularly varying in zero with the exponent 1.

Theorem

Let X and Y be independent rv's with common df F from the class $\mathcal{F}$. Then the equality in distribution $X \stackrel{d}{=} k(X,Y)$ is valid iff $X \in \mathcal{N}(0,\sigma^2)$ for some variance $\sigma^2 > 0$.

Introduction

Let $X_1, \ldots, X_n$ be independent observations with zero mean and df F, and let F_n be the usual empirical df based on this sample. We are interested in testing the composite hypothesis $H_0 : F \in \mathcal{N}(0, \sigma^2)$ for some unknown variance $\sigma^2 > 0$ against the alternatives H_1, under which the hypothesis H_0 is false. We build the U-*empirical* df H_n using the formula

$$H_n(t) = (C_n^2)^{-1} \sum_{1 \le i < j \le n} \mathbf{1}\{k(X_i, X_j) < t\}, \ t \in R^1.$$

Consider two statistics:

$$I_n = \int_{R^1} (H_n(t) - F_n(t)) dF_n(t),$$

$$D_n = \sup_{t \in R^1} |H_n(t) - F_n(t)|.$$

Limiting distribution of the statistic I_n

It is well-known that non-degenerate U- and V-statistics are asymptotically normal (Hoeffding, 1948). Let show that I_n belongs to this class.

The statistic I_n is asymptotically equivalent to the U-statistic of degree 3 with centered kernel

$$\Psi(x,y,z) = \frac{1}{3}\left(\mathbf{1}\{k(x,y) < z\} + \mathbf{1}\{k(x,z) < y\} + \mathbf{1}\{k(y,z) < x\}\right) - 1/2. \quad (1)$$

Let calculate the projection of the kernel (1)

$$\psi(x) := E(\Psi(X,Y,Z)|X = x) =$$
$$= \frac{1}{3}\left(\mathbb{P}(k(x,Y) < Z) + \mathbb{P}(k(x,Z) < Y) + \mathbb{P}(k(Y,Z) < x)\right) - 1/2.$$

After some calculations we obtain that the projection of the kernel Ψ is $\psi(x) = \frac{1}{3}(\Phi(x) - \frac{1}{2})$ with the variance $\sigma^2 = \int_{R^1} \psi^2(x) d\Phi(x) = \frac{1}{108}$.

Hence the kernel (1) is non-degenerate. By Hoeffding's theorem

$$\sqrt{n} I_n \xrightarrow{d} \mathcal{N}(0, \frac{1}{12}).$$

Limiting distribution of the statistic D_n

The rv $H_n(t) - F_n(t)$ for fixed t is asymptotically equivalent to U-statistic with the kernel depending on $t \in R^1$

$$\Xi(x, y; t) = \mathbf{1}\{k(x, y) < t\} - \frac{1}{2}(\mathbf{1}\{x < t\} + \mathbf{1}\{y < t\}).$$

The projection of this kernel for fixed t is equal to

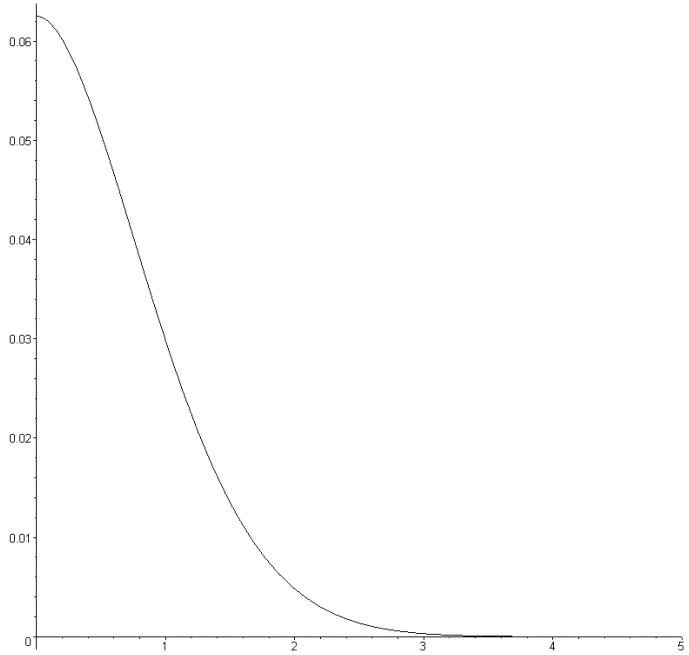
$$\xi(x; t) = \begin{cases} -\frac{1}{2}(\mathbf{1}\{x < t\} + \Phi(t)), \text{if } t < -2|x|; \\ \\ \Phi\left(\frac{t|x|}{\sqrt{4x^2 - t^2}}\right) - \frac{1}{2}(\mathbf{1}\{x < t\} + \Phi(t)), \text{ if } -2|x| \leq t \leq 2|x|; \\ \\ 1 - \frac{1}{2}(\mathbf{1}\{x < t\} + \Phi(t)), \text{ if } t > 2|x|. \end{cases}$$

