Pricing Bermudan options - A non parametric estimation approach

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Northern Triangular Seminar

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Work in progress with Ali Hamdi, KTH

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Introduction

- Let $\{X, Y\}$ be a pair of r.v. such that Y is square integrable,

- We are seeking an estimator $f_{D_M}(X)$ of the conditional expectation

$$f(X) = \mathbb{E}\left[Y \mid X\right]$$

given a set $D_M := \{x_m, y_m\}_{m=1}^{M}$ of independent samples from $\{X, Y\}$

Problem: Any parametrized assumption of the solution demand a priori knowledge

- Solution: Smoothed least squares.

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Least squares

- Given a priori information about the shape of $\mathbb{E}[Y \mid X]$ we may approximate the conditional expectation from the data D_M by asserting a set of vectors that should hold the key features $\mathbb{E}[Y \mid X]$, call them $\{v_k(X)\}_{k=1}^n$.

- The least squares solution is

$$\sum_{k=1}^{n} \hat{c}_k v_k(X)$$

where

$$\hat{c} = \underset{c \in \mathbb{R}^n}{\operatorname{argmin}} \sum_{m=1}^{M} \left(y_k - \sum_{k=1}^{n} c_k v_k(x_k) \right)^2$$

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Smoothing of least squares solutions

- Assuming no a priori information about the shape of $\mathbb{E}[Y \mid X]$ we are tempted to set $f_{D_M}(x_m) = c_m$ where $c \in \mathbb{R}^M$. This gives ill-posedness since then, the least squares solution is $c_m = y_m, \forall 1 \leq m \leq M$.

- We may demand for the solution to be sufficiently smooth, by penalizing large deviations between nearby points. The smoothed least squares then looks on the form

$$\min_{c \in \mathbb{R}^n} \sum_{m=1}^{M} \left(y_k - f_{D_M}(x_k)\right)^2 + \alpha \left\| \mathcal{A} f_{D_M} \right\|^2 \qquad (1)$$

where $\mathcal{A}$ is some operator put on the solution, e.g. the finite difference scheme for a second derivative

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Approximation of conditional expectation

- Search for an approximation $f_{D_M}(X)$ of the conditional expectation $\mathbb{E}[Y \mid X]$ given data $D_M := \{x_m, y_m\}_{m=1}^M$

- Choose n equidistantly placed points $\{x_k\}_{k=1}^n$ which forms the grid Π in the support of X ($\text{supp}\{X\}$)

- Let $A_k := \{x \in \text{supp}\{X\} : \|s_k - x\| \leq \|s_{k'} - x\|, \forall k' \neq k\}$, called the Voronoi teselation of Π

- Define $f_{D_M}(X) := \sum_{k=1}^n c_k I_{A_k}(X)$

- Assumption is the size of the grid

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properties of f_{D_M}

- f_{D_M} is a quantization of supp$\{Y\}$ which has a regularizing property in itself
- The least squares solution $\hat{f}_{D_M}$ gives the coefficients $\hat{c}_{k,D_M} = \frac{\sum_{m=1}^{M} y_m I_{A_k}(x_k)}{\sum_{m=1}^{M} I_{A_k}(x_k)}$, needs larges samples to converge
- Needs to study the linkage between the points $f_{D_M}(x_m)$
- Smoothing can be used to connect nearby function values
- Need to know if convergence still holds

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Convergens of regularized least squares

Proposition 1 *Assume that* $G(\Pi, \cdot)$ *is some operator on the estimator* $f_{D_M}(X)$ *and that* $\|G(\Pi, x)\|^2 < \infty$ *for any finite* Π *and* x*. Let*

$$\sum_{m=1}^{M} \left(y_k - \hat{f}_{D_M}(x_k) \right)^2 + \alpha \left\| \mathcal{A} \hat{f}_{D_M} \right\|^2 =$$

$$\min \sum_{m=1}^{M} \left(y_k - f_{D_M}(x_k) \right)^2 + \alpha \left\| \mathcal{A} f_{D_M} \right\|^2$$

If $M \to \infty$,

$$\hat{f}_{D_M}(x; \Pi, G, \alpha) \longrightarrow \sum_{k=1}^{n} \mathbb{E}\left[Y \mid X \in A_k \right] I_{A_k}(x) \quad a.s.,$$

for any $\alpha \in \mathbb{R}^+$.