Limiting distribution of statistic D_n

The variance of the projection has the form

$$\sigma_\xi^2(t) = E\xi^2(X;t) = \begin{cases} 2\int_{t/2}^{\infty} \Phi^2\left(\frac{tx}{\sqrt{4x^2-t^2}}\right) d\Phi(x) - \int_{t/2}^{t} \Phi\left(\frac{tx}{\sqrt{4x^2-t^2}}\right) d\Phi(x) - \\ -\frac{1}{4}\Phi(t) - \frac{1}{4}\Phi^2(t) + \Phi(t/2) - \frac{1}{2}, \text{ if } t \geq 0; \\ \\ 2\int_{-t/2}^{\infty} \Phi^2\left(\frac{tx}{\sqrt{4x^2-t^2}}\right) d\Phi(x) - \int_{-t}^{\infty} \Phi\left(\frac{tx}{\sqrt{4x^2-t^2}}\right) d\Phi(x) + \\ +\frac{1}{4}\Phi(t) - \frac{1}{4}\Phi^2(t), \text{ if } t < 0. \end{cases}$$

Taking in consideration the symmetry of the function $\sigma_\xi^2(t)$, we conclude that $\sup_{t\in\mathbb{R}} \sigma_\xi^2(t) = \sigma_\xi^2(0) = \frac{1}{16}$. This value is important when calculating large deviation asymptotics.



Limiting distribution of the statistic D_n

Limiting distribution of the statistic D_n is unknown. Using the methods of Silverman (1983), one can show that the U-empirical process

$$\eta_n(t) = \sqrt{n}\left(H_n(t) - F_n(t)\right), \; t \in R^1,$$

converges weakly as $n \to \infty$ to some centered Gaussian process $\eta(t)$ with complicated covariance. Consequently the sequence of statistics $\sqrt{n}D_n$ converges in distribution to $\sup_t |\eta(t)|$, which has very complicated distribution (currently unknown). But the critical values for statistics D_n can be found via simulating their sample distribution.

Large deviations and local efficiency of statistics I_n

The kernel Ψ is not only centered but bounded. Therefore from theorem of Nikitin and Ponikarov(1999) describing large deviations of non-degenerate U- and V-statistics we have for $a > 0$

$$\lim_{n\to\infty} n^{-1} \ln \mathbb{P}(I_n > a) = -f(a),$$

where the function f is continuous for sufficiently small $a > 0$, and, moreover,

$$f(a) = 6a^2(1 + o(1)), \text{ as } a \to 0.$$

Large deviations and local efficiency of statistics I_n

Let calculate local Bahadur efficiency of statistics I_n for some alternatives.

- Location alternative.
  Under H_1 the observations have the df $\Phi(x-\theta), \theta \geq 0$. Using the Law of Large Numbers for U-statistics, we see that the limit in probability of I_n under H_1 is equal to

$$b_I(\theta) = \mathbb{P}_\theta(k(X,Y) < Z) - \frac{1}{2}.$$

  Hence, as $\theta \to 0$ we have $b_I(\theta) \sim (2\sqrt{\pi})^{-1}\theta$. Therefore the local exact slope of the sequence I_n as $\theta \to 0$ admits the representation

$$c_I(\theta) \sim 12 b_I^2(\theta) \sim \frac{3}{\pi}\theta^2 \sim 0.955\ \theta^2.$$

  The theoretical upper bound is $2K(\theta) \sim \theta^2$ as $\theta \to 0$ (Muliere, Nikitin, 2002). It follows that the local Bahadur efficiency of our test is rather high and is equal to

$$eff^B(I) := \lim_{\theta \to 0}\{c_I(\theta)/2K(\theta)\} = 3/\pi \approx 0.9549.$$

Large deviations and local efficiency of statistics I_n

Let calculate local Bahadur efficiency of statistics I_n for some alternatives.