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Convergens of the conditional expectation

Let Π be a grid in $A \subseteq \mathbb{R}^d$ and let

$$\{A_k\}_{k=1}^n, \quad n \geq 1$$

be the Voronoi tessellation which is generated by Π. Define the *outer sets* to be the sets $\{A_k\}_{k \in N_O}$, where

$$N_O = \left\{ k \in (1, \ldots, n) : \sup_{x,y \in A_k} \|x - y\| = \infty \right\}, \quad (2)$$

and define the *inner sets* to be the sets $\{A_k\}_{k \in N_I}$, where

$$N_I = \left\{ k \in (1, \ldots, n) : \sup_{x,y \in A_k} \|x - y\| < \infty \right\}, \quad (3)$$

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Convergens of the conditional expectation

Proposition 2 *In the setting of the previous proposition, let*

$$\{A_{k,n}\}_{k=1}^{n}, \quad n \geq 1$$

be the Voronoi tessellations on $\operatorname{supp}(X)$ *generated by a sequence of grids* Π_n*. Let* $N_{I,n}$ *and* $N_{O,n}$ *be the indexes of the inner and outer sets respectively of tessellation* n*, given by* (2) *and* (3)*. Assume that*

1. f *is twice differentiable, where* $f(X) = \mathbb{E}[Y \mid X]$,
2. f *is locally Lipschitz on* $\left\{x \in \mathbb{R}^d : \|x\| > a\right\}$, *for some* a,
3. $P\left(X \in \cup_{k \in N_{O,n}} A_{k,n}\right) \to 0$, *as* $n \to \infty$,
4. $\max\limits_{k \in N_{I,n}} \sup\limits_{x,y \in A_{k,n}} \|x - y\| \to 0$, *as* $n \to \infty$.

Then, if we first let $M \to \infty$*, we have that*

$$\hat{f}_D(X; \Pi_n, G, \alpha) \to \mathbb{E}[Y \mid X]$$

in L^2*, as* $n \to \infty$*, for any,* G *and* α*.*

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Specifying X and Y

- Consider the markov process S on the space $\{\Omega, \mathcal{F}, \mathbb{Q}\}$ satisfying the SDE

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t^{\mathbb{Q}}$$

where $W_t^{\mathbb{Q}}$ is a BM under the probability measure $\mathbb{Q}$, $r_t = 0$ (for convenience) and $\sigma \in \mathbb{R}$

- Our data set D_M consists of M independent samples of $\{S_{T_0}, S_{T_1}, S_{T_2}\}, T_0 < T_1 < T_2$ where S_{T_0} is deterministic.

- Under $\mathbb{Q}$ every T-claim is an $\mathcal{F}$-martingale, i.e. $V(t, S_T) := \mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \mid \mathcal{F}_t]$ is a $\mathcal{F}_t$-martingale under $\mathbb{Q}$

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Bermudan contract

- A bermudan contract gives the holder the right to the payoff $\Phi(S_T)$ at one time $T \in \mathcal{T} := \{T_1, T_2\}$ which is up to the holder to decide.

- Optimal stopping problem: the value of the contract is

$$V(t, S_t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[\Phi(S_\tau) \mid S_t \right], \ t < T_1$$

- Optimal to redeem the contract at times

$$\{\tau \in \mathcal{T} : \Phi(S_\tau) = V(\tau, S_\tau)\}$$

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Approach for finding solution

- $V(T_2, S_{T_2}) = \Phi(S_{T_2})$
- $V(T_1, S_{T_1}) = \max\{\Phi(S_{T1}), \mathbb{E}^{\mathbb{Q}}[V(T_2, S_{T_2}) \mid S_{T1}]\}$
- $V(T_0, S_{T_0}) = \mathbb{E}^{\mathbb{Q}}[V(T_1, S_{T_1}) \mid S_{T0}]$
- Snell Envelope: Optimal to stop for

$$\{\tau \in \mathcal{T} : \Phi(S_{Ti-1}) \geq \mathbb{E}[V(T_n, S_{T_n}) \mid S_{Tn-1}]\}$$

- Searching for $V(T_{i-1}, S_{T_{i-1}}) := \mathbb{E}[V(T_i, S_{T_i}) \mid S_{i-1}]$, where $V(T_2, S_{T_2}) = \Phi(S_{T_2})$.

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PDE formulation for the solution

• Since the value process $V(t, S_t)$ is an $\mathcal{F}_t$-martingale under $\mathbb{Q}$, assuming sufficiently nice processes S_t, the value process must satisfy

$$(\partial_t + \mathcal{A})\, V(t, s) = 0, \quad T_{i-1} < t < T_i$$

where $\mathcal{A}$ is the infinitesimal operator for S.