- Skew alternative in Azzalini's sense.
  Suppose that the observations have the density $2\varphi(x)\Phi(\theta x), \theta \geq 0,\ x \in R^1$. The calculations give

$$b_I(\theta) \sim \frac{1}{\pi\sqrt{2}}\theta, \theta \to 0.$$

  The local exact slope has the representation $c_I(\theta) \sim \frac{6}{\pi^2}\theta^2,\ \theta \to 0$. The upper bound $2K(\theta)$ for exact slopes under skew alternative is known as $2K(\theta) \sim \frac{2}{\pi}\theta^2$ (Muliere and Nikitin, 2002). Consequently, the local Bahadur efficiency of our test for skew alternative is again equal to $3/\pi \approx 0.9549$.

Large deviations and local efficiency of statistics I_n

Let calculate local Bahadur efficiency of statistics I_n for some alternatives.

- Contamination alternative. The observations have the df

$$G(x,\theta) = (1-\theta)\Phi(x) + \theta\Phi^2(x). \tag{2}$$

We get $b_I(\theta) \sim 0.1667\ \theta, \theta \to 0$. The local exact slope as $\theta \to 0$ admits the representation $c_I(\theta) \sim 0.3334\ \theta^2$. It is known that $2K(\theta) \sim \frac{4}{5}\ \theta^2$ (Litvinova 2004). Therefore the local Bahadur efficiency of our test for the contamination alternative is equal to 0.4167.

Large deviations and local efficiency of statistic D_n

The family of kernels $\{\Xi(x, y; t)\}, t \in R^1$ is not only centered but bounded. Hence using the result on large deviations of the supremum of a family of U-statistics (Nikitin, 2008), we obtain

$$\lim_{n\to\infty} n^{-1} \ln \mathbb{P}(D_n > a) = -f(a),$$

where the function f is continuous for sufficiently small $a > 0$, and, moreover,

$$f(a) = 2a^2(1 + o(1)), \text{ as } a \to 0.$$

Large deviations and local efficiency of statistic D_n

Let calculate the local Bahadur efficiency of D_n for some alternatives.

- Location alternative.
  According to Glivenko-Cantelli theorem for U-empirical df's the limit in probability of D_n is

$$b_D(\theta) = \sup_t |\mathbb{P}_\theta(k(X,Y) < t) - \Phi(t-\theta)|.$$

  Hence, as $\theta \to 0$ we have $b_D(\theta) \sim \frac{\theta}{\sqrt{2\pi}}$. Therefore the local exact slope admits the representation

$$c_D(\theta) \sim \frac{2}{\pi}\theta^2 \sim 0.6366 \cdot \theta^2.$$

  We know that in this case $2K(\theta) \sim \theta^2$ as $\theta \to 0$. It follows that the local Bahadur efficiency of our test is $2/\pi \approx 0.6366$.

Large deviations and local efficiency of statistic D_n

Let calculate the local Bahadur efficiency of D_n for some alternatives.

- Skew alternative.
  It is easy to see that for any t

$$b_D(\theta) \sim \frac{\theta}{\pi}, \ \theta \to 0.$$

  Therefore the local exact slope as $\theta \to 0$ looks like

$$c_D(\theta) \sim \frac{4}{\pi^2}\theta^2.$$

  We have already seen that for the skew alternative $2K(\theta) \sim \frac{2\theta^2}{\pi}$ as $\theta \to 0$. Consequently, the local Bahadur efficiency of our test is again equal to $2/\pi \approx 0.6366$.

Large deviations and local efficiency of statistic D_n

Let calculate the local Bahadur efficiency of D_n for some alternatives.

- Contamination alternative.
  We get easily that
  $$b_D(\theta) \sim \theta/4, \ \theta \to 0.$$
  Hence we have the asymptotics $c_D(\theta) \sim \theta^2/4$. As seen before, in this case $2K(\theta) \sim \frac{4}{5}\theta^2$ as $\theta \to 0$. Hence, the local Bahadur efficiency of our test is 0.3125.

Concluding remarks

We can remark that the efficiencies of our two tests for the *composite* hypothesis H_0 of normality under location alternative (respectively $3/\pi$ and $2/\pi$) coincide with the efficiencies under the same alternative of classical tests based on the Chapman-Moses statistic ω_n^1 and Kolmogorov statistic when testing the *simple* hypothesis of normality (they are known since 1970-s).

It would be interesting to find the theoretical explanation of this empirical observation.

Tests of exponentiality based on Rossberg's characterization, and their efficiencies

We propose two new tests of exponentiality based on Rossberg's characterization of exponential distribution:

Let $X_1, \ldots, X_n$ be non-negative i.i.d. rv's. Then for any j statistics $X_{j+s,n} - X_{j,n}$ and $X_{s,n-j}$ are identically distributed iff the sample has the exponential distribution.

Its simplest formulation is as follows:

Theorem

Let X_1, X_2 and X_3 be i.i.d. rv's with continuous df F. Two statistics $X_{2,3} - X_{1,3}$ and $\min(X_1, X_2)$ are identically distributed iff F is the exponential df.

Introduction

We are testing the hypothesis

$$H_0 : F \text{ is exponential with the density } \lambda e^{-\lambda x}, x \geq 0, \lambda > 0$$

against the alternative hypothesis $H_1 : F$ is non-exponential df.
We construct two $U-$statistical df H_n and G_n using the formulas

$$H_n(t) = \binom{n}{3}^{-1} \sum_{1 \leq i < j < k \leq n} \mathbf{1}\{X_{2,\{X_i,X_j,X_k\}} - X_{1,\{X_i,X_j,X_k\}} < t\}, \quad t \geq 0,$$

$$G_n(t) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \mathbf{1}\{\min(X_i, X_j) < t\}, \quad t \geq 0.$$

Consider two statistics for testing of H_0:

$$S_n = \int_0^\infty (H_n(t) - G_n(t))\, dF_n(t),$$

$$R_n = \sup_{t \geq 0} \mid H_n(t) - G_n(t) \mid .$$

Limiting distribution of the statistic S_n

The statistic S_n is asymptotically equivalent to the U-statistic of degree 4 with centered kernel

$$\begin{aligned}\Phi(X_i,X_j,X_k,X_l) = &\\ &\frac{1}{4}\mathbf{1}\{X_{2,\{i,j,k\}} - X_{1,\{i,j,k\}} < X_l\} + \frac{1}{4}\mathbf{1}\{X_{2,\{j,k,l\}} - X_{1,\{j,k,l\}} < X_i\}\\ &+ \frac{1}{4}\mathbf{1}\{X_{2,\{k,l,i\}} - X_{1,\{k,l,i\}} < X_j\} + \frac{1}{4}\mathbf{1}\{X_{2,\{l,i,j\}} - X_{1,\{l,i,j\}} < X_k\})\\ &- \frac{1}{12}\sum_{\substack{j_1,j_2,j_3\in(i,j,k,l)\\ j_1\neq j_2\neq j_3}} \mathbf{1}\{\min(X_{j_1},X_{j_2}) < X_{j_3}\}. \qquad (3)\end{aligned}$$

Limiting distribution of the statistic S_n

Let calculate the projection of the kernel (3)

$$\varphi(x) := E(\Phi(X,Y,Z,W)|X=x) = \frac{3}{4}P(X_{2,\{X,Y,Z\}} - X_{1,\{X,Y,Z\}} < W \mid X = x)$$
$$+ \frac{1}{4}P(X_{2,\{Y,Z,W\}} - X_{1,\{Y,Z,W\}} < x) - \frac{3}{12}P(\min(Y,Z) < x)$$
$$- \frac{3}{12}P(\min(Y,Z) < W) - \frac{6}{12}P(\min(x,Y) < Z).$$