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PDE formulation for the solution

- We want to demand for our estimate $f_{D_M}(X)$ to satisfy

$$(\partial_t + \mathcal{A})\, f_{D_M^i} = 0,$$

where $D_M^i := \{s_{T_{i-1},m}, s_{T_i,m}\}_{m=1}^M$, to add known information about the solution apart from the simulations.

- We formulate

$$\hat{c}_{D_M^i} = \underset{c\in\mathbb{R}^n}{\operatorname{argmin}} \sum_{m=1}^{M} \left(V(T_i, s_{T_i,m}) - f_{D_M^i}(s_{T_{i-1},m})\right)^2 + \alpha \left\|(\Delta + B)\, f_{D_M^i}\right\|^2$$

and $\hat{f}_{D_M^i}(X) = \sum_{k=1}^n \hat{c}_{k,D_M^i} I_{A_k}(X)$ where Δ is the discretized time derivative, B is the discretized infinitesimal operator for S. α is the smoothing parameter.

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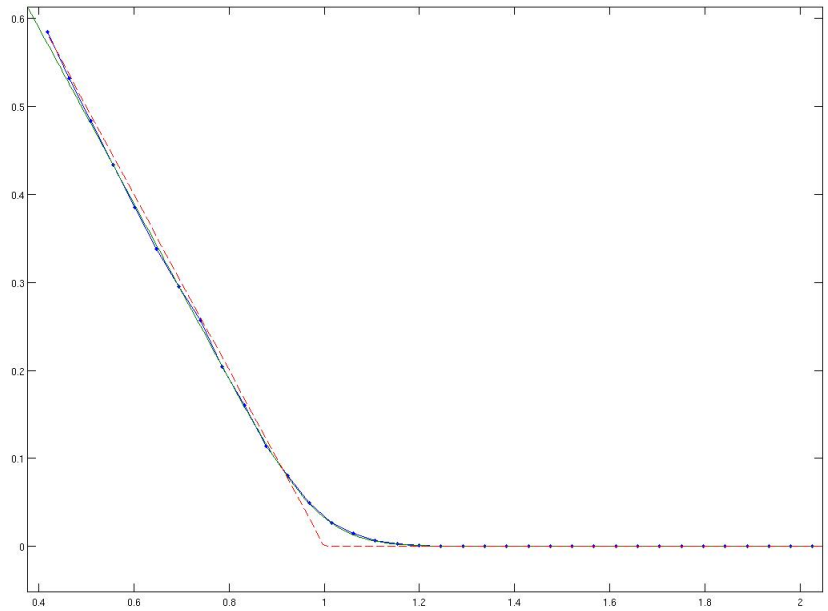
Bermudan put option example



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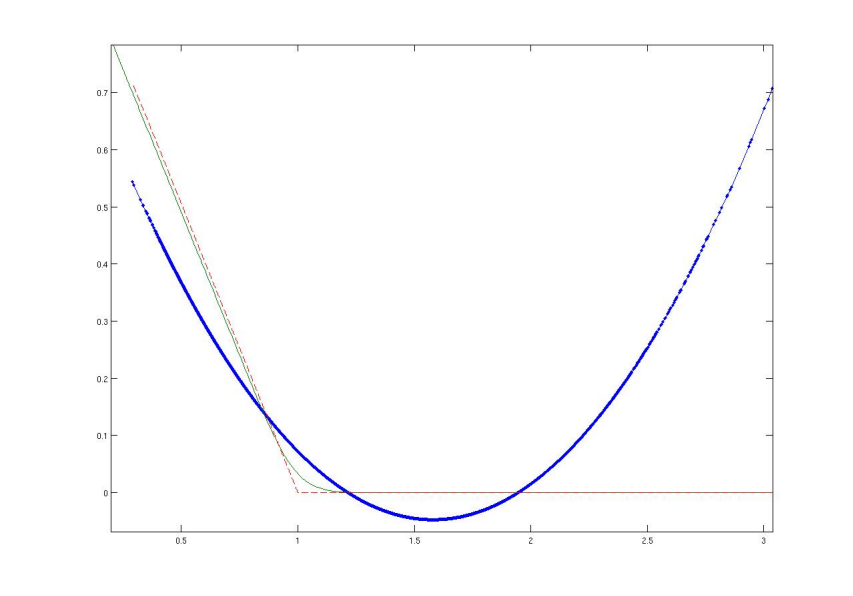
Comparison: Bermudan put option by a priori vector assumption



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Discussion

- Loosen the a priori assumptions
- Introduce relation between PDE and Monte Carlo methods
- Stable in terms of the smoothing parameter α
- Not yet investigated how convergence is affected by the smoothing

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