After some calculations we obtain that the projection of the kernel Φ is

$$\varphi(x) = \frac{1}{4}\left((3x - \frac{1}{2})e^{-2x} - \frac{1}{6}\right)$$

with the variance

$$\sigma_\varphi^2 = \int_0^\infty \varphi^2(x)d\Phi(x) = \frac{13}{4500}.$$

Hence the kernel (3) is non-degenerate. By Hoeffding's theorem

$$\sqrt{n}S_n \xrightarrow{d} \mathcal{N}(0, \frac{52}{1125}).$$

Limiting distribution of the statistic R_n

The rv $H_n(t) - G_n(t)$ for fixed t is asymptotically equivalent to U-statistic with the kernel depending on $t \geq 0$

$$\Omega(X,Y,Z;t) = \mathbf{1}\{X_{2,\{X,Y,Z\}} - X_{1,\{X,Y,Z\}} < t\} - \frac{1}{3}\mathbf{1}\{\min(X,Y) < t\} - \\ -\frac{1}{3}\mathbf{1}\{\min(Y,Z) < t\} - \frac{1}{3}\mathbf{1}\{\min(X,Z) < t\}.$$

The projection of this kernel for fixed t is equal to

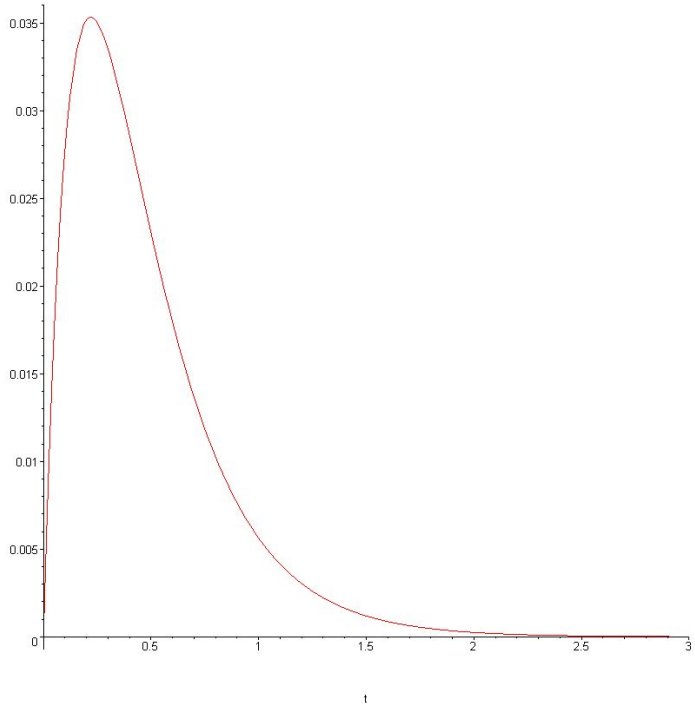
$$\omega(x;t) = \mathbf{1}\{x > t\}(1 - \frac{1}{3}e^{-t} + e^{-2x}(e^t - 1)) + \mathbf{1}\{x < t\}(1 - e^{-2x}) + \\ +e^{-2x}(1 - e^{-2t}) - 1 + \frac{1}{3}e^{-2t}.$$

Limiting distribution of statistic R_n

The variance of the projection has the form

$$\sigma_\omega^2(t) = E\omega^2(X;t) = \frac{4}{45}e^{-3t}(-2e^{-3t}+e^{-t}+1).$$

We conclude that
$\sigma_\omega^2 = \sup_{t\geq 0}\sigma_\omega^2(t) \approx 0.03532.$
This value is important when calculating large deviation asymptotics.



Limiting distribution of statistic R_n

Limiting distribution of the statistic R_n is unknown. Using the methods of Silverman (1983), one can show that the U-empirical process

$$\eta_n(t) = \sqrt{n}\left(H_n(t) - G_n(t)\right), \ t \geq 0,$$

converges weakly as $n \to \infty$ to some centered Gaussian process $\eta(t)$ with complicated covariance. Consequently the sequence of statistics $\sqrt{n}R_n$ converges in distribution to $\sup_{t\geq 0}|\eta(t)|$, which has very complicated distribution (currently unknown). But the critical values for statistics R_n can be found via simulating their sample distribution.

Large deviations and local efficiency of statistics S_n

The kernel Φ is not only centered but bounded, therefore from theorem of Nikitin and Ponikarov we have for $a > 0$

$$\lim_{n\to\infty} n^{-1} \ln \mathbb{P}(S_n > a) = -f(a),$$

where the function f is continuous for sufficiently small $a > 0$, and, moreover,

$$f(a) = \frac{1125}{104} a^2 (1 + o(1)), \text{ при } a \to 0.$$

Large deviations and local efficiency of statistics S_n

Let calculate local Bahadur efficiency of statistics S_n for some alternatives.

- Weibull alternative.
  Under H_1 the observations have the density $(1+\theta)x^{\theta}\exp(-x^{1+\theta}), x \geq 0$. Using the Law of Large Numbers for U-statistics, we see that the limit in probability of S_n under H_1 is equal to

$$b_S(\theta) = P_\theta(X_{2,3} - X_{1,3} < X_4) - P_\theta(X_{1,2} < X_3).$$

  Hence, as $\theta \to 0$ we have $b_S(\theta) \sim \frac{2}{9}\theta$. Therefore the local exact slope of the sequence S_n as $\theta \to 0$ admits the representation

$$c_S(\theta) = b_S(\theta)^2/(16\sigma_\varphi^2) = \frac{125}{117}\theta^2 \sim 1.068\theta^2.$$

  The theoretical upper bound is $2K(\theta) = \pi^2\theta^2/6$ as $\theta \to 0$ (Litvinova, 2004). It follows that the local Bahadur efficiency of our test is equal to

$$eff^B(S) := \lim_{\theta\to 0}\{c_S(\theta)/2K(\theta)\} \approx 0.6495.$$

Large deviations and local efficiency of statistics S_n

Let calculate local Bahadur efficiency of statistics S_n for some alternatives.

- Makeham alternative.

  Suppose that the observations have the density $(1+\theta(1-e^{-x}))\exp(-x-\theta(e^{-x}-1+x))$, $x \geq 0$. The calculations give

  $$b_S(\theta) \sim \frac{1}{24}\theta, \theta \to 0.$$

  The local exact slope has the representation $c_S(\theta) \sim 0.0376\theta^2,\ \theta \to 0$. The upper bound $2K(\theta)$ for exact slopes under Makeham alternative is known as $2K(\theta) \sim \theta^2/12$ (Litvinova, 2004). Consequently, the local Bahadur efficiency of our test for Makeham alternative is equal to 0.4507.

Large deviations and local efficiency of statistics S_n

Let calculate local Bahadur efficiency of statistics S_n for some alternatives.

- Linear failure rate alternative.

  The observations have the density $(1+\theta x)e^{-x-\frac{1}{2}\theta x^2}, x \geq 0$. We get

  $$b_S(\theta) \sim \frac{2}{27}\theta, \theta \to 0.$$

  The local exact slope as $\theta \to 0$ admits the representation $c_S(\theta) \sim 0.1187\theta^2$. It is known that $2K(\theta) \sim \theta^2$ (Litvinova, 2004). Therefore the local Bahadur efficiency of our test for the contamination alternative is equal to 0.1187.

Large deviations and local efficiency of statistic R_n.

The family of kernels $\{\Omega(X, Y, Z; t)\}, t \geq 0$ is not only centered but bounded. Hence using the result on large deviations of the supremum of a family of U-statistics (Nikitin, 2008), we obtain

$$\lim_{n \to \infty} n^{-1} \ln \mathbb{P}(R_n > a) = -f(a),$$

where the function f is continuous for sufficiently small $a > 0$, and, moreover,

$$f(a) = 1.5729a^2(1 + o(1)), \text{ as } a \to 0.$$

Large deviations and local efficiency of statistic R_n

Let calculate the local Bahadur efficiency of R_n for some alternatives.

- Weibull alternative.

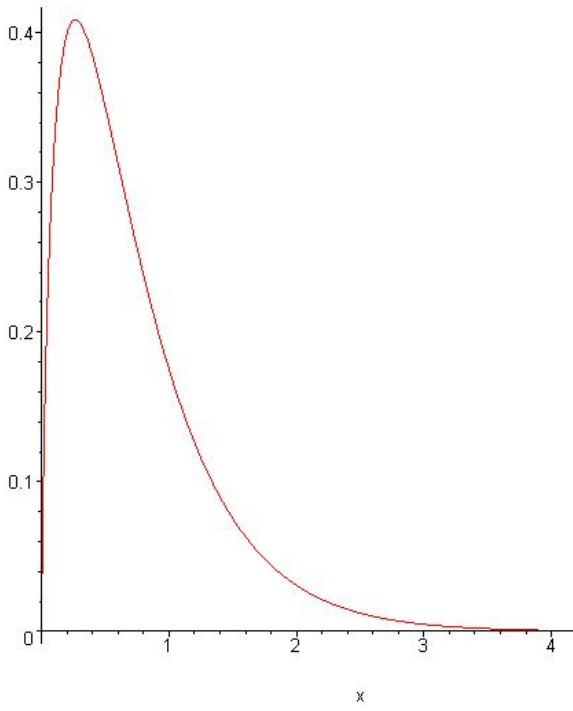
Limit in probability of R_n is

$$b_R(\theta) = \sup_{t \geq 0} |P_\theta(X_{2,3} - X_{1,3} < t) - P_\theta(X_{1,2} < t)|.$$

Hence, as $\theta \to 0$ we have

$$b_R(t, \theta) \sim \left(\frac{2}{3} e^{-2t} (\ln t + \gamma + \ln 3) + \frac{2}{3} Ei(1, 3t) e^t \right) \theta,$$

where $\gamma \approx 0.5772157$, and $Ei(k, z) = \int_1^\infty e^{-zt} t^{-k} dt$.
We conclude that $b_R(\theta) = \sup_{t \geq 0} b_R(t, \theta) \sim 0.4088\theta$, $\theta \to 0$.



Therefore the local exact slope admits the representation as $\theta \to 0$

$$c_R(\theta) \sim 0.5258\theta^2.$$

We know that in this case $2K(\theta) \sim \pi^2\theta^2/6$ as $\theta \to 0$. It follows that the local Bahadur efficiency of our test is 0.3196.

Large deviations and local efficiency of statistic R_n

Let calculate the local Bahadur efficiency of R_n for some alternatives.

- Makeham alternative.
  It is easy to see that for any t

$$b_R(t,\theta) \sim \frac{1}{2}e^{-2t}(1-e^{-t})\theta,\ \theta \to 0.$$

After simple computation we have
$b_R(\theta) = \sup_{t\geq 0} b_R(t,\theta) = b_R(\ln 3 - \ln 2, \theta) \sim 0.(074)\theta,\ \theta \to 0.$Therefore the local exact slope as $\theta \to 0$ looks like

$$c_R(\theta) \sim 0.01735\theta^2.$$

We have already seen that for the Makeham alternative $2K(\theta) \sim \theta^2/12$ as $\theta \to 0$. Consequently, the local Bahadur efficiency of our test is equal to 0.2071.

Large deviations and local efficiency of statistic R_n

Let calculate the local Bahadur efficiency of R_n for some alternatives.

- Linear failure rate alternative.
  We get easily that

$$b_R(t,\theta) \sim \frac{2}{3} t e^{-2t} \theta, \quad b_R(\theta) = \sup_{t \geq 0} b_R(t,\theta) = b_R(\frac{1}{2},\theta) \sim \frac{\theta}{3e}.$$

  Hence we have the asymptotics $c_R(\theta) \sim 0.0473\theta^2$. As seen before, in this case $2K(\theta) \sim \theta^2$ as $\theta \to 0$. Hence, the local Bahadur efficiency of our test is 0.0473.

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- K.Yu.Volkova. On asymptotic efficiency of exponentiality tests based on Rossberg's characterization.// Zapiski POMI, 2009, v.368, 95-109.

Thank you for your attention